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The matrix arithmetic-geometric mean inequality revisited

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Twenty years ago we formulated and proved a matrix version of the arithmetic-geometric mean inequality [13]. This seems to have stimulated several authors who have found different proofs, equivalent statements, extensions, and generalisations in different directions. In this article we survey these developments, and discuss other closely related matters.

While our main focus is on the arithmetic-geometric mean inequality, the article can also serve as an introduction to the basic ideas and typical problems of the flourishing subject of matrix inequalities.

1. Notations

Let $\mathbb{M}(n)$ be the space of $n \times n$ complex matrices. If A is a Hermitian element of $\mathbb{M}(n)$, then we enumerate its eigenvalues as $\lambda_1(A) \geq \cdots \geq \lambda_n(A)$. If A is arbitrary, then its singular values are enumerated as $s_1(A) \geq \cdots \geq s_n(A)$. These are the eigenvalues of the positive (semidefinite) matrix $|A| = (A^*A)^{1/2}$. If A and B are Hermitian matrices, and $A - B$ is positive, then we say that

$$B \leq A. \tag{1.1}$$

Weyl's monotonicity theorem [9, p.63] says that the relation (1.1) implies

$$\lambda_j(B) \leq \lambda_j(A) \quad \text{for all } \ 1 \leq j \leq n. \tag{1.2}$$

The condition (1.2) is equivalent to the following: there exists a unitary matrix U such that

$$B \leq UAU^*. \tag{1.3}$$

We say that the n-tuple $\{\lambda_j(B)\}$ is *weakly majorised* by $\{\lambda_j(A)\}$, if we have

$$\sum_{j=1}^{k} \lambda_j(B) \leq \sum_{j=1}^{k} \lambda_j(A) \quad \text{for all } \ 1 \leq k \leq n. \tag{1.4}$$

In a condensed notation, the family of inequalities (1.4) is expressed as

$$\{\lambda_j(B)\} \prec_w \{\lambda_j(A)\}. \tag{1.5}$$

A norm $|||\cdot|||$ on $\mathbb{M}(n)$ is said to be *unitarily invariant* if $|||UAV||| = |||A|||$ for all A and for all unitary matrices U and V. Special examples of such norms are the *Schatten p-norms*

$$\|A\|_p := \left[\sum_{j=1}^{n} [s_j(A)]^p\right]^{1/p}, \quad 1 \leq p \leq \infty, \tag{1.6}$$

Here it is understood that $\|A\|_\infty = s_1(A)$. This norm called the *operator norm* is denoted simply by $\|A\|$. The Schatten 2-norm, also called the *Hilbert-Schmidt norm* is somewhat special. It can be calculated easily from the entries of the matrix:

$$\|A\|_2 = \left(\sum_{i,j} |a_{ij}|^2 \right)^{1/2}. \tag{1.7}$$

By the Fan dominance theorem [9], $\{s_j(B)\} \prec_w \{s_j(A)\}$ is equivalent to the statement

$$|||B||| \leq |||A||| \quad \text{for every unitarily invariant norm.} \tag{1.8}$$

Arguments with block matrices are often useful. If A, B, C, D are elements of $\mathbb{M}(n)$, then $\begin{bmatrix} A & C \\ D & B \end{bmatrix}$ is an element of $\mathbb{M}(2n)$. The block diagonal matrix $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ is denoted as $A \oplus B$. Block matrices of higher order and those in which the diagonal blocks are square matrices of different sizes are defined in the obvious way.

2. Levels of matrix inequalities

After defining the object $|A|$, the authors of the famous text [35] warn

> "*The reader should be wary of the emotional connotations of the symbol* $|\cdot|$."

and go on to point out that, among other things, the prospective triangle inequality

$$|A + B| \leq |A| + |B|, \tag{2.1}$$

is not true in general.

Inequalities for positive numbers could have several plausible extensions to positive matrices. Only some of them turn out to be valid. Let us illustrate this by an example. If a and b are positive numbers, then we have the inequality

$$|a - b| \leq a + b. \tag{2.2}$$

A natural extension of this to positive matrices A and B could be

$$|A - B| \leq A + B. \tag{2.3}$$

This, however, is not always true. If we choose

$$A = \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix},$$

then

$$|A - B| = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}, \quad A + B = \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix},$$

and the putative inequality $|A-B| \le A+B$ is violated. This having failed, one may wonder whether the weaker assertion

$$s_j(A-B) \le s_j(A+B), \quad \text{for all } 1 \le j \le n, \tag{2.4}$$

is always true. In the example above $A-B$ has singular values $\{3,3\}$ and $A+B$ has $\{9,1\}$. Thus $s_2(A-B)$ is bigger than $s_2(A+B)$ and (2.4) is violated. An assertion weaker than (2.4) would be

$$|||A-B||| \le |||A+B|||, \tag{2.5}$$

for all unitarily invariant norms. This turns out to be true.

If an inequality like (2.3), (2.4) or (2.5) is valid we call it an extension of (2.2) at Level $1, 2$ or 3, respectively.

A proof of (2.5) goes as follows. Since $\pm(A-B) \le A+B$, the inequality (2.5) is a consequence of the following:

Lemma 2.1 Let X and Y be Hermitian matrices such that $\pm Y \le X$. Then $|||Y||| \le |||X|||$ for every unitarily invariant norm.

Proof. Choose an orthonormal basis $e_1, e_2, \ldots, e_n$, such that $Ye_j = \mu_j e_j$, where $|\mu_1| \ge |\mu_2| \ge \cdots |\mu_n|$. Then for $1 \le k \le n$, we have

$$\sum_{j=1}^{k} s_j(Y) = \sum_{j=1}^{k} |\mu_j| = \sum_{j=1}^{k} |\langle e_j, Ye_j\rangle| \le \sum_{j=1}^{k} \langle e_j, Xe_j\rangle.$$

By Ky Fan's maximum principle [9, p.24] the sum on the extreme right is bounded by $\sum_{j=1}^{k} s_j(X)$. So the assertion of the Lemma follows from the Fan dominance theorem. ■

If x and y are real numbers such that $\pm y \le x$, then $|y| \le x$. If X and Y are Hermitian matrices and $\pm Y \le X$, then the lemma says that the Level 3 inequality $|||Y||| \le |||X|||$ is true. The higher Level 2 inequality $s_j(Y) \le s_j(X)$, $1 \le j \le n$, is not always true.

3. The arithmetic-geometric mean inequality (AGMI).

The familiar AGMI for positive real numbers a and b can be written either as $\sqrt{ab} \le (a+b)/2$, or as $ab \le (a^2+b^2)/2$. We seek an attractive version for positive matrices A and B. If A and B commute, then AB is positive, and the inequality $AB \le (A^2+B^2)/2$ is true. In the general case AB is not positive. So, a possible "Level 1 AGMI" would be the assertion

$$|AB| \le \frac{A^2+B^2}{2}.$$

This turns out to be false. If

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

then

$$|AB| = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix}, \quad \frac{1}{2}\left(A^2 + B^2\right) = \begin{bmatrix} 3/2 & 1 \\ 1 & 1 \end{bmatrix},$$

and the putative inequality is not true.

A Level 2 version of the inequality does hold and was proved by us in [13]. For positive matrices A and B

$$2\, s_j(AB) \leq s_j\,\left(A^2 + B^2\right) \quad \text{for } \ 1 \leq j \leq n. \tag{3.1}$$

We stated this in another form: for all $n \times n$ matrices A and B

$$2\, s_j(A^*B) \leq s_j(AA^* + BB^*), \quad \text{for } \ 1 \leq j \leq n. \tag{3.2}$$

It is clear that if A and B are positive, then (3.2) reduces to (3.1). If A and B are arbitrary, then we use their polar decompositions $A = PU$, $B = RV$, in which P and R are positive, and U and V unitary, to obtain (3.2) from (3.1).

Our original proof, with a simplification suggested by X. Zhan, goes as follows. Let $X = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}$. Then

$$XX^* = \begin{bmatrix} AA^* + BB^* & 0 \\ 0 & 0 \end{bmatrix}, \quad X^*X = \begin{bmatrix} A^*A & A^*B \\ B^*A & B^*B \end{bmatrix}.$$

Let $U = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$. Then the off-diagonal part of X^*X can be expressed as

$$Y = \begin{bmatrix} 0 & A^*B \\ B^*A & 0 \end{bmatrix} = \frac{X^*X - U(X^*X)U^*}{2}.$$

The matrix $U(X^*X)U^*$ is positive. Hence, this implies the inequality $Y \leq \frac{1}{2}X^*X$. By Weyl's monotonicity principle,

$$\lambda_j(Y) \leq \frac{1}{2}\lambda_j(X^*X) \quad \text{for all } \ j = 1, 2, \ldots, 2n. \tag{3.3}$$

But the eigenvalues of X^*X are the same as those of XX^* which, in turn, are the eigenvalues of $AA^* + BB^*$ together with n zeros. The eigenvalues of Y are the singular values of A^*B together with their negatives. Hence the inequality (3.2) follows from (3.3). The use of block matrices in this argument is typical of several other proofs of this and related results.

X. Zhan [38] and Y. Tao [36] have shown that the AGMI is equivalent to other interesting matrix inequalities for which they have found different proofs. We discuss these ideas in brief.

Let A and B be positive matrices, and let $X = \begin{bmatrix} A^{1/2} & B^{1/2} \\ 0 & 0 \end{bmatrix}$. Then

$$XX^* = \begin{bmatrix} A+B & 0 \\ 0 & 0 \end{bmatrix}, \quad X^*X = \begin{bmatrix} A & A^{1/2}B^{1/2} \\ B^{1/2}A^{1/2} & B \end{bmatrix}.$$

The block diagonal matrix $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = A \oplus B$ is a *pinching* of the matrix X^*X, and hence $|||A \oplus B||| \leq |||X^*X|||$ for every unitarily invariant norm [9, p.97]. On the other hand $|||X^*X||| = |||XX^*|||$, and hence

$$|||A \oplus B||| \leq |||(A+B) \oplus 0|||. \tag{3.4}$$

This inequality could be abbreviated to $|||A \oplus B||| \leq |||A+B|||$, provided we are careful in interpreting such inequalities between matrices of different sizes. Thus if X is of smaller size than Y, an inequality like $|||X||| \leq |||Y|||$ really means that $|||X \oplus 0||| \leq |||Y|||$ where the zero block is added to make the size of $X \oplus 0$ the same as that of Y.

In [13] we observed that for all positive matrices A and B we have

$$|||A - B||| \leq |||A \oplus B||| \tag{3.5}$$

for all unitarily invariant norms. In view of (3.4) this is an improvement on the inequality (2.5). Zhan [38] showed that the Level 3 inequality (3.5) can be enhanced to a Level 2 version,

$$s_j(A - B) \leq s_j(A \oplus B), \quad 1 \leq j \leq n, \tag{3.6}$$

and further, this statement is *equivalent* to the AGMI (3.1). We show how (3.6) follows from (3.2). Let

$$X = \begin{bmatrix} A^{1/2} & 0 \\ -B^{1/2} & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} A^{1/2} & 0 \\ B^{1/2} & 0 \end{bmatrix}$$

Then,

$$X^*Y = \begin{bmatrix} A-B & 0 \\ 0 & 0 \end{bmatrix}, \quad XX^* + YY^* = \begin{bmatrix} 2A & 0 \\ 0 & 2B \end{bmatrix}.$$

So, the inequality (3.2) (applied to X and Y in place of A and B) says that

$$s_j\left((A-B) \oplus 0\right) \leq s_j(A \oplus B), \quad 1 \leq j \leq 2n.$$

This is the inequality (3.6). In turn, this implies another proposition proved by Tao [36]. Suppose the block matrix $X = \begin{bmatrix} A & C \\ C^* & B \end{bmatrix}$ is positive. Let $U = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$. Then the matrix $Y = UXU^* = \begin{bmatrix} A & -C \\ -C^* & B \end{bmatrix}$ is also positive. Therefore, by (3.6) we have

$$s_j(X - Y) \leq s_j(X \oplus Y) \quad \text{for } 1 \leq j \leq 2n.$$

But $X - Y = \begin{bmatrix} 0 & 2C \\ 2C^* & 0 \end{bmatrix}$. So the singular values of $X - Y$ are the singular values of $2C$, each repeated twice. The matrices X and Y, being unitarily equivalent, have the same singular values, and therefore the singular values of $X \oplus Y$ are the singular values of X, each repeated twice. Thus we have shown that:

$$\text{If } X = \begin{bmatrix} A & C \\ C^* & B \end{bmatrix} \text{ is positive, then}$$

$$2\, s_j(C) \leq s_j(X), \text{ for } 1 \leq j \leq n. \tag{3.7}$$

Tao proved (3.7) and showed that this implies the AGMI (3.1). To see this implication, let A and B be positive matrices, and let $T = \begin{bmatrix} A & 0 \\ B & 0 \end{bmatrix}$. Then

$$X := TT^* = \begin{bmatrix} A^2 & AB \\ BA & B^2 \end{bmatrix}, \quad Y := T^*T = \begin{bmatrix} A^2 + B^2 & 0 \\ 0 & 0 \end{bmatrix}.$$

From (3.7) we have the inequality

$$2\, s_j(AB) \leq s_j(X) = s_j(Y) = s_j(A^2 + B^2),$$

for $1 \leq j \leq n$. That is the AGMI (3.1).

We have shown that the statements (3.1), (3.2), (3.6) and (3.7) can be derived from each other. Zhan [38] and Tao [36] have given alternative proofs of (3.6) and (3.7). Yet another proof by Zhan [40] is remarkably simple, and goes as follows. If H is any Hermitian matrix, then

$$s_j(H) = \lambda_j(H \oplus -H), \quad 1 \leq j \leq n. \tag{3.8}$$

Apply this to the matrix $H = A - B$, and note that $(A - B) \oplus (B - A) \leq A \oplus B$. Using Weyl's monotonicity principle we get (3.6). Another interesting proof of (3.1) is given by Aujla and Bourin [8].

The formulation (3.1) of AGMI is somewhat delicate. For example, another possible formulation could have been

$$s_j\,(AB + BA) \leq s_j\,(A^2 + B^2).$$

This is not always true. If we choose

$$A = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

then

$$|AB + BA|^2 = a^2 \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}, \quad A^2 + B^2 = \begin{bmatrix} a^2 + 2 & 2 \\ 2 & 2 \end{bmatrix}.$$

Clearly, $s_2(A^2+B^2) \le 2$, but the two singular values of $AB+BA$ are of the same order of magnitude as a.

In the same vein, the positioning of the stars in (3.2) is just right. Any change destroys the inequality.

4. Stronger Level 3 inequalities

A corollary of (3.1) is the norm inequality

$$|||A^{1/2}B^{1/2}||| \le \frac{1}{2}|||A+B||| \tag{4.1}$$

for all unitarily invariant norms. Stronger versions of this were obtained by R. Bhatia and C. Davis [11]. Among other things they showed that for all positive matrices A, B and for all X, we have

$$|||A^{1/2}XB^{1/2}||| \le \frac{1}{2}|||AX+XB|||. \tag{4.2}$$

For the operator norm alone this inequality had been proved earlier by A. McIntosh [32] and used by him to obtain simple proofs of some famous inequalities of E. Heinz in perturbation theory. Following [11] different proofs of (4.2) were found by F. Kittaneh [26], R. A. Horn [24], R. Mathias [34] and others. One of these ideas has been particularly fruitful, and we explain it briefly.

The insertion of the matrix X in (4.2) greatly enhances the scope of the inequality (4.1). At the same time it leads to a simpler proof. The trick is that the general case of (4.2) follows from its very special case when $A = B$. (In this case the original inequality (4.1) is a tautology.) In order to prove that

$$|||A^{1/2}XA^{1/2}||| \le \frac{1}{2}|||AX+XA|||, \tag{4.3}$$

we may assume that $A = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$, a diagonal matrix. The (i,j) entry of the matrix $A^{1/2}XA^{1/2}$ is $\sqrt{\lambda_i\lambda_j}\, x_{ij}$, and this is bounded in absolute value by $\frac{1}{2}(\lambda_i+\lambda_j)x_{ij}$, the (i,j) entry of $\frac{1}{2}(AX+XA)$. There is one unitarily invariant norm $\|\cdot\|_2$ for which entrywise domination $|s_j| \le |t_{ij}|$ implies $\|S\|_2 \le \|T\|_2$. So, the inequality (4.3) is obviously true for this norm. For other norms a more elaborate argument is required. Let $S \circ T$ be the entrywise product $s_{ij}t_{ij}$. If S is a positive (semidefinite) matrix, then by Theorem 5.5.19 in [25] we have

$$|||S \circ T||| \le \max_i s_{ii}\, |||T|||.$$

Let Y be the matrix with entries

$$y_{ij} = \frac{2\sqrt{\lambda_i\lambda_j}}{\lambda_i+\lambda_j}.$$

Then

$$A^{1/2}XA^{1/2} = Y \circ \left[\frac{1}{2}(AX+XA)\right].$$

So, if we show that Y is positive, then the inequality (4.3) would follow. The matrix Y is equal to $2\,A^{1/2}CA^{1/2}$ where $c_{ij} = \frac{1}{\lambda_i+\lambda_j}$. It is a well-known fact that C is positive, and hence so is Y.

The passage from (4.3) to (4.2) is affected by a block matrix argument. Let

$$\tilde{A} = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}, \quad \tilde{X} = \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix},$$

and apply (4.3) to this pair. The inequality (4.2) follows.

Bhatia and Davis [11] proved a strong generalisation of (4.2). In the light of later work this can be interpreted as follows. For $0 \le \nu \le 1$, the family of *Heinz means* of positive numbers a and b is defined as

$$H_\nu(a,b) = \frac{a^\nu b^{1-\nu} + a^{1-\nu} b^\nu}{2}.$$

Then $H_\nu(a,b) = H_{1-\nu}(a,b)$; $H_{1/2}(a,b) = \sqrt{ab}$; $H_0(a,b) = H_1(a,b) = \frac{1}{2}(a+b)$. It can be seen that

$$\sqrt{ab} \le H_\nu(a,b) \le \frac{1}{2}(a+b). \tag{4.4}$$

Thus H_ν is a family of means that *interpolates* between the geometric and the arithmetic means. A matrix version of (4.4) proved in [11] says

$$2\,|||A^{1/2}XB^{1/2}||| \le |||A^\nu XB^{1-\nu} + A^{1-\nu}XB^\nu||| \le |||AX + XB|||. \tag{4.5}$$

The argument that we adumbrated for proving (4.3) can be adapted to this situation. For example, to prove the second inequality in (4.5), we need to prove that the matrix Z with entries

$$z_{ij} = \frac{\lambda_i^\nu \lambda_j^{1-\nu} + \lambda_i^{1-\nu}\lambda_j^\nu}{\lambda_i + \lambda_j},$$

is positive. This is more complicated than the matrix Y considered earlier. In [15] and [31] the authors establish a connection between such matrices and the theory of positive definite functions. Using this they prove many inequalities involving different means. This theme is carried out further in [19] and [20]. It has led to a wealth of new results. A convenient summary and a list of references can be obtained from the monographs [21], [40], and the recent book [10].

A norm inequality equivalent to (4.2) was proved by Corach, Porta and Recht [16] with a different motivation. See [28] for a further discussion. Other papers with different emphases and viewpoints include [3, 4, 5, 18].

In Section 3 we gave a Level 2 version of the inequality (4.1). The inequality (4.2) cannot be raised to this higher level. Choose 2×2 positive definite matrices A and X such that $AX+XA$ is also positive. Then the diagonal entries of $A^{1/2}XA^{1/2}$ are equal to those $\frac{1}{2}(AX + XA)$. The sum of these diagonal entries is equal to the trace of these matrices, which is the sum of their singular values. Since $s_1(A^{1/2}XA^{1/2}) \le \frac{1}{2}s_1(AX+XA)$, we must have $s_2(A^{1/2}XA^{1/2}) \ge \frac{1}{2}s_2(AX+XA)$.

If we do not insist on inserting the factor X, then there are Level 2 stronger versions of the AGMI. These are given in the next section. There seems to be a delicate balance between raising the level and inserting X.

5. Stronger Level 2 inequalities

The inequality $ab \le (a^2+b^2)/2$ has a generalisation in Young's inequality that is often used in analysis. This says that

$$ab \le \frac{a^p}{p} + \frac{b^q}{q},$$

where p and q are conjugate indices; i.e. they are positive numbers such that $1/p+1/q = 1$. T. Ando [2] obtained an analogous extension of our matrix AGMI (3.1); he showed that

$$s_j(AB) \le s_j\left(\frac{A^p}{p} + \frac{B^q}{q}\right) \quad \text{for all } 1 \le j \le n, \tag{5.1}$$

where A and B are positive matrices, and $1/p+1/q = 1$. Of course, this implies that

$$|||AB||| \le \left|\left|\left|\frac{A^p}{p} + \frac{B^q}{q}\right|\right|\right|. \tag{5.2}$$

It has been pointed out by Ando [1] that the stronger version

$$|||AXB||| \le \left|\left|\left|\frac{A^pX}{p} + \frac{XB^q}{q}\right|\right|\right|$$

does *not* hold in general. H. Kosaki [31] showed that an inequality weaker than this:

$$|||AXB||| \le \frac{|||A^pX|||}{p} + \frac{|||XB^q|||}{q} \tag{5.3}$$

does hold, and used this to give another proof of the multiplicative inequality

$$|||AXB||| \le |||A^pX|||^{1/p}\ |||XB^q|||^{1/q} \tag{5.4}$$

proved earlier by Kittaneh [27] and by Bhatia and Davis [12].

Young's inequality in the infinite-dimensional setting has been investigated by D. Farenick *et al* in [6, 17, 33]. Another paper on the inequality is [22].

In another direction, the second of the inequalities (4.4) has a Level 2 matrix version. In response to a conjecture by X. Zhan [39], K. Audenaert [7] has recently proved the inequality

$$s_j\left(A^\nu B^{1-\nu} + A^{1-\nu}B^\nu\right) \le s_j(A+B),\ 1 \le j \le n, \tag{5.5}$$

for $0 \le \nu \le 1$.

Audenaert's proof depends on the following general theorem about matrix monotone functions. We give a short and simple proof of it.

Theorem [7] Let f be a matrix monotone function on $[0,\infty)$. Then for all positive matrices A and B

$$A\,f(A)+B\,f(B)\geq\frac{1}{2}\,(A+B)^{1/2}\,(f(A)+f(B))\,(A+B)^{1/2}. \tag{5.6}$$

Proof. The function f is also matrix concave, and $g(t)=t\,f(t)$ is matrix convex. (See Theorems V.25 and V.29 in [9]) The matrix convexity of g implies the inequality

$$\frac{Af(A)+B\,f(B)}{2}\geq\frac{A+B}{2}\;f\left(\frac{A+B}{2}\right).$$

The expression on the right hand side is equal to $\frac{1}{2}(A+B)^{1/2}f\left(\frac{A+B}{2}\right)(A+B)^{1/2}$. The matrix concavity of f implies that

$$f\left(\frac{A+B}{2}\right)\geq\frac{f(A)+f(B)}{2}.$$

Combining these two inequalities we get (5.6). ■

We now show how (5.5) is derived from (5.6). The proof cleverly exploits the fact that the matrices XY and YX have the same eigenvalues.

Let $f(t)=t^r$, $0\leq r\leq 1$. This function is matrix monotone. Hence from (5.6) we have

$$2\,\lambda_j\left(A^{1+r}+B^{1+r}\right)\geq\lambda_j\left((A+B)(A^r+B^r)\right). \tag{5.7}$$

Except for trivial zeros the eigenvalues of $(A+B)(A^r+B^r)$ are the same as those of the matrix

$$\begin{bmatrix} A^{1/2} & B^{1/2}\\ 0 & 0\end{bmatrix}\begin{bmatrix} A^{1/2} & 0\\ B^{1/2} & 0\end{bmatrix}\begin{bmatrix} A^r+B^r & 0\\ 0 & 0\end{bmatrix},$$

and in turn, these are the same as the eigenalues of

$$\begin{aligned}&\begin{bmatrix} A^{1/2} & 0\\ B^{1/2} & 0\end{bmatrix}\begin{bmatrix} A^r+B^r & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix} A^{1/2} & B^{1/2}\\ 0 & 0\end{bmatrix}\\ &=\begin{bmatrix} A^{1/2}(A^r+B^r)A^{1/2} & A^{1/2}(A^r+B^r)B^{1/2}\\ B^{1/2}(A^r+B^r)A^{1/2} & B^{1/2}(A^r+B^r)B^{1/2}\end{bmatrix}.\end{aligned}$$

So, the inequalities (3.7) and (5.7) together give

$$\begin{aligned}\lambda_j(A^{1+r}+B^{1+r}) &\geq s_j\left(A^{1/2}(A^r+B^r)B^{1/2}\right)\\ &= s_j\left(A^{1/2+r}B^{1/2}+A^{1/2}B^{1/2+r}\right)\end{aligned}$$

Replacing A and B by $A^{1/1+r}$ and $B^{1/1+r}$, respectively, we get from this

$$s_j(A+B)\geq s_j\geq\left(A^{\frac{2r+1}{2r+2}}B^{\frac{1}{2r+2}}+A^{\frac{1}{2r+2}}B^{\frac{2r+1}{2r+2}}\right),\quad 0\leq r\leq 1.$$

In other words

$$s_j(A+B)\geq s_j\left(A^{\nu}B^{1-\nu}+A^{1-\nu}B^{\nu}\right),\quad 1/2\leq\nu\leq 3/4,$$

and we have proved (5.5) for this special range.

Again, except for trivial zeros, the eigenvalues of $(A+B)(A^r+B^r)$ are the same as those of

$$\begin{bmatrix} A^{r/2} & 0 \\ B^{r/2} & 0 \end{bmatrix} \begin{bmatrix} A+B & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A^{r/2} & B^{r/2} \\ 0 & 0 \end{bmatrix}.$$

If we repeat the arguments above we have, at the end, the inequality (5.5) for $\frac{3}{4} \le \nu \le 1$.

This establishes (5.5) for $\frac{1}{2} \le \nu \le 1$, and by symmetry for all ν in $[0,1]$.

Here it is interesting to note that the first inequality in (4.4) fails to have a Level 2 matrix extension. Audenaert [7] gives an example of 3×3 matrices A and B for which

$$s_2\left(A^{1/2}B^{1/2}\right) > s_2\left(H_\nu(A,B)\right) \quad \text{for } 0 < \nu < 0.13.$$

Another generalisation of (3.1) that can be proved using these ideas is

$$2\, s_j\left(A^{1/2}(A+B)^r B^{1/2}\right) \le s_j\left((A+B)^{r+1}\right) \quad \text{for } r \ge 0. \tag{5.8}$$

The special case $r = 1$ was proved by Bhatia and Kittaneh [14], and Tao [36] has proved this for all positive integers r. Using the polar decomposition $X = UP$ one sees that $(XX^*)^{r+1} = X(X^*X)^r X^*$ for every matrix X. Let $X = \begin{bmatrix} A^{1/2} & 0 \\ B^{1/2} & 0 \end{bmatrix}$. Then

$$\begin{aligned} (XX^*)^{r+1} &= X(X^*X)^r X^* \\ &= \begin{bmatrix} A^{1/2} & 0 \\ B^{1/2} & 0 \end{bmatrix} \begin{bmatrix} (A+B)^r & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A^{1/2} & B^{1/2} \\ 0 & 0 \end{bmatrix}, \\ &= \begin{bmatrix} A^{1/2}(A+B)^r A^{1/2} & A^{1/2}(A+B)^r B^{1/2} \\ B^{1/2}(A+B)^r A^{1/2} & B^{1/2}(A+B)^r B^{1/2} \end{bmatrix} \end{aligned}$$

So using (3.7) we get for $1 \le j \le n$,

$$2\, s_j\left(A^{1/2}(A+B)^r B^{1/2}\right) \le s_j(XX^*)^{r+1} = s_j(X^*X)^{r+1} = s_j\left((A+B)^{r+1}\right),$$

as claimed.

An X-version of (3.5) has been recently proved by Kittaneh [29, 30]: if A and B are positive, and X arbitrary, then

$$|||AX - XB||| \le ||X||\; |||A \oplus B|||. \tag{5.9}$$

Note that this implies in particular, that $||AX - XA|| \le ||X||\, ||A||$, a significant improvement on the inequality $||AX - XA|| \le 2\, ||X||\, ||A||$ that a raw use of the triangle inequality leads to. A simple proof goes as follows. Let U be any unitary matrix, then using unitary invariance of the norm, and (3.5) we get

$$|||AU - UB||| = |||A - UBU^*||| \le |||A \oplus UBU^*||| = |||A \oplus B|||.$$

Now let X be any contraction, i.e. $||X|| \le 1$. Then there exist unitary matrices U and V such that $X = \frac{1}{2}(U+V)$. Hence

$$|||AX - XB||| \le \frac{1}{2}\left(|||AU - UB||| + |||AV - VB|||\right) \le |||A \oplus B|||.$$

Finally, if X is any matrix, then $X/||X||$ is a contraction and the inequality above leads to (5.9).

Improving upon this, Kittaneh [30] has obtained an X-version of (3.6) as well. This says that for A, B positive and X arbitrary we have

$$s_j(AX - XB) \le ||X||\ s_j(A \oplus B), \quad 1 \le j \le n. \tag{5.10}$$

The reader can find more inequalities of this type and their applications in [30].

6. Another level of matrix inequalities

If the Level 2 inequality $s_j(Y) \le s_j(X)$, $1 \le j \le n$, fails, we may still have a weaker inequality

$$s_j(Y \oplus 0) \le s_j(X \oplus X), \quad 1 \le j \le 2n.$$

We express this in an abbreviated form as

$$s_j(Y) \le s_j(X \oplus X), \quad 1 \le j \le n, \tag{6.1}$$

and note that this is equivalent to saying

$$s_j(Y) \le s_{\lfloor \frac{j+1}{2} \rfloor}(X), \quad 1 \le j \le n, \tag{6.2}$$

where $\lfloor r \rfloor$ denotes the integer part of r. Some examples of such inequalities germane to ones discussed above are presented below.

Lemma 6.1 let X and Y be Hermitian matrices such that $\pm Y \le X$. Then

$$s_j(Y) \le s_j(X \oplus X), \quad 1 \le j \le n.$$

Proof. The condition $\pm Y \le X$ implies that $Y \oplus (-Y) \le X \oplus X$. Using (3.8) and Weyl's monotonicity principle we have for $1 \le j \le n$,

$$s_j(Y) \le \lambda_j(X \oplus X) = s_j(X \oplus X). \qquad \blacksquare$$

This leads to another version of the AGMI:

Proposition 6.2 For all A, $B \in \mathbb{M}(n)$ we have

$$s_j\left(A^*B + B^*A\right) \le s_j\left((A^*A + B^*B) \oplus (A^*A + B^*B)\right), \quad 1 \le j \le n. \tag{6.3}$$

Proof. Since $(A \pm B)^*(A \pm B) \ge 0$, we have $\pm(A^*B + B^*A) \le A^*A + B^*B$. The inequality (6.3) follows from Lemma 6.1.$\blacksquare$

If A and B are Hermitian, this reduces to

$$s_j(AB+BA) \le s_j\left((A^2+B^2)\oplus(A^2+B^2)\right),\ 1\le j\le n. \tag{6.4}$$

See the discussion at the end of Section 3. Hirzallah and Kittaneh [23] have shown that we also have

$$s_j(AB^*+BA^*) \le s_j\left((A^*A+B^*B)\oplus(A^*A+B^*B)\right), \quad 1\le j\le n. \tag{6.5}$$

We have remarked at the beginning of Section 2 that a Level 1 triangle inequality (2.1) is not true. Even a Level 3 inequality

$$|||A+B||| \le |||\ |A|+|B|\ ||| \tag{6.6}$$

is not always true for 2×2 matrices. If

$$A=\begin{bmatrix}1&0\\0&0\end{bmatrix},\quad B=\begin{bmatrix}0&1\\0&0\end{bmatrix},$$

then $A+B$ has singular values $\left\{\sqrt{2},0\right\}$, whereas $|A|+|B|=I$ and its singular values are $\{1,1\}$.

For these matrices, there does not exist any unitary U with the property

$$|A+B| \le U(|A|+|B|)U^*. \tag{6.7}$$

A well-known theorem of R. C. Thompson [37] says that given any A,B in $\mathbb{M}(n)$, there exist unitary matrices U and V such that

$$|A+B| \le U|A|U^*+V|B|V^*. \tag{6.8}$$

We prove another version of the triangle inequality:

Theorem 6.2 Let A,B be any two $n\times n$ matrices. Then

$$s_j(A+B) \le s_j\left((|A|+|B|)\oplus(|A^*|+|B^*|)\right), \tag{6.9}$$

for $1\le j\le n$.

Proof. The matrices $\begin{bmatrix}|X| & \pm X^*\\ \pm X & |X^*|\end{bmatrix}$ are positive for every $X\in\mathbb{M}(n)$; see [9,p.10]. Hence $\begin{bmatrix}|A|+|B| & \pm(A+B)^*\\ \pm(A+B) & |A^*|+|B^*|\end{bmatrix}$ are positive, and therefore,

$$\pm\begin{bmatrix}0 & (A+B)^*\\ A+B & 0\end{bmatrix} \le \begin{bmatrix}|A|+|B| & 0\\ 0 & |A^*|+|B^*|\end{bmatrix}.$$

So, using Lemma 6.1 we get

$$\begin{aligned}&s_j\left((A+B)\oplus(A+B)^*\right)\\ &\quad\le\ s_j\left((|A|+|B|)\oplus(|A^*|+|B^*|)\oplus(|A|+|B|)\oplus(|A^*|+|B^*|)\right),\end{aligned}$$

for $j=1,2,\ldots,2n$. Now note that $s_j(X)=s_j(X^*)$, and $s_j(Y\oplus Y)\le s_j(X\oplus X)$ for all j if and only if $s_j(Y)\le s_j(X)$ for all j. Hence the last inequality above is equivalent to (6.9). ■

Corollary 6.3 If A and B are $n\times n$ normal matrices, then for all $j=1,2,\ldots,n$,

$$\begin{aligned} s_j(A+B) &\le s_j\left((|A|+|B|)\oplus(|A|+|B|)\right) \\ &= s_{\lfloor\frac{j+1}{2}\rfloor}(|A|+|B|). \end{aligned} \tag{6.10}$$

We remark that for normal matrices A and B the Level 3 inequality (6.6) is true, but the Level 2 inequality (6.7) is not true even for Hermitian matrices. If

$$A=\begin{bmatrix}1&1\\1&1\end{bmatrix},\quad B=\begin{bmatrix}0&0\\0&-2\end{bmatrix},$$

then $s_2(A+B)=\sqrt{2}$, and this is bigger than $s_2(|A|+|B|)=2-\sqrt{2}$.

Another well-known and very useful result is the *pinching inequality*. Let $A=[A_{ij}]$ be an $m\times m$ block matrix where the diagonal blocks $A_{11},\ldots,A_{mm}$ are square matrices of sizes $n_1,\ldots,n_m$, with $n_1+\cdots+n_m=n$. The block diagonal matrix $\mathcal{C}(A)=A_{11}\oplus\cdots\oplus A_{mm}$ is called a pinching (or m-pinching) of A. The pinching inequality says

$$|||\mathcal{C}(A)|||\le|||A|||$$

for every unitary invariant norm. A Level 2 version of this inequality is not true. The identity matrix is a pinching of $A=\begin{bmatrix}1&1\\1&1\end{bmatrix}$, and $s_2(I)=1$ whereas $s_2(A)=0$. However, we do have the following:

Theorem 6.4 Let $\mathcal{C}(A)$ be an m-pinching of an $n\times n$ matrix A. Then for $j=1,2,\ldots,n$,

$$s_j(\mathcal{C}(A)) \le s_j(A\oplus A\oplus+\cdots\oplus A) \quad (m \text{ copies}). \tag{6.11}$$

Proof. Every m-pinching can be expressed as $\mathcal{C}(A)=\frac{1}{m}\sum_{k=0}^{m-1}U^{*k}AU^k$ where U is a unitary matrix; see [10, p.88]. Recently it has been shown in [23] that if $X_0,\ldots,X_{m-1}$ are any elements of $\mathbb{M}(n)$, then $\frac{1}{m}s_j\left(\sum_{i=0}^{m-1}X_i\right)\le s_j\left(X_0\oplus\cdots\oplus X_{m-1}\right)$. Combining these two facts we obtain (6.11). ■

7. Other versions of the AGMI

The AGMI for positive numbers a and b could be written in different ways as

(i) $\sqrt{ab}\le\frac{a+b}{2}$,

(ii) $ab \leq \frac{a^2+b^2}{2}$,

(iii) $ab \leq \left(\frac{a+b}{2}\right)^2$.

Each of these three can be obtained from another. However, they suggest different plausible matrix versions. For example, instead of the formulation (3.1) we could ask whether

$$s_j^{1/2}(AB) \leq \frac{1}{2} s_j(A+B), \quad 1 \leq j \leq n. \tag{7.1}$$

The level 3 inequality that would follow from this is

$$|||\ |AB|^{1/2}\ ||| \leq \frac{1}{2}\,|||A+B|||, \tag{7.2}$$

for all unitarily invariant norms. The Level 2 inequality suggested by (iii) above is $s_j(AB) \leq \frac{1}{4} s_j^2(A+B)$, and this is no different from (7.1). The Level 3 inequality suggested by (iii) is

$$|||AB||| \leq \frac{1}{4}\,|||(A+B)^2|||. \tag{7.3}$$

for all unitarily invariant norms. It turns out that this is a weaker statement than (7.2).

Bhatia and Kittaneh [14] considered all these different formulations. They proved the inequality (7.3) for all unitarily invariant norms. This is equivalent to saying that (7.2) is true for all Q-norms, a class that includes all Schatten p-norms for $p \geq 2$. They showed that (7.2) is valid also for the trace norm ($p = 1$). Further they showed that the Level 2 inequality (7.1) is true for the case $n = 2$. Other cases of this remain open.

Finally we remark that there is a large body of work on a positive matrix valued geometric mean of A and B with several connections to problems in matrix theory, electrical networks, physics, and geometry. The interested reader could see Chapters 4-6 of [10].

References

[1] T. Ando, Majorizations and inequalities in matrix theory, Linear Algebra Appl. 199 (1994), 17-67.

[2] T. Ando, Matrix Young inequalities, Oper. Theory Adv. Appl. 75 (1995), 33-38.

[3] E. Andruchow, G. Corach, and D. Stojanoff, Geometric operator inequalities, Linear Algebra Appl. 258 (1997), 295–310.

[4] E. Andruchow, G. Corach, and D. Stojanoff, On Heinz inequality, Math. Japon. 46 (1997), 143-145.

[5] E. Andruchow, G. Corach, and D. Stojanoff, Geometrical significance of the Löwner-Heinz inequality, Proc. Amer. Math. Soc. 128 (2000), 1031-1037.

[6] M. Argerami and D.R. Farenick, Young's inequality in trace-class operators, Math. Ann. 325 (2003), 727–744.

[7] K. Audenaert, A singular value inequality for Heinz means, Linear Algebra Appl. 422 (2007), 279-283.

[8] J.S. Aujla and J.-C. Bourin, Eigenvalue inequalities for convex and log-convex functions, Linear Algebra Appl. 424 (2007), 25-35.

[9] R. Bhatia, Matrix Analysis, Springer-Verlag, New York, 1997.

[10] R. Bhatia, Positive Definite Matrices, Princeton University Press, Princeton, 2007.

[11] R. Bhatia and C. Davis, More matrix forms of the arithmetic-geometric mean inequality, SIAM J. Matrix Anal. Appl. 14 (1993), 132-136.

[12] R. Bhatia and C. Davis, A Cauchy-Schwarz inequality for operators with applications, Linear Algebra Appl. 223/224 (1995), 119-129.

[13] R. Bhatia and F. Kittaneh, On the singular values of a product of operators, SIAM J. Matrix Anal. Appl. 11 (1990), 272-277.

[14] R. Bhatia and F. Kittaneh, Notes on matrix arithmetic-geometric mean inequalities, Linear Algebra Appl. 308 (2000), 203-211.

[15] R. Bhatia and K.R. Parthasarathy, Positive definite functions and operator inequalities, Bull. London Math. Soc. 32 (2000), 214-228.

[16] G. Corach, H. Porta, and L. Recht, An operator inequality, Linear Algebra Appl. 142 (1990), 153-158.

[17] J. Erlijman, D.R. Farenick, and R. Zeng, Young's inequality in compact operators, Oper. Theory Adv. Appl. 130 (2001), 171–184.

[18] J. Fujii, M. Fujii, T. Furuta, and R. Nakamoto, Norm inequalities equivalent to Heinz inequality, Proc. Amer. Math. Soc. 118 (1993), 827-830.

[19] F. Hiai and H. Kosaki, Comparison of various means for operators, J. Funct. Anal. 163 (1999), 300–323.

[20] F. Hiai and H. Kosaki, Means for matrices and comparison of their norms, Indiana Univ. Math. J. 48 (1999), 899–936.

[21] F. Hiai and H. Kosaki, Means of Hilbert Space Operators, Lecture Notes in Mathematics, 1820, Springer-Verlag, Berlin, 2003.

[22] O. Hirzallah and F. Kittaneh, Matrix Young inequalities for the Hilbert-Schmidt norm, Linear Algebra Appl. 308 (2000), 77-84.

[23] O. Hirzallah and F. Kittaneh, Inequalities for sums and direct sums of Hilbert space operators, Linear Algebra Appl. 424 (2007), 71-82.

[24] R.A. Horn, Norm bounds for Hadamard products and the arithmetic-geometric mean inequality for unitarily invariant norms, Linear Algebra Appl. 223/224 (1995), 355-361.

[25] R. A. Horn and C. R. Johnson, Topics in Matrix Analysis, Cambridge University Press, 1991.

[26] F. Kittaneh, A note on the arithmetic-geometric mean inequality for matrices, Linear Algebra Appl. 171 (1992), 1-8.

[27] F. Kittaneh, Norm inequalities for fractional powers of positive operators, Lett. Math. Phys. 27 (1993), 279-285.

[28] F. Kittaneh, On some operator inequalities, Linear Algebra Appl. 208/209 (1994), 19-28.

[29] F. Kittaneh, Norm inequalities for commutators of positive operators and applications, Math. Z., in press.

[30] F. Kittaneh, Inequalities for commutators of positive operators, J. Funct. Anal. 250 (2007), 132-143.

[31] H. Kosaki, Arithmetic-geometric mean and related inequalities for operators, J. Funct. Anal. 156 (1998), 429-451.

[32] A. McIntosh, Heinz inequalities and perturbation of spectral families, Macquarie Mathematical Reports, 79-0006, 1979.

[33] S.M. Manjegani and D.R. Farenick, Young's inequality in operator algebras, J. Ramanujan Math. Soc. 20 (2005), 107-124.

[34] R. Mathias, An arithmetic-geometric-harmonic mean inequality involving Hadamard products, Linear Algebra Appl. 184 (1993), 71-78.

[35] M. Reed and B. Simon, Methods of Modern Mathematical Physics I: Functional Analysis, Academic Press, New York, 1980.

[36] Y. Tao, More results on singular value inequalities, Linear Algebra Appl. 416 (2006), 724-729.

[37] R.C Thompson, Convex and concave functions of singular values of matrix sums, Pacific J. Math. 66 (1976), 285-290.

[38] X. Zhan, Singular values of differences of positive semidefinite matrices, SIAM J. Matrix. Anal. Appl. 22 (2000), 819-823.

[39] X. Zhan, Matrix Inequalities, Lecture Notes in Mathematics, 1790, Springer-Verlag, Berlin, 2002.

[40] X. Zhan, On some matrix inequalities, Linear Algebra Appl. 376 (2004), 299-303.