Estimating the parameters of Burst-type signals

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ESTIMATING THE PARAMETERS OF BURST-TYPE SIGNALS

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ABSTRACT. In this paper, we study a model which exhibits burst-type features such as ECG signals, under certain condition. The model is proposed by Sharma and Sircar (2001) and we call it burst-type signals. It is a generalization of the fixed amplitude sinusoidal model. The amplitudes take a certain deterministic function. We assume that the error random variables are independent and identically distributed. The least square method is proposed to estimate the unknown parameters. We show that the least squares estimators are strongly consistent and find their asymptotic distribution as Gaussian. Some numerical results based on simulations results are reported for illustrative purposes.

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1. INTRODUCTION

Estimation of parameters of a parametric model is of prime interest in almost all problems involved in statistical modelling. In signal detection, once the model in decided, the next step is the estimation of parameters from the given set of observations. The present article addresses the estimation problem of parameters in the following model.

$$y(t) = \sum_{i=1}^{q} A_i \exp[b_i\{1 - \cos(\alpha t + c_i)\}] \cos(\theta_i t + \phi_i) + e(t), \quad t = 1, \ldots, N, \quad (1)$$

where for $i = 1, \ldots, q$, A_i is the amplitude of the carrier wave; b_i and c_i are the gain part and the phase part of the exponential modulation signal; θ_i is the carrier angular frequency, α is the modulation angular frequency and ϕ_i is the phase corresponding to the carrier angular frequency. The number of components present in the signal is denoted by q. The error random variables $\{e(t)\}$ are assumed to be independent and identically

distributed (i.i.d.) with mean zero and finite variance. The model (1) is a sinusoidal model with time-dependent amplitude like $\sum_{i=1}^{q} s_i(t) \cos(\theta_i t + \phi_i) + e(t)$. Here $s_i(t)$ is taking the particular exponential form $\exp[b_i\{1 - \cos(\alpha t + c_i)\}]$ multiplied by a constant A_i. The modulation angular frequency α is assumed to be same through all components which ensures the presence of burst like signal.

The model (1) is proposed by Sharma and Sircar (2001). The authors used the complex-valued model corresponding to (1). In real life, we mostly deal with real-valued observations, hence it is suggested in Sharma and Sircar (2001) to estimate the imaginary part of each of them using Hilbert transform. Then, one has complex-valued data (as observed, estimated by Hilbert transform) and the techniques of complex model can be implemented. Sharma and Sircar (2001) used the proposed model in describing a segment of real electrocardiograph (ECG) signal. In an earlier article, Mukhopadhyay and Sircar (1996) proposed a similar kind of model to analyze an ECG signal. Actually it is the same model as (1) with a different representation of parameters. Also some of the parameters themselves are related with certain relationships. In both these papers the authors analyzed ECG data using some ad-hoc estimation procedures of the unknown parameters. Here we consider the particular form as the real model given in (1) (real in the sense of carrier part). The maim aim of this paper is to provide a proper estimation procedure of the unknown parameters and study their properties in a systematic manner.

ECG signals like many other signals exhibit burst-type features. Similar structures have been observed in the plot of data generated by equation (1) for different set of values. For an example, see Fig. 2 in section 4. Model (1) was proposed to employ one or more amplitude modulated sinusoidal signals with the aim of modelling different features of an ECG output signal separately. Following Sharma and Sircar (2001), we term (1) as the burst-type signal. For another type of amplitude modulated sinusoidal model, the readers are referred to Sircar and Syali (1996) and Nandi, Iyer and Kundu (2004).

We discuss the problem of parameter estimation of the burst-type signal in presence of i.i.d. noise. We implement the least square method for estimation and study the properties of the estimators. It is known that the constant amplitude multiple sinusoidal model does not satisfy the sufficient conditions of Jennrich (1969) or Wu (1981) for the LSEs to be consistent. Model (1), being a more complicated general model does not satisfy them. However, the special structure of the model allows us to establish the strong consistency and the large sample distribution of the LSEs of the unknown parameters.

The paper is organized as follows. In section 2, we state the asymptotic properties of the LSEs for single burst-type signal ($q = 1$). The results for general q are discussed in section 3. Numerical results are presented in section 4 and we conclude the paper in section 5. All proofs are provided in Appendix A.

2. Asymptotic Distribution of LSEs for Single Burst-Type Signal

In this section, we consider the case when the number of signals $q = 1$ and write the model (1) as

$$y(t) = A \exp[b\{1 - \cos(\alpha t + c)\}] \cos(\theta t + \phi) + e(t), \quad t = 1, \ldots, N. \tag{2}$$

It is assumed that $|b| \leq J$, therefore, $e^{b\cos(\alpha t)} \leq e^{|b|} \leq e^J = K$(say) a finite constant, $\forall \, t$ and the frequencies $\alpha, \theta \in (0, \pi)$; the phases $c, \phi \in (-\pi, \pi)$; $A \in \mathcal{R}$ is a finite constant; and $e(t)$ is i.i.d. with mean zero and finite variance σ^2. Thus, $\{y(t)\}$ is a sequence of mean non-stationary random variables. Our problem is to estimate the unknown parameters, A, b, α, c, θ and ϕ from a given sample of size N.

Define the parameter vector $\boldsymbol{\eta} = (A, b, \alpha, c, \theta, \phi)$ and $\boldsymbol{\eta}^0$ denote the true value of $\boldsymbol{\eta}$. The LSE of $\boldsymbol{\eta}$, $\hat{\boldsymbol{\eta}}$ for the model (2) minimizes the following residual sum of squares

$$Q(\boldsymbol{\eta}) = \sum_{t=1}^{N} \left[y(t) - A \exp[b\{1 - \cos(\alpha t + c)\}] \cos(\theta t + \phi) \right]^2, \tag{3}$$

with respect to $\boldsymbol{\eta}$. In the following we first state the consistency property of $\hat{\boldsymbol{\eta}}$ in the following Theorem 2.1.

THEOREM 2.1. Let $\boldsymbol{\eta}^0 = (A^0, b^0, \alpha^0, c^0, \theta^0, \phi^0)$, the true parameter value, be an interior point of the parameter space $\{(-\infty, \infty) \times (-\log(K), \log(K)) \times (0, \pi) \times (-\pi, \pi) \times (0, \pi) \times (-\pi, \pi)\}$, where K is a large positive real number, so that $\exp\{|b^0|\} < K$. If the error random variables $e(t)$s are i.i.d., then $\hat{\boldsymbol{\eta}}$, the LSE of $\boldsymbol{\eta}^0$, is a strongly consistent estimator of $\boldsymbol{\eta}^0$.

PROOF OF THEOREM 2.1: See in Appendix A.

In rest of this section, we develop the joint asymptotic distribution of the LSEs of the unknown parameters for single component model (2). We use the usual Taylor series expansion. We denote the first derivative vector of $Q(\boldsymbol{\eta})$ as $Q'(\boldsymbol{\eta})$ which is of order 1×6 and the 6×6 matrix of second order derivatives as $Q''(\boldsymbol{\eta})$. Now expanding $Q'(\hat{\boldsymbol{\eta}})$ around $\boldsymbol{\eta}^0$ by multivariate Taylor series up to first order terms, we have

$$Q'(\hat{\boldsymbol{\eta}}) - Q'(\boldsymbol{\eta}^0) = (\hat{\boldsymbol{\eta}} - \boldsymbol{\eta}^0) Q''(\bar{\boldsymbol{\eta}}), \tag{4}$$

where $\bar{\boldsymbol{\eta}}$ is a point between $\hat{\boldsymbol{\eta}}$ and $\boldsymbol{\eta}^0$. Now define a diagonal matrix of order six as follows:

$$\mathbf{D} = \mathrm{diag}\{N^{-\frac{1}{2}}, N^{-\frac{1}{2}}, N^{-\frac{3}{2}}, N^{-\frac{1}{2}}, N^{-\frac{3}{2}}, N^{-\frac{1}{2}}\}. \tag{5}$$

Since $Q'(\hat{\boldsymbol{\eta}}) = 0$, (4) can be written as

$$(\hat{\boldsymbol{\eta}} - \boldsymbol{\eta}^0)\mathbf{D}^{-1} = -\left[\mathbf{Q}'(\boldsymbol{\eta^0})\mathbf{D}\right]\left[\mathbf{D}\mathbf{Q}''(\bar{\boldsymbol{\eta}})\mathbf{D}\right]^{-1}. \tag{6}$$

We can write (6) because $\left[\mathbf{D}\mathbf{Q}''(\bar{\boldsymbol{\theta}})\mathbf{D}\right]$ is an invertible matrix *a.e.* for large N. From Theorem 2.1, it follows that $\hat{\boldsymbol{\eta}}$ converges *a.s.* to $\boldsymbol{\eta}^0$ and since each element of $Q''(\boldsymbol{\eta})$ is a continuous function of $\boldsymbol{\theta}$, therefore,

$$\lim_{N\to\infty}\left[\mathbf{D}\mathbf{Q}''(\bar{\boldsymbol{\eta}})\mathbf{D}\right] = \lim_{N\to\infty}\left[\mathbf{D}\mathbf{Q}''(\boldsymbol{\theta^0})\mathbf{D}\right] = 2\mathbf{\Sigma}(\boldsymbol{\eta}^0) \quad \text{(say)}. \tag{7}$$

In obtaining the exact form of the limit matrix $\mathbf{\Sigma}(\boldsymbol{\eta})$, let us write $\boldsymbol{\eta} = (A, \boldsymbol{\xi})$, where $\boldsymbol{\xi} = (b, \alpha, c, \theta, \phi)$. Then $\mathbf{\Sigma}(\boldsymbol{\eta}) = e^{2b^0}\mathbf{\Delta}(\boldsymbol{\eta}^0)$, where

$$\mathbf{\Delta}(\boldsymbol{\eta}) = \begin{bmatrix} \delta_1(0) & A\delta_5(0) & Ab\delta_6(1) & Ab\delta_6(0) & A\delta_7(1) & A\delta_7(0) \\ A\delta_5(0) & A^2\delta_2(0) & A^2b\delta_8(1) & A^2b\delta_8(0) & -A^2\delta_9(1) & -A^2\delta_9(0) \\ Ab\delta_6(1) & A^2b\delta_8(1) & A^2b^2\delta_3(2) & A^2b^2\delta_3(1) & -A^2b\delta_{10}(2) & -A^2b\delta_{10}(1) \\ Ab\delta_6(0) & A^2b\delta_8(0) & A^2b^2\delta_3(1) & A^2b^2\delta_3(0) & -A^2b\delta_{10}(1) & -A^2b\delta_{10}(0) \\ Ab\delta_7(1) & -A^2\delta_9(1) & -A^2b\delta_{10}(2) & -A^2b\delta_{10}(1) & A^2\delta_4(2) & A^2\delta_4(1) \\ A\delta_7(0) & -A^2\delta_9(0) & -A^2b\delta_{10}(1) & -A^2b\delta_{10}(0) & A^2\delta_4(1) & A^2\delta_4(0) \end{bmatrix}, \tag{8}$$

where $\delta_k(m) = \delta_k(\boldsymbol{\eta}, m)$, $m = 0, 1, 2$, $k = 1, \ldots, 10$ are defined in Appendix B. Now let us consider the random vector $Q'(\boldsymbol{\eta}^0)\mathbf{D}$ of order 1×6,

$$Q'(\boldsymbol{\eta}^0)\mathbf{D} = \begin{bmatrix} -\frac{2}{\sqrt{N}}\sum_{t=1}^{N} e(t)\exp\{b^0(1 - \cos(\alpha^0 t + c^0))\}\cos(\theta^0 t + \phi^0) \\[2mm] -\frac{2}{\sqrt{N}}A^0\sum_{t=1}^{N} e(t)\exp\{b^0(1 - \cos(\alpha^0 t + c^0))\}(1 - \cos(\alpha^0 t + c^0))\cos(\theta^0 t + \phi^0) \\[2mm] -\frac{2}{N^{\frac{3}{2}}}A^0 b^0\sum_{t=1}^{N} te(t)\exp\{b^0(1 - \cos(\alpha^0 t + c^0))\}\sin(\alpha^0 t + c^0)\cos(\theta^0 t + \phi^0) \\[2mm] -\frac{2}{\sqrt{N}}A^0 b^0\sum_{t=1}^{N} e(t)\exp\{b^0(1 - \cos(\alpha^0 t + c^0))\}\sin(\alpha^0 t + c^0)\cos(\theta^0 t + \phi^0) \\[2mm] \frac{2}{N^{\frac{3}{2}}}A^0\sum_{t=1}^{N} te(t)\exp\{b^0(1 - \cos(\alpha^0 t + c^0))\}\sin(\theta^0 t + \phi^0) \\[2mm] \frac{2}{\sqrt{N}}A^0\sum_{t=1}^{N} e(t)\exp\{b^0(1 - \cos(\alpha^0 t + c^0))\}\sin(\theta^0 t + \phi^0) \end{bmatrix}.$$

All the elements of $Q'(\boldsymbol{\eta}^0)\mathbf{D}$ satisfy the Lindeberg-Feller's condition, therefore it converges to a 6-variate normal distribution. Using the limits given in Appendix B, it follows that

$$Q'(\boldsymbol{\eta}^0)\mathbf{D} \to \mathcal{N}_6\left(\mathbf{0}, 4\sigma^2\mathbf{\Sigma}(\boldsymbol{\eta}^0)\right). \tag{9}$$

Therefore, using (7) and (9) in (6), we have the asymptotic distribution as

$$(\hat{\boldsymbol{\eta}} - \boldsymbol{\eta}^0)\mathbf{D}^{-1} \to \mathcal{N}_6\Big(\mathbf{0}, \sigma^2\boldsymbol{\Sigma}^{-1}(\boldsymbol{\eta}^0)\Big). \tag{10}$$

Now using the inequality (25), we have shown in Appendix B, that $\delta_k(\boldsymbol{\xi}, p) = 0$ for $k = 6, \ldots, 10$ and $\delta_5(\boldsymbol{\xi}, p) = \delta_1(\boldsymbol{\xi}, p)$, $p = 0, 1, 2, \ldots$ Thus

$$\boldsymbol{\Delta}(\boldsymbol{\eta}) = \begin{bmatrix} \boldsymbol{\Delta}_1(\boldsymbol{\eta}) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Delta}_2(\boldsymbol{\eta}) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{\Delta}_3(\boldsymbol{\eta}) \end{bmatrix},$$

where

$$\boldsymbol{\Delta}_1(\boldsymbol{\eta}) = \begin{bmatrix} \delta_1(0) & A\delta_1(0) \\ A\delta_1(0) & A^2\delta_2(0) \end{bmatrix}, \quad \boldsymbol{\Delta}_2(\boldsymbol{\eta}) = A^2b^2\begin{bmatrix} \delta_3(2) & \delta_3(1) \\ \delta_3(1) & \delta_3(0) \end{bmatrix}, \quad \boldsymbol{\Delta}_3(\boldsymbol{\eta}) = A^2\begin{bmatrix} \delta_4(2) & \delta_4(1) \\ \delta_4(1) & \delta_4(0) \end{bmatrix}. \tag{11}$$

Thus, the asymptotic variance-covariance matrix of $(\hat{\boldsymbol{\eta}} - \boldsymbol{\eta}^0)\mathbf{D}^{-1}$ is

$$\sigma^2\boldsymbol{\Sigma}^{-1}(\boldsymbol{\eta}^0) = \sigma^2 e^{-2b^0}\boldsymbol{\Delta}^{-1}(\boldsymbol{\eta}^0) = \sigma^2 e^{-2b^0}\begin{bmatrix} \boldsymbol{\Delta}_1^{-1}(\boldsymbol{\eta}) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Delta}_2^{-1}(\boldsymbol{\eta}) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{\Delta}_3^{-1}(\boldsymbol{\eta}) \end{bmatrix} \tag{12}$$

with

$$\boldsymbol{\Delta}_1^{-1}(\boldsymbol{\eta}) = \frac{1}{\delta_2(0) - \delta_1(0)}\begin{bmatrix} \frac{\delta_2(0)}{\delta_1(0)} & -\frac{1}{A} \\ -\frac{1}{A} & \frac{1}{A^2} \end{bmatrix}, \quad \boldsymbol{\Delta}_2^{-1}(\boldsymbol{\eta}) = \frac{1}{A^2b^2[\delta_3(2)\delta_3(0) - \delta_3(1)^2]}\begin{bmatrix} \delta_3(0) & -\delta_3(1) \\ -\delta_3(1) & \delta_3(2) \end{bmatrix},$$

and

$$\boldsymbol{\Delta}_3^{-1}(\boldsymbol{\eta}) = \frac{1}{A^2[\delta_4(2)\delta_4(0) - \delta_4(1)^2]}\begin{bmatrix} \delta_4(0) & -\delta_4(1) \\ -\delta_4(1) & \delta_4(2) \end{bmatrix}.$$

So (12) implies that the pairs of parameters (A, b), (α, c) and (θ, ϕ) are asymptotically independent to each other whereas the parameters in each pair are asymptotically dependent. In comparison to the constant amplitude frequency model, in present case, the amplitude is a deterministic function of the time variable t, which is equal to $A\exp\{b(1 - \cos(\alpha t + c))\}$. Thus, it depends on parameters A, b, α and c. For constant amplitude case least square estimator of amplitude is independent of that of frequency and phase. Along the same line, we observe that for model (2) that the estimators of amplitude parameters $\hat{A}, \hat{b}, \hat{\alpha}$ and $\hat{c}$ are independent of frequency and phase estimators, $\hat{\theta}$ and $\hat{\phi}$.

Remark 1. The rate of convergence of each of A, b, c and ϕ is $O_p(N^{-1/2})$ whereas for the carrier angular frequency θ as well as for the modulating frequency α, the rate of convergence is $O_p(N^{-3/2})$. In comparison to constant amplitude sinusoidal model, α is a parameter of the amplitude. In this case it is observed that its LSE has the same convergence rate as that of the carrier angular frequency θ as α, multiplied by t

is appearing as the argument of a cosine function in the time varying amplitude of the model.

3. Theoretical Properties of LSEs for general p

In this section, we provide the asymptotic results of the LSEs for model (1). Let us write $\boldsymbol{\psi}_k = (A_k, b_k, c_k, \theta_k, \phi_k)$, $k = 1, \ldots, q$ and $\boldsymbol{\psi} = (\boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_q, \alpha)$ be the parameter vector. Then the LSE of $\boldsymbol{\psi}$ is obtained by minimizing the residual sum of squares which can be defined similarly as (3). Let $\hat{\boldsymbol{\psi}}$ and $\boldsymbol{\psi}^0$ denote the least squares estimator and the true value of $\boldsymbol{\psi}$. The consistency of $\hat{\boldsymbol{\psi}}$ follows similarly as the consistency of $\hat{\boldsymbol{\eta}}$, considering the parameter vector as $\boldsymbol{\psi}$ instead of $\boldsymbol{\eta}$. We state the asymptotic distribution of $\hat{\boldsymbol{\psi}}$ here. The proof involves routine calculations and the use of multiple Taylor series and a central limit theorem similarly as in section 2.

For asymptotic distribution of $\hat{\boldsymbol{\psi}}$, following the notation used in previous section, we write $\boldsymbol{\xi}_j = (b_j, \alpha, c_j, \theta_j, \phi_j)$, $j = 1, \ldots, q$; $\delta_k(\boldsymbol{\xi}_j, p) = \delta_k^j(p)$, $k = 1(1)4$, $j = 1(1)q$, $p = 0, 1, 2, \ldots$. Now let $\mathbf{D}_q$ be a diagonal matrix of order $(5q + 1)$, defined as follows;

$$
\mathbf{D}_q = \begin{bmatrix}
\mathbf{D}_1 & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{D}_1 & \cdots & \mathbf{0} & \mathbf{0} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{D}_1 & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & N^{-\frac{3}{2}}
\end{bmatrix},
$$

where $\mathbf{D}_1 = \text{diag}\{N^{-\frac{1}{2}}, N^{-\frac{1}{2}}, N^{-\frac{1}{2}}, N^{-\frac{3}{2}}, N^{-\frac{1}{2}}\}$. Then the asymptotic distribution of $\hat{\boldsymbol{\psi}}$ is

$$
(\hat{\boldsymbol{\psi}} - \boldsymbol{\psi}^0)\mathbf{D}_q^{-1} \xrightarrow{d} \mathcal{N}_{5q+1}\left(\mathbf{0}, \sigma^2 \mathbf{G}_q^{-1}(\boldsymbol{\psi}^0)\right), \tag{13}
$$

$$
\mathbf{G}_q(\boldsymbol{\psi}) = \begin{bmatrix}
e^{2b_1}\boldsymbol{\Gamma}(\boldsymbol{\psi}_1) & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{w}(\boldsymbol{\psi}_1) \\
\mathbf{0} & e^{2b_2}\boldsymbol{\Gamma}(\boldsymbol{\psi}_2) & \cdots & \mathbf{0} & \mathbf{w}(\boldsymbol{\psi}_2) \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\mathbf{0} & \cdots & \mathbf{0} & e^{2b_q}\boldsymbol{\Gamma}(\boldsymbol{\psi}_q) & \mathbf{w}(\boldsymbol{\psi}_q) \\
\mathbf{w}'(\boldsymbol{\psi}_1) & \mathbf{w}'(\boldsymbol{\psi}_2) & \cdots & \mathbf{w}'(\boldsymbol{\psi}_q) & f^*
\end{bmatrix}.
$$

Here $f^* = \sum_{j=1}^{q} e^{2b_j} A_j^2 b_j^2 \delta_3^j(2)$ and $\mathbf{w}'(\boldsymbol{\psi}_j) = \begin{pmatrix} 0 & 0 & e^{2b_j} A_j^2 b_j^2 \delta_3^j(1) & 0 & 0 \end{pmatrix}$. The submatrix $\boldsymbol{\Gamma}(\boldsymbol{\psi}_j)$ is obtained by deleting third row and third column of matrix $\boldsymbol{\Delta}(\boldsymbol{\eta})$ and

replacing (A, b, c, θ, ϕ) by $(A_j, b_j, c_j, \theta_j, \phi_j)$. Thus,

$$\boldsymbol{\Gamma}(\boldsymbol{\psi}_j) = \begin{bmatrix} \delta_1^j(0) & A_j\delta_1^j(0) & 0 & 0 & 0 \\ A_j\delta_1^j(0) & A_j^2\delta_2^j(0) & 0 & 0 & 0 \\ 0 & 0 & A_j^2 b_j^2 \delta_3^j(0) & 0 & 0 \\ 0 & 0 & 0 & A_j^2\delta_4^j(2) & A_j^2\delta_4^j(1) \\ 0 & 0 & 0 & A_j^2\delta_4^j(1) & A_j^2\delta_4^j(0) \end{bmatrix}.$$

The inverse matrix $\mathbf{G}_q^{-1}(\boldsymbol{\psi}^0)$ is given by

$$\mathbf{G}_q^{-1}(\boldsymbol{\psi}) =$$

$$\begin{pmatrix} e^{-2b_1}\boldsymbol{\Gamma}(\boldsymbol{\psi}_1)^{-1} + \mathbf{F}_{11} & \mathbf{F}_{12} & \cdots & \mathbf{F}_{1q} & e^{-2b_1}\boldsymbol{\Gamma}(\boldsymbol{\psi}_1)\mathbf{w}(\boldsymbol{\psi}_1) \\ \mathbf{F}_{21} & e^{-2b_2}\boldsymbol{\Gamma}(\boldsymbol{\psi}_2)^{-1} + \mathbf{F}_{22} & \cdots & \mathbf{F}_{2q} & e^{-2b_2}\boldsymbol{\Gamma}(\boldsymbol{\psi}_2)\mathbf{w}(\boldsymbol{\psi}_2) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{F}_{q1} & \mathbf{F}_{q2} & \cdots & e^{-2b_q}\boldsymbol{\Gamma}(\boldsymbol{\psi}_q)^{-1} + \mathbf{F}_{qq} & e^{-2b_q}\boldsymbol{\Gamma}(\boldsymbol{\psi}_q)\mathbf{w}(\boldsymbol{\psi}_q) \\ e^{-2b_1}\boldsymbol{\Gamma}(\boldsymbol{\psi}_1)\mathbf{w}(\boldsymbol{\psi}_1) & e^{-2b_2}\boldsymbol{\Gamma}(\boldsymbol{\psi}_2)\mathbf{w}(\boldsymbol{\psi}_2) & \cdots & e^{-2b_q}\boldsymbol{\Gamma}(\boldsymbol{\psi}_q)\mathbf{w}(\boldsymbol{\psi}_q) & 1/d^* \end{pmatrix}$$

where $d^* = \sum_{j=1}^q e^{2b_j} A_j^2 b_j^2 \left[\delta_3^j(2) - \frac{\delta_3^j(1)^2}{\delta_3^j(0)} \right]$ and $\mathbf{F}_{jk}$, $j, k = 1, \ldots, q$ is a 5×5 symmetric matrix whose all elements are zero except $(3,3)$ element which is equal to $\frac{1}{d^*}\frac{\delta_3^j(1)\delta_3^k(1)}{\delta_3^j(0)\delta_k^k(0)}$.

In the previous section, we have observed that for $q = 1$, the estimator of α only asymptotically depends on the estimator of c_1, but for $q > 1$, the estimator of α depends on all c_j, $j = 1, \ldots, q$ for large N. This is expected also as in the model, the modulating angular frequency is same for each component.

4. NUMERICAL EXPERIMENTS

In this section, for illustration, we present some numerical results based on simulations. We consider the model (1) with $q = 4$. Data are generated using the following values;

$$A_1 = 5.70166706 \times 10^{-5}, \quad A_2 = 3.3049426 \times 10^{-25},$$

$$A_3 = 1.002 \times 10^{-3}, \quad A_4 = 3.7575 \times 10^{-4}$$

$$b_1 = 4.989495798, \, b_2 = 28.886554622, \quad b_3 = 2.62605042, \, b_4 = 2.62605042,$$

$$c_1 = .1904, \, c_2 = 2.05632, \quad c_3 = 5.9024, \, c_4 = 3.2368, \tag{14}$$

$$\theta_1 = .07616, \, \theta_2 = .26656, \quad \theta_3 = .03808, \, \theta_4 = .03808,$$

$$\phi_1 = 1.166198163, \, \phi_2 = 18.007071552 \quad \phi_3 = 10.928948246 \, \phi_4 = 6.378392654$$

$$\alpha = .03808.$$

These parameter values have been obtained from Muthopadhyay and Sircar (1996). We consider the case when carrier frequencies are harmonics of the modulation angular

frequency i.e. θ_i's are integer multiples of α. As ECG signal is periodic, θ_i has to be some integer multiple of α. We use the sample size $N = 100$ for simulations study. The error random variables are independent and identically distributed $\mathcal{N}(0, \sigma^2)$. We have reported results for $\sigma^2 = .00001$ and $.0001$. We generate the data using (1) and the true parameter values as mentioned above. The LSEs of different parameters are estimated by minimizing the residual sum of squares given in (3). The minimization has been carried out by using the simplex method and for that purposes, routines given in Press et al. (1987) have been used. The true parameter values are taken as the initial estimates. Though with $q = 4$, the parameter set contains 21 parameters and the optimization is taken place in quite a higher dimensional space, the final results are quite satisfactory. We replicate the procedure of data generation and estimation of parameters 1000 times, then calculate the average estimate (AVEST) and the mean squared error (MSE) of each parameter. We summarize results in Tables 1 and 2 for error variance .00001 and .0001 respectively. In section 3, we have developed the asymptotic distribution of LSEs when $q > 1$, which may be used for interval estimation for finite samples . As the limiting distribution in section 3 involves several limiting quantities, for samples of moderate size, we wish to see how the percentile bootstrap (boot-p) method - a well-used form of parametric bootstrap, works. In Nandi, Iyer and Kundu (2002) and Kundu and Nandi (2006), similar bootstrapping method has been used for interval estimation in case of sinusoidal frequency model and real chirp signal model respectively. In each replication of our experiment, we generate 1000 bootstrap resamples using the estimated parameters and then the bootstrap confidence intervals using the bootstrap quantiles at 95% nominal level. Thus, from the replicated experiment, we have 1000 intervals for each parameter. Then we estimate the 95% boot-p coverage probability by calculating the proportion covering the true value of the parameter. We report them as B-CVP in Tables 1 and 2. In addition, we also report the average length of the boot-p confidence intervals as B-AVLEN.

We observe that the average estimates are quite good, which is reflected in the fact that biases (not reported) are quite small in absolute values. The MSEs are reasonably small. Their dependence on the magnitude of the constant amplitude A^0 is quite clear. The asymptotic variances of all parameters, except A, are reciprocally proportional to A^{0^2} which is also visible in MSEs to some extent. As the error variance increases the biases and MSEs increase. The bootstrap coverage percentages attain the nominal level in all cases, except in c_1 and A_3. B-CVPs are quite high and close to 1 in both cases considered here. The average confidence length is quite small, depending on the magnitude of the true parameter value and the order of the asymptotic variance is reflected in the length of the interval in each case.

Apart from the replicated experiment, we consider the same model in data analysis format. For that, we generate the data of sample size $N = 500$ and the error variance $\sigma^2 = .05$. The generated data are plotted when no noise is present, in Fig. 1. The corrupted version of the same data set with $\sigma^2 = .05$ are presented in Fig. 2. Now we estimate the parameters by minimizing the residual sum of square and plugging them in the model, we estimate the signal. It is plotted in Fig. 3. Now if the level of noise increases to $\sigma^2 = 1.0$, then the form of the original signal (Fig. 1) is totally distorted. We wanted to see, whether in this case it is possible to extract the signal. The signal with noise level $\sigma^2 = 1.0$ is plotted in Fig. 4 and one can see that the form is totally destroyed. Now we estimate the LSEs and the estimated signal is plotted in Fig. 5. In both cases, the LSEs are able to estimate quite satisfactorily and the estimated graphs (Figs. 3 and 5) match well with the no-noise signal. In simulated experiments, we have used percentile bootstrap method for interval estimation. In single data example, we used the same method. Using 1000 bootstrap resamples, we estimate the 95% confidence intervals of each parameter in case of $\sigma^2 = .05$ and $\sigma^2 = 1.0$. They are reported in Table 3 along with their point estimates. We see that bias is negligible in most of the cases when $\sigma^2 = .05$. Though, it is not true, in each parameter estimate with larger noise level, the estimated signal is quite good. The boot-p confidence intervals in each case include the true parameter value and the order of asymptotic variance is reflected in the length of the interval.
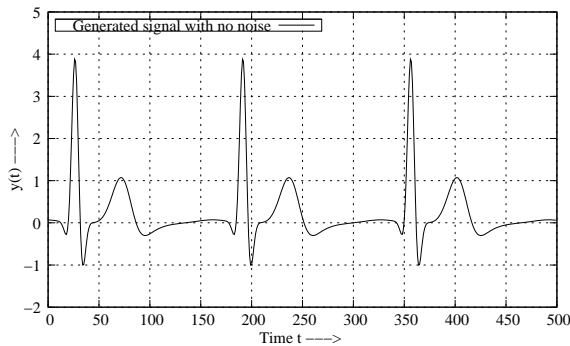

FIGURE 1. The signal using model (14) subjected to no noise.
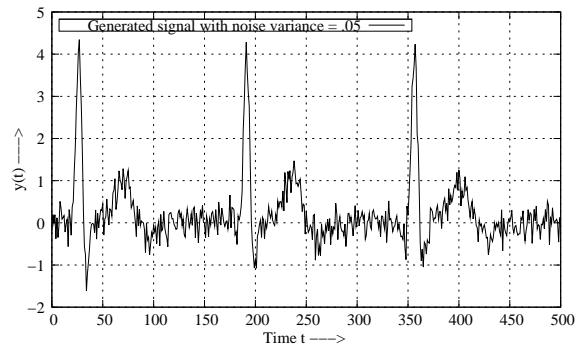
FIGURE 2. The same signal as in Fig. 1 corrupted by noise (variance=.05).

5. CONCLUSION

In the present article, we study a comparatively new model. The model is proposed by Sharma and Sircar (2001) to analyse ECG data. We propose the least square estimators to estimate the unknown parameters. The model is a particular functional amplitude

TABLE 1. True value, average estimate, MSE, Boot-p coverage probability and average length of LSE of different parameter for model (1) with $p = 4$ when sample size is $N = 100$ and $\sigma^2 = .00001$.

PARA	T-VALUE	AVEST	MSE	B-AVLEN	B-CVP
A_1	5.70166706e-5	5.70145239e-5	1.02906152e-16	3.8605424e-8	0.957
b_1	4.989495798	4.9894557	3.11893842e-7	2.19602301e-3	0.942
c_1	.1904	.190409496	1.05839382e-9	1.20837358e-4	1.0
θ_1	.07616	.0761569068	8.81921092e-10	1.13200498e-4	0.950
ϕ_1	1.166198163	1.16615772	1.38404461e-8	40067191e-4	0.978
A_2	3.3049426e-25	3.31255587e-25	0.	1.32071863e-26	0.924
b_2	28.886554622	28.8854427	2.6715863e-5	1.9776715e-2	0.959
c_2	2.05632	2.0562737	4.35365415e-8	7.97438901e-4	0.981
θ_2	.26656	.266568691	3.24213345e-9	7.97438901e-4	0.981
ϕ_2	18.007071552	18.0068378	2.85143915e-6	6.59841439e-3	0.957
A_3	1.002e-3	1.00206817e-3	1.42561622e-13	1.72421574e-6	1.0
b_3	2.62605042	2.6258955	3.92893924e-7	2.37937062e-3	0.958
c_3	5.9024	5.90223742	1.37313691e-6	4.46954789e-3	0.961
θ_3	.03808	0.0381004699	1.96623624e-8	5.32256323e-4	0.924
ϕ_3	10.928948246	10.9286032	2.77993604e-6	6.36323635e-3	0.958
A_4	3.7575e-4	3.75736476e-4	4.33552713e-15	2.54767286e-7	0.974
b_4	2.62605042	2.62593865	2.47176132e-7	1.90590811e-3	0.965
c_4	3.2368	3.23656392	9.79961783e-7	3.74561013e-3	0.969
θ_4	.03808	.0380559228	2.22348078e-8	3.74561013e-3	0.969
ϕ_4	6.378392654	6.37809134	3.65015421e-6	7.19337584e-3	0.963
α	.03808	.0380816758	8.213831e-11	3.51257004e-5	0.936

sinusoidal model and the form is quite complicated specially when the number compo-
nents is large. However, we observe that the LSEs satisfy the large sample properties,
strong consistency and asymptotic normality, under the assumptions that errors are i.i.d.
In numerical experiments, we find that one can use the percentile bootstrap method for
interval estimation. We have used the true values as the starting estimates in simula-
tions, but given a data set, it is extremely important to guess the initial estimates. We
feel that the use of different combination of periodogram function may be used for this
purpose. Further work is required in this direction and these computational aspects will
be addressed elsewhere. Another point we would like to mention that we have assumed
that the number of components is known. So the estimation of the number of burst

TABLE 2. True value, average estimate, MSE, Boot-p coverage probability and average length of LSE of different parameter for model (1) with $p = 4$ when sample size is $N = 100$ and $\sigma^2 = .0001$.

PARA	T-VALUE	AVEST	MSE	B-AVLEN	B-CVP
A_1	5.70166706e-5	5.70133125e-5	1.15013776e-15	1.25403702e-7	0.963
b_1	4.989495798	4.9893713	3.14257386e-6	7.05490913e-3	0.938
c_1	.1904	.190418169	1.13296146e-8	3.31763964e-4	0.998
θ_1	.07616	.0761515722	8.63750405e-9	3.634319e-4	0.955
ϕ_1	1.166198163	1.16611457	1.86317749e-7	1.43747276e-3	0.977
A_2	3.3049426e-25	3.32128243e-25	0.	4.37729703e-26	0.942
b_2	28.886554622	28.8843803	2.63705937e-4	6.29487708e-2	0.960
c_2	2.05632	2.0562005	4.93348864e-7	2.79974216e-3	0.988
θ_2	.26656	.266580075	3.70137982e-8	7.67455786e-4	0.952
ϕ_2	18.007071552	18.0065517	3.29326722e-5	2.22241636e-2	0.964
A_3	1.002e-3	1.0022046e-3	1.29145674e-12	4.84236125e-6	0.997
b_3	2.62605042	2.62576246	4.06719164e-6	7.53721222e-3	0.966
c_3	5.9024	5.90209675	1.36529898e-5	1.41755259e-2	0.966
θ_3	.03808	.0381309576	1.99165257e-7	1.75511267e-3	0.931
ϕ_3	10.928948246	10.9282236	3.26206246e-5	2.13409178e-2	0.963
A_4	3.7575e-4	3.7571229e-4	5.05222491e-14	8.09413677e-7	0.983
b_4	2.62605042	2.6258409	2.63120751e-6	6.04074216e-3	0.964
c_4	3.2368	3.23635507	9.69904522e-6	1.20087676e-2	0.967
θ_4	.03808	.0380305238	1.88419975e-7	1.6579018e-3	0.983
ϕ_4	6.378392654	6.37799883	3.98849552e-5	2.26221029e-2	0.968
α	.03808	.0380841866	8.76801465e-10	1.21157573e-4	0.943

components is of utmost important. LSEs for burst-type signals have the desirable theoretical properties. At the same time, obtaining LSEs is difficult; so the estimation of the number of signals as well as developing an efficient algorithm need to be addressed.

REFERENCES

[1] Jennrich, R.I. (1969), "Asymptotic properties of the nonlinear least squares estimators", *Annals of Mathematical Statistics* **40**, 633-643.

[2] Kudnu, D. and Nandi, S. (2006) "Parameter estimation of chirp signals in presence of stationary noise", To appear in *Statistica Sinica*.

[3] Mukhopadhyay, S. and Sircar, P. (1996), "Parametric modelling ecg signal, *Medical & and Biological Engineering & computing*, **34**, 171-174.

TABLE 3. LSEs and percentile bootstrap confidence interval of different parameters ($N = 500$)

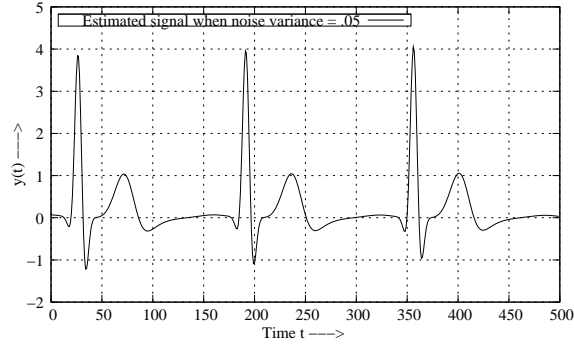| PARA	T-VALUE	Error variance = .05		Error variance = 1.0	
		LSE	Boot-p C.I.	LSE	Boot-p C.I.
A_1	5.70166706e-5	5.72215577e-5	(5.664791e-5, 5.744407e-5)	5.69990552e-5	(5.596786e-5 5.844120e-5
b_1	4.989495798	4.97478247	(4.95373821, 5.02667046)	4.97899866	(4.8581996, 5.17925692)
c_1	.1904	.189251259	(.17937465, .20895347)	.225356966	(.158369198, .269945771)
θ_1	.07616	.0762672499	(.0757925957, .0767704621)	.0772908255	(.0760189742, .0785392374)
ϕ_1	1.166198163	1.15923095	(1.15439439, 1.18735075)	1.17153347	(1.08730626, 1.26344573)
A_2	3.3049426e-25	3.03041114e-25	(2.454487e-25, 6.315634e-25)	2.49631977e-25	(9.907965e-26, 1.900617e-24)
b_2	28.886554622	28.9446011	(28.6586857, 29.1089344)	29.0836334	(28.293335, 29.7137547)
c_2	2.05632	2.03544688	(2.02223277, 2.0569036)	1.99407959	(1.94885516, 2.07028937)
θ_2	.26656	.266473562	(.266269267, .266789019)	.266281039	(.265390217, .267506808)
ϕ_2	18.007071552	18.0454769	(17.9902306, 18.1049805)	18.121191	(17.9078636, 18.3518963)
A_3	1.002e-3	1.00651465e-3	(9.948860e-4, 1.008619e-3)	1.00142101e-3	(9.821707e-4, 1.029121e-3)
b_3	2.62605042	2.61257982	(2.59002972, 2.68200302)	2.65862751	(2.51707864, 2.97121692)
c_3	5.9024	5.91638947	(5.80828619, 6.0510087)	5.60666466	(5.63202095, 6.38537598)
θ_3	.03808	.037919607	(.0364742056, .0421706513)	.0372068435	(.0294211712, .0505509488)
ϕ_3	10.928948246	11.1142969	(10.8126116, 11.1670628)	11.1489267	(10.365386, 11.7023048)
A_4	3.7575e-4	3.76711047e-4	(3.73048475e-4, 3.7917291e-4)	3.74084688e-4	(3.684895e-4, 3.852599e-4)
b_4	2.62605042	2.61216354	(2.58843017, 2.70266461)	2.64941788	(2.53093982, 2.94608474)
c_4	3.2368	3.21962428	(3.19550252, 3.30128908	3.19691062	(3.11355329, 3.50221658)
θ_4	.03808	.039557945	(.0351168662, .041667372)	.0316661075	(.028909985, .0547059737)
ϕ_4	6.378392654	6.3666358	(6.29677153, 6.46921301)	6.61410189	(6.16404438, 6.79063654)
α	.03808	.0381681249	(.0381025411, .038249664)	.0383719206	(.0381485969, .0387147069)

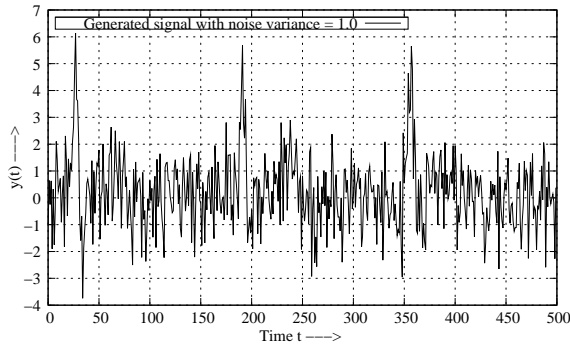FIGURE 3. Estimated signal from the signal given in Fig 2.

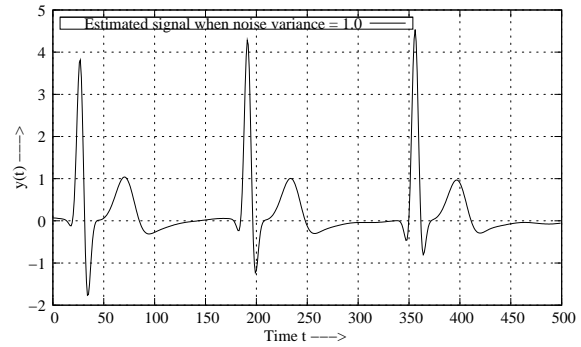FIGURE 4. The same signal as in Fig. 1 corrupted by noise (Error variance =1.0).

FIGURE 5. Estimated signal from the signal given in Fig 4.

[4] Nandi, S. Kundu, D. and Iyer, S.K. (2004), "Amplitude modulated model for analyzing non stationary speech signals" *Statistics*, Vol. 38, No. 5, 439-456.

[5] Nandi, S., Iyer, S.K. and Kundu, D. (2002), "Estimation of frequencies in presence of heavy tail errors", *Statistics and Probability Letters*, **58**, 265-282.

[6] Sharma, R. K. and Sircar, P. (2001) "Parametric modelling of burst-type signals", *Journal of the Franklin Institute*, **338**, 817-832.

[7] Sircar, P. and Syali, M.S. (1996), "Complex AM signal model for non-stationary signals", *Signal Processing*, Vol. 53, 35-45.

[8] Wu, C.F.J. (1981), "Asymptotic theory of the nonlinear least squares estimation", *Annals of Statistics* **9**, 501-513.

APPENDIX A

In Appendix A, we provide the proofs of the results for model (1) (with $q = 1$), stated in section 2.

Lemma 1. *Let $X(1), X(2), \ldots$ be i.i.d. random variables with mean zero and finite second moment and b is a real number such that $e^{|b|} \leq K$. Define $\Pi = (0, \pi) \times (0, \pi) \in \mathcal{R}^2$. Then as $n \to \infty$,*

$$\sup_{(\alpha,\theta)\in\Pi} \frac{1}{N} \sum_{t=1}^{N} X(t) \exp\{b \cos(\alpha t)\} \cos(\theta t) \overset{a.s.}{\to} 0, \quad as \quad N \to \infty \tag{15}$$

Proof of Lemma 1: Define

$$Z(t) = \begin{cases} X(t) & \text{if} \quad |X(t)| \leq t^{\frac{1}{2}} \\ 0 & \text{if} \quad |X(t)| > t^{\frac{1}{2}} \end{cases}$$

Then

$$
\begin{aligned}
\sum_{t=1}^{\infty} P[Z(t) \neq X(t)] &= \sum_{t=1}^{\infty} P[|X(t)| > t^{\frac{1}{2}}] \\
&= \sum_{t=1}^{\infty} \sum_{2^{t-1} \leq n \leq 2^t} P[|X(1)| > n^{\frac{1}{2}}] \\
&\leq \sum_{t=1}^{\infty} 2^t P[|X(1)| > 2^{\frac{t-1}{2}}] \\
&\leq \sum_{t=1}^{\infty} 2^t \sum_{j=t}^{\infty} P[2^{\frac{j-1}{2}} \leq |X(1)| < 2^{\frac{j}{2}}] \\
&\leq \sum_{j=1}^{\infty} P[2^{\frac{j-1}{2}} \leq |X(1)| < 2^{\frac{j}{2}}] \sum_{t=1}^{j} 2^t \\
&\leq c \sum_{j=1}^{\infty} 2^{j-1} P[2^{\frac{j-1}{2}} \leq |X(1)| < 2^{\frac{j}{2}}] \leq cE|X(1)|^2 < \infty.
\end{aligned}
$$

So, $P[Z(t) \neq X(t) \text{ i.o.}] = 0$ and $Z(t)$ and $X(t)$ are equivalent random variables. Thus

$$\sup_{(\alpha,\theta)\in\Pi} \frac{1}{N} \sum_{t=1}^{N} X(t) \exp\{b \cos(\alpha t)\} \cos(\theta t) \overset{a.s.}{\to} 0 \Leftrightarrow \sup_{(\alpha,\theta)\in\Pi} \frac{1}{N} \sum_{t=1}^{N} Z(t) \exp\{b \cos(\alpha t)\} \cos(\theta t) \overset{a.s.}{\to} 0.$$
$$\tag{16}$$

Let $U_t = Z(t) - E(Z(t))$. Then

$$\sup_{(\alpha,\theta)\in\Pi} \left| \frac{1}{N} \sum_{t=1}^{N} Z(t) \exp\{b \cos(\alpha t)\} \cos(\theta t) \right| \leq e^{|b|} \frac{1}{N} \sum_{t=1}^{N} |Z(t)| \to 0.$$

Thus, it is enough to show that

$$\sup_{(\alpha,\theta)\in\Pi} \frac{1}{N} \sum_{t=1}^{N} U_t \exp\{b \cos(\alpha t)\} \cos(\theta t) \overset{a.s.}{\to} 0. \tag{17}$$

For any fixed $\epsilon > 0$, assume that $0 \leq h \leq \frac{1}{2N^{1/2}K}$. Then $|hU_t \cos(\theta t)e^{b\cos(\alpha t)}| \leq \frac{1}{2}$. Now, using $e^{|x|} \leq 2e^x$ and $e^x \leq 1 + x + 2x^2$ for $|x| \leq \frac{1}{2}$, we have

$$P\left[\left|\frac{1}{N}\sum_{t=1}^{N}U_t\cos(\theta t)e^b\cos(\alpha t)\right| \geq \epsilon\right] \leq e^{-hN\epsilon}\prod_{t=1}^{N}E\left(\exp\{|hU_t\cos(\theta t)e^{b\cos(\alpha t)}|\}\right)$$

$$\leq 2e^{-hN\epsilon}\prod_{t=1}^{N}E\left(\exp\{hU_t\cos(\theta t)e^{b\cos(\alpha t)}\}\right)$$

$$\leq 2e^{-hN\epsilon}\prod_{t=1}^{N}(1+2h^2\sigma^2K^2)$$

$$\leq 2e^{-hN\epsilon+2Nh^2\sigma^2K^2}.$$

Take $h = \frac{1}{2N^{1/2}K}$ in the above inequality.

$$P\left[\left|\frac{1}{N}\sum_{t=1}^{N}U_t\cos(\theta t)e^b\cos(\alpha t)\right| \geq \epsilon\right] \leq 2e^{-\frac{1}{2}N^{\frac{1}{2}}K^{-1}\epsilon+\frac{1}{2}\sigma^2} \leq ce^{-\frac{1}{2}N^{\frac{1}{2}}K^{-1}\epsilon}.$$

Let $L = N^2$. Choose $(\alpha_1, \theta_1), \ldots, (\alpha_L, \theta_L)$ such that for each $(\alpha, \theta) \in \Pi$, we have a (α_j, θ_j) satisfying $|\alpha_j - \alpha| \leq \frac{\pi}{N^2}$ and $|\theta_j - \theta| \leq \frac{\pi}{N^2}$. Now let us consider

$$\left|\cos(\theta t)e^{b\cos(\alpha t)} - \cos(\theta_j t)e^{b\cos(\alpha_j t)}\right|$$

$$= \left|\cos(\theta t)e^{b\cos(\alpha t)} - \cos(\theta_j t)e^{b\cos(\alpha t)} + \cos(\theta_j t)e^{b\cos(\alpha t)} - \cos(\theta_j t)e^{b\cos(\alpha_j t)}\right|$$

$$\leq \left|e^{b\cos(\alpha t)}\right||\cos(\theta t) - \cos(\theta_j t)| + |\cos(\theta_j t)|\left|e^{b\cos(\alpha t)} - e^{b\cos(\alpha_j t)}\right|$$

$$\leq Kt|\theta_j - \theta| + Kt|b||\alpha_j - \alpha|.$$

So,

$$\left|\frac{1}{N}\sum_{t=1}^{N}U_t\left(\cos(\theta t)e^{b\cos(\alpha t)} - \cos(\theta_j t)e^{b\cos(\alpha_j t)}\right)\right| \leq \frac{2}{N}\sum_{t=1}^{N}t^{\frac{1}{2}}Kt(|\theta_j - \theta| + |b||\alpha_j - \alpha|)$$

$$\leq \frac{2}{N}\sum_{t=1}^{N}t^{\frac{1}{2}}Kt\frac{\pi}{N^2}(1+|b|)$$

$$\leq 2K(1+|b|)\frac{\pi}{\sqrt{N}} \rightarrow 0, \quad \text{as} \quad N \rightarrow \infty.$$

Therefore, for large N,

$$P\left[\sup_{\alpha,\theta}\left|\frac{1}{N}\sum_{t=1}^{N}U_t\cos(\theta t)e^{b\cos(\alpha t)}\right| \geq 2\epsilon\right] \leq P\left[\max_{j\leq N^2}\left|\frac{1}{N}\sum_{t=1}^{N}U_t\cos(\theta_j t)e^{b\cos(\alpha_j t)}\right| \geq \epsilon\right]$$

$$\leq cN^2e^{-\frac{1}{2}N^{\frac{1}{2}}K^{-1}\epsilon}.$$

Since $\sum_{n=1}^{\infty} n^2 e^{-\frac{1}{2}n^{\frac{1}{2}}K^{-1}\epsilon} < \infty$, we have the following

$$\sup_{(\alpha,\theta)\in\Pi} \frac{1}{N} \sum_{t=1}^{N} X(t) \exp\{b\cos(\alpha t)\} \cos(\theta t) \overset{a.s.}{\to} 0, \tag{18}$$

as $N \to \infty$, using Borel Canteli Lemma. ∎

Lemma 2. *Let us denote the set*

$$S_{\epsilon,M} = \left\{ \boldsymbol{\eta} : |\boldsymbol{\eta} - \boldsymbol{\eta}^0| > 6\epsilon, |A| \le M \right\}.$$

If for any $\epsilon > 0$ and for some $M < \infty$,

$$\liminf_{N\to\infty} \inf_{\boldsymbol{\eta}\in S_{\epsilon,M}} \frac{1}{N} \left[Q(\boldsymbol{\eta}) - Q(\boldsymbol{\eta}^0) \right] > 0 \quad a.s.$$

then $\hat{\boldsymbol{\eta}}$ is a strongly consistent estimator of $\boldsymbol{\eta}^0$.

Proof of Lemma 2: It is simple, so it is not provided here. ∎

Proof of Theorem 1:

In this proof, we denote $\hat{\boldsymbol{\eta}}$ by $\hat{\boldsymbol{\eta}}_N = (A_N, b_N, \alpha_N, c_N, \theta_N, \phi_N)$, to emphasize that $\hat{\boldsymbol{\eta}}$ depends on N. Let us assume that $\hat{\boldsymbol{\eta}}_N$ is not a consistent estimator for $\boldsymbol{\eta}^0$. Then either:
CASE I: For all sub sequences $\{N_k\}$ of $\{N\}$, $|\hat{A}_{N_k}| \to \infty$. This implies

$$\frac{1}{N_k} \left[Q(\hat{\boldsymbol{\eta}}_{N_k}) - Q(\boldsymbol{\eta}^0) \right] \to \infty.$$

But as $\hat{\boldsymbol{\eta}}_{N_k}$ is the LSE of $\boldsymbol{\eta}^0$ with sample size N_k, we have

$$Q(\hat{\boldsymbol{\eta}}_{N_k}) - Q(\boldsymbol{\eta}^0) < 0,$$

which leads to a contradiction. So $\hat{\boldsymbol{\eta}}_N$ is a strongly consistent estimator of $\boldsymbol{\eta}^0$.

CASE II: For at least one sub sequence $\{N_k\}$ of $\{N\}$, $\hat{\boldsymbol{\eta}}_{N_k} \in S_{\epsilon,M}$, for some $\epsilon > 0$ and for an $0 < M < \infty$. Now we write

$$\frac{1}{N} \left[Q(\boldsymbol{\eta}) - Q(\boldsymbol{\eta}^0) \right] = f(\boldsymbol{\eta}) + g(\boldsymbol{\eta}),$$

where

$$\begin{aligned}
f(\boldsymbol{\eta}) &= \frac{1}{N} \sum_{t=1}^{N} \Big[A^0 \exp\{b^0(1 - \cos(\alpha^0 t + c^0))\} \cos(\theta^0 t + \phi^0) \\
&\quad - A \exp\{b(1 - \cos(\alpha t + c))\} \cos(\theta t + \phi) \Big]^2 \\
g(\boldsymbol{\eta}) &= \frac{2}{N} \sum_{t=1}^{N} e(t) \Big[A^0 \exp\{b^0(1 - \cos(\alpha^0 t + c^0))\} \cos(\theta^0 t + \phi^0) \\
&\quad - A \exp\{b(1 - \cos(\alpha t + c))\} \cos(\theta t + \phi) \Big].
\end{aligned}$$

Using Lemma 1, we have

$$\lim_{N\to\infty} \sup_{\eta\in S_{\epsilon,M}} g(\eta) = 0, \quad a.s. \tag{19}$$

Define sets S_ϵ^i, $i = 1,\ldots,6$ as follows:

$$S_{\epsilon,M}^i = \left\{ \eta : |\eta_i - \eta_i^0| > \epsilon, |A| \le M \right\}, \tag{20}$$

where η_i, $i = 1,\ldots,6$ stands for the elements of η, that is, A, b, α, c, θ and ϕ. Note that $S_{\epsilon,M} \subset \cup_{i=1}^6 S_{\epsilon,M}^i = S$ (say). Therefore,

$$\liminf_{N\to\infty} \inf_{S_{\epsilon,M}} \frac{1}{N} \left[Q(\eta) - Q(\eta^0) \right] \ge \liminf_{N\to\infty} \inf_{S} \frac{1}{N} \left[Q(\eta) - Q(\eta^0) \right]. \tag{21}$$

Next, our aim is to show that

$$\liminf_{N\to\infty} \inf_{S_{\epsilon,M}^i} \frac{1}{N} \left[Q(\eta) - Q(\eta^0) \right] = \liminf_{N\to\infty} \inf_{S_{\epsilon,M}^i} f(\eta) > 0, \quad a.s. \tag{22}$$

for $i = 1,\ldots,6$ which implies (using (21)) that

$$\liminf_{N\to\infty} \inf_{S_{\epsilon,M}} \frac{1}{N} \left[Q(\eta) - Q(\eta^0) \right] > 0, \quad a.s. \tag{23}$$

So, for $i = 1$,

$$\liminf_{N\to\infty} \inf_{S_{\epsilon,M}^1} f(\eta)$$

$$= \liminf_{N\to\infty} \inf_{|A-A^0|>\epsilon} \frac{1}{N} \sum_{t=1}^N \Big[A^0 \exp\{b^0(1 - \cos(\alpha^0 t + c^0))\} \cos(\theta^0 t + \phi^0)$$

$$- A \exp\{b(1 - \cos(\alpha t + c))\} \cos(\theta t + \phi) \Big]^2$$

$$= \lim_{N\to\infty} \inf_{|A-A^0|>\epsilon} \frac{1}{N} \sum_{t=1}^N (A - A^0)^2 \exp\{2b^0(1 - \cos(\alpha^0 t + c^0))\} \cos^2(\theta^0 t + \phi^0)$$

$$\ge e^{2b^0} \epsilon^2 \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^N \exp\{-2b^0 \cos(\alpha^0 t + c^0))\} \cos^2(\theta^0 t + \phi^0)$$

$$\ge e^{2b^0} e^{-|2b^0|} \epsilon^2 \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^N \cos^2(\theta^0 t + \phi^0) = \frac{c_{b^0} \epsilon^2}{2} > 0 \quad a.s.$$

where $c_b = 1$, if $b > 0$ and $c_b = e^{-|4b^0|}$. Using similar technique, the inequality (22) can be shown for other i also and that proves the theorem.

Appendix B

The following limits have been used to obtain the asymptotic distribution of the LSE $\hat{\boldsymbol{\eta}}$ of $\boldsymbol{\eta}^0$.

$$\lim_{N \to \infty} \frac{1}{N^p + 1} \sum_{t=1}^{N} t^p \exp\{-2b \cos(\alpha t + c)\} \cos^2(\theta t + \phi) = \delta_1(\boldsymbol{\xi}, p)$$

$$\lim_{N \to \infty} \frac{1}{N^p + 1} \sum_{t=1}^{N} t^p \exp\{-2b \cos(\alpha t + c)\}(1 - \cos(\alpha t + c))^2 \cos^2(\theta t + \phi) = \delta_2(\boldsymbol{\xi}, p)$$

$$\lim_{N \to \infty} \frac{1}{N^p + 1} \sum_{t=1}^{N} t^p \exp\{-2b \cos(\alpha t + c)\} \sin^2(\alpha t + c) \cos^2(\theta t + \phi) = \delta_3(\boldsymbol{\xi}, p)$$

$$\lim_{N \to \infty} \frac{1}{N^p + 1} \sum_{t=1}^{N} t^p \exp\{-2b \cos(\alpha t + c)\} \sin^2(\theta t + \phi) = \delta_4(\boldsymbol{\xi}, p)$$

$$\lim_{N \to \infty} \frac{1}{N^p + 1} \sum_{t=1}^{N} t^p \exp\{-2b \cos(\alpha t + c)\}(1 - \cos(\alpha t + c)) \cos^2(\theta t + \phi) = \delta_5(\boldsymbol{\xi}, p) \qquad (24)$$

$$\lim_{N \to \infty} \frac{1}{N^p + 1} \sum_{t=1}^{N} t^p \exp\{-2b \cos(\alpha t + c)\} \sin(\alpha t + c) \cos^2(\theta t + \phi) = \delta_6(\boldsymbol{\xi}, p)$$

$$\lim_{N \to \infty} \frac{1}{N^p + 1} \sum_{t=1}^{N} t^p \exp\{-2b \cos(\alpha t + c)\} \sin(\theta t + \phi) \cos(\theta t + \phi) = \delta_7(\boldsymbol{\xi}, p)$$

$$\lim_{N \to \infty} \frac{1}{N^p + 1} \sum_{t=1}^{N} t^p \exp\{-2b \cos(\alpha t + c)\} \sin(\alpha t + c)\}(1 - \cos(\alpha t + c)) \cos^2(\theta t + \phi) = \delta_8(\boldsymbol{\xi}, p)$$

$$\lim_{N \to \infty} \frac{1}{N^p + 1} \sum_{t=1}^{N} t^p \exp\{-2b \cos(\alpha t + c)\}(1 - \cos(\alpha t + c)) \sin(\theta t + \phi) \cos(\theta t + \phi) = \delta_9(\boldsymbol{\xi}, p)$$

$$\lim_{N \to \infty} \frac{1}{N^p + 1} \sum_{t=1}^{N} t^p \exp\{-2b \cos(\alpha t + c)\} \sin(\alpha t + c) \sin(\theta t + \phi) \cos(\theta t + \phi) = \delta_{10}(\boldsymbol{\xi}, p)$$

for $p = 0, 1, 2.....$

Note that

$$\exp\{-2|b|\} \leq \exp\{-2b \cos(\alpha t + c)\} \leq \exp\{2|b|\}. \qquad (25)$$

Using it in the first sequence listed above with $p = 0$, we have

$$e^{-2|b|} \frac{1}{N} \sum_{t=1}^{N} \cos^2(\theta t + \phi) \leq \frac{1}{N} \sum_{t=1}^{N} \exp\{-2b \cos(\alpha t + c)\} \cos^2(\theta t + \phi) \leq e^{2|b|} \frac{1}{N} \sum_{t=1}^{N} \cos^2(\theta t + \phi).$$

$$(26)$$

Now taking limit as $\lim_{N \to \infty}$, we get

$$\frac{e^{-2|b|}}{2} \leq \delta_1(\psi, 0) \leq \frac{e^{2|b|}}{2}. \tag{27}$$

For notational simplicity, $\delta_k(\psi, p) = \delta_k(p)$, $k = 1, \ldots, 10$ has been used in obtaining the asymptotic distribution of the LSEs.

Using the inequality given in (25), in $\delta_6(\boldsymbol{\xi}, p)$, we have

$$\delta_6(\boldsymbol{\xi}, p) \quad \leq \left\{\geq \quad \right\} e^{|2b|} \left\{ e^{-|2b|} \right\} \lim_{N \to \infty} \frac{1}{N^{p+1}} \sum_{t=1}^{N} t^p \sin(\alpha t + c) \cos^2(\theta t + \phi)$$

$$\rightarrow \quad e^{|2b|} \left\{ e^{-|2b|} \right\} \times 0.$$

This implies that

$$0 \leq \delta_6(\boldsymbol{\xi}, p) \leq 0$$

$$\Rightarrow \quad \delta_6(\boldsymbol{\xi}, p) \to 0, \quad \text{for all} \quad p \quad \text{and} \quad \boldsymbol{\xi}.$$

In a similar way, we find that $\delta_k(\boldsymbol{\xi}, p) \to 0$ for all p and $\boldsymbol{\xi}$ for $k = 7, \ldots, 10$ and $\delta_5(\boldsymbol{\xi}, p) = \delta_1(\boldsymbol{\xi}, p)$.

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