isid/ms/2009/03
March 18, 2009
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Autonomous geometro-statistical formalism for quantum mechanics I : Noncommutative symplectic geometry and Hamiltonian mechanics

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Considering the problem of autonomous development of quantum mechanics in the broader context of solution of Hilbert's sixth problem (which relates to joint axiomatization of physics and probability theory), a formalism is evolved in this two-part work which facilitates the desired autonomous development and satisfactory treatments of quantum-classical correspondence and quantum measurements. This first part contains a detailed development of superderivation based differential calculus and symplectic structures and of noncommutative Hamiltonian mechanics (NHM) which combines elements of noncommutative symplectic geometry and noncommutative probability in an algebraic setting. The treatment of NHM includes, besides its basics, a reasonably detailed treatment of symplectic actions of Lie groups and noncommutative analogues of the momentum map, Poincaré-Cartan form and the symplectic version of Noether's theorem. Consideration of interaction between systems in the NHM framework leads to a division of physical systems into two 'worlds' — the 'commutative world' and the 'noncommutative world' [in which the systems have, respectively, (super-)commutative and non-(super-)commutative system algebras] — with no consistent description of interaction allowed between two systems belonging to different 'worlds'; for the 'noncommutative world', the formalism dictates the introduction of a universal Planck type constant as a consistency requirement.

I. INTRODUCTION

The traditional formalism of quantum mechanics (QM) has some unsatisfactory features; of relevance here is the fact that, while applying QM, we generally *quantize* classical systems. Being the parent theory, QM must stand on its own and should not depend on its daughter theory, classical mechanics (CM), for its very formulation. A related fact is that the 'languages' employed in the traditional treatments of QM and CM are very different; this obscures their parent-daughter relationship. The main objective of the present two-part work is to remove these deficiencies in the development of QM and evolve an autonomous formalism for it which permits a transparent treatment of quantum-classical correspondence and quantum measurements.

[The insistence on an autonomous development of quantum physics is not just a matter of aesthetic satisfaction. It is the author's view that, if we have such a formalism, many problems of theoretical physics, when reformulated in the autonomous quantum framework, will be easier to solve. To get a feel for this, suppose that, instead of having the elaborate (Poincar*é* group based) formalism for special relativity, we had only somehow found some working rules to *relativize* nonrelativistic equations. The several important results obtained by using Lorentz-covariant formalisms [for example, the covariant renormalization programme of quantum field theory (QFT) and the theorems of axiomatic QFT] would either have not been obtained or obtained using a highly cumbersome formalism.]

The conceptual development of a fundamentally new theory often takes place around a unifying principle. For example, Maxwell-Lorentz electrodynamics unifies electricity and magnetism, special relativity unifies the concepts of space and time and general relativity unifies space-time geometry and gravitation. Is there a unifying principle underlying QM ? Noting that we all the time employ Schr*ö*dinger wave functions for statistical averaging as well as describing dynamics of atomic systems, it was proposed by the author, in an article entitled 'Towards an autonomous formalism for quantum mechanics' [arXiv : quant-ph/0207104; referred to henceforth as TD02], that this principle is 'unification of probability and dynamics' (UPD). This principle is very close to the theme of Hilbert's sixth problem (Hilbert 1902), henceforth referred to as Hilprob6, whose statement reads :

"To treat in the same manner, by means of axioms, the physical sciences in which mathematics plays an important part; in the first rank are the theory of probabilities and mechanics."

It appears reasonable to have a somewhat augmented version of Hilprob6; the following formal statement is being hereby proposed for this :

"To evolve an axiomatic scheme covering all physics including the probabilistic framework employed for the treatment of statistical aspects of physical phenomena."

It is useful (indeed, fruitful, as we shall see) to consider the problem of evolving an autonomous formalism for QM in the broader context of solution of Hilprob6. A solution of

Hilprob6 must provide for a satisfactory treatment of the dynamics of the universe and its subsystems. Since all physics is essentially mechanics, the formalism underlying such a solution must be an elaborate scheme of mechanics (with elements of probability incorporated). Keeping in view the presently understood place of QM in the description of nature, such a scheme of mechanics must incorporate, at least as a subdiscipline or in some approximation, an ambiguity-free autonomous development of QM . One expects that, such a development will, starting with some appealing basics, connect smoothly to the traditional Hilbert space QM and facilitate a satisfactory treatment of measurements.

Wightman's (1976) article describes Hilbert's own work related to Hilprob6 and works relating to axiomatic treatment of QM and quantum field theory (QFT). It appears fair to say that none of the works described by Wightman nor any of the works relating to the foundations of QM that appeared later (Holevo 1982; Ludwig 1985; Bell 1988; Bohm and Hiley 1993; Peres 1993; Busch, Grabouski and Lahti 1995) provides a formalism satisfying the conditions stated above. In these two papers, we shall evolve a formalism of the desired sort which does the needful for (non-relativistic) QM and holds promise to provide a base for a solution of (the augmented) Hilprob6.

Physics is concerned with observations, studying correlations between observations, theorizing about those correlations and making theory-based conditional predictions/retrodictions about observations. We are accustomed to describing observations as geometrical facts. The desired all-embracing formalism must, therefore, have an all-embracing underlying geometry. An appealing choice for the same is noncommutative geometry (NCG) (Connes 1994; Dubois-Violette 1991; Madore 1995; Landi 1997; Gracia Bondia, Varilly and Figuerra 2001). Non-commutativity is the hallmark of QM. Indeed, the central point made in Heisenberg's (1925) paper that marked the birth of QM was that the kinematics underlying QM must be based on a non-commutative algebra of observables. This idea was developed into a scheme of mechanics — called matrix mechanics — by Born, Jordan, Dirac and Heisenberg (Born and Jordan 1925; Dirac 1926; Born, Heisenberg and Jordan 1926). The proper geometrical framework for the construction of the quantum Poisson brackets of matrix mechanics is provided by non-commutative symplectic structures (Dubois-Violette 1991,1995,1999; Dubois-Violette, Kerner and Madore 1990; Djemai 1995). The NCG scheme employed in these works (referred to henceforth as DVNCG) is a straightforward generalization of the scheme of commutative differential geometry in which the algebra $C^\infty(M)$ of smooth functions on a manifold M is replaced by a general (not necessarily commutative) complex associative *-algebra $\mathcal{A}$ and the Lie algebra $\mathcal{X}(M)$ of smooth vector fields on M by the Lie algebra Der($\mathcal{A}$) of derivations of $\mathcal{A}$.

We shall exploit the fact that the *-algebras of the type employed in DVNCG also provide a general framework for an observable-state based treatment of quantum probability (Meyer 1995). This allows us to adopt the strategy of combining elements of noncommutative symplectic geometry and noncommutative probability in an algebraic framework ; this would be a reasonably deep level realization of UPD in the true spirit of Hilprob6.

The scheme based on normed algebras (Jordan, von Neumann and Wigner 1934; Segal

1947, 1963; Haag and Kastler 1964; Haag 1992;Emch 1972, 1984; Bratteli and Robinson 1979, 1981), although it has an observable - state framework of the type mentioned above, does not serve our needs because it is not suitable for a sufficiently general treatment of noncommutative symplectic geometry. Iguri and Castagnino (1999) have analyzed the prospects of a more general class of algebras (nuclear, barreled b*-algebras) as a mathematical framework for the formulation of quantum principles prospectively better than that of the normed algebras. These algebras accommodate unbounded observables at the abstract level. Following essentially the 'footsteps' of Segal (1947), they obtain results parallel to those in the C*-algebra theory — an extremal decomposition theorem for states, a functional representation theorem for commutative subalgebras of observables and an extension of the classical GNS theorem. In a sense, this work is complementary to the present one where the emphasis is on the development of noncommutative Hamiltonian *mechanics*.

Development of such a mechanics in the algebraic framework requires some augmentation of DVNCG; in particular, one needs the noncommutative analogues of the push-forward and pull-back mappings induced by diffeomorphisms between manifolds on vector fields and differential forms. These were introduced by the author in an article entitled " Noncommutative geometry and unified formalism for classical and quantum mechanics" (IIT Kanpur preprint, 1993; henceforth referred to as TD93) and in TD02. In fact, in section 4 of TD02, a scheme of mechanics of the above mentioned sort was introduced. This work, however, was restricted to providing the proper geometrical setting for the matrix mechanics mentioned above and did not introduce states in the algebraic setting; it, therefore, falls short of a proper realization of the strategy mentioned above. In the present work, this deficiency has been removed and a proper integration of noncommutative symplectic geometry and noncommutative probability has been achieved. Moreover, to accommodate fermionic objects on an equal footing with the bosonic ones, the scheme developed here is based on superalgebras.

In TD02, a generalization of DVNCG based on algebraic pairs $(\mathcal{A}, \mathcal{X})$ where $\mathcal{A}$ is a *-algebra as above and $\mathcal{X}$ is a Lie subalgebra of $\mathrm{Der}(\mathcal{A})$ was introduced. (This is noncommutative analogue of working on a leaf of a foliation; see section II D.) Such a generalization will be needed in the algebraic treatment of general quantum systems admitting superselection rules (see section III in part II).

In the next section, a superalgebraic version of DVNCG is presented which incorporates the improvement in the definition of noncommutative differential forms introduced in (Dubois-Violette 1995,1999) [i.e. demanding $\omega(..., KX, ...) = K\omega(..., X, ...)$ where K is in the center of the algebra; for notation, see section II] and the augmentation and generalization of DVNCG mentioned above. In section III, a straightforward development of noncommutative symplectic geometry and Hamiltonian mechanics is presented. It includes, besides an observable-state based algebraic treatment of mechanics, a treatment of symplectic actions of Lie groups and noncommutative generalizations of the momentum map , Poincare-Cartan form and the symplectic version of Noether's theorem.

Section IV contains the treatment of two coupled systems in the framework of noncommutative Hamiltonian mechanics (NHM). An important result obtained there is that the tensor product of the two system algebras admits the 'canonically induced' symplectic structure if and only if either both the superalgebras are supercommutative or both non-supercommutative with a 'quantum symplectic structure' [i.e. one which gives a Poisson bracket which is a (super-)commutator up to multiplication by a constant ($i\lambda^{-1}$) where λ is a real-valued constant of the dimension of action] characterized by a *universal* parameter λ. It follows that the formalism, firstly, prohibits a quantum-classical interaction, and, secondly, has a natural place for the Planck constant as a universal constant — just as special relativity has a natural place for a universal speed. In fact, the situation here is somewhat better because, whereas, in special relativity, the existence of a universal speed is *postulated*, in NHM, the existence of a universal Planck-like constant is *dictated/predicted* by the formalism.

The formalism of NHM needs to be augmented to make a satisfactory autonomous treatment of quantum systems possible. These developments along with the promised autonomous treatment of quantum systems including quantum measurements will be presented in part II.

Note. The author had earlier presented this work in a somewhat easy-paced write-up in a single paper entitled "Supmech : the geometro-statistical formalism underlying quantum mechanics"(arxiv : 0807.3604 v3; henceforth referred to as TD08). The length of that article, however, created problems for its publication. The present two-part work is an improved (mainly in the treatment of quantum systems) and reorganized version of TD08 and supersedes that work.

II. SUPERDERIVATION-BASED DIFFERENTIAL CALCULUS

Note. In most applications, the non-super version of the formalism developed below is adequate; this can be obtained by simply putting,in the formulas below, all the epsilons representing parities equal to zero.

A. Superalgebras and superderivations

We start by recalling a few basic concepts in superalgebra (Manin 1988; Berezin 1987; Leites 1980; Scheunert 1979). A *supervector space* is a (complex) vector space $V = V^{(o)} \oplus V^{(1)}$; a vector $v \in V$ can be uniquely expressed as a sum $v = v_0 + v_1$ of even and odd vectors; they are assigned parities $\epsilon(v_0) = 0$ and $\epsilon(v_1) = 1$. Objects with definite parity are called homogeneous. We shall denote the parity of a homogeneous object w by $\epsilon(w)$ or ϵ_w according to convenience. A *superalgebra* $\mathcal{A}$ is a supervector space which is an associative algebra with identity (denoted as I); it becomes a **-superalgebra* if an antilinear *-operation or involution $* : \mathcal{A} \to \mathcal{A}$ is defined such that

$$(AB)^* = \eta_{AB} B^* A^*, \quad (A^*)^* = A, \quad I^* = I \text{ where } \eta_{AB} = (-1)^{\epsilon_A \epsilon_B}.$$

An element $A \in \mathcal{A}$ will be called *hermitian* if $A^* = A$. The *supercommutator* of two elements A,B of $\mathcal{A}$ is defined as $[A, B] = AB - \eta_{AB} BA$. We shall also employ the notations $[A, B]_\mp = AB \mp BA$. A superalgebra $\mathcal{A}$ is said to be *supercommutative* if the supercommutator of every pair of its elements vanishes. The *graded center* of $\mathcal{A}$, denoted as $Z(\mathcal{A})$, consists of those elements of $\mathcal{A}$ which have vanishing supercommutators with all elements of $\mathcal{A}$; it is clearly a supercommutative superalgebra. Writing $Z(\mathcal{A}) = Z^{(0)}(\mathcal{A}) \oplus Z^{(1)}(\mathcal{A})$, the object $Z^{(0)}(\mathcal{A})$ is the traditional center of $\mathcal{A}$. A *(*-)homomorphism* of a superalgebra $\mathcal{A}$ into $\mathcal{B}$ is a linear mapping $\Phi : \mathcal{A} \rightarrow \mathcal{B}$ which preserves products, identity elements, parities (and involutions) :

$$\Phi(AB) = \Phi(A)\Phi(B), \quad \Phi(I_{\mathcal{A}}) = I_{\mathcal{B}}, \quad \epsilon(\Phi(A)) = \epsilon(A), \quad \Phi(A^*) = (\Phi(A))^*;$$

if it is, moreover, bijective, it is called a *(*-)isomorphism.* A *Lie superalgebra* is a supervector space $\mathcal{L}$ with a *superbracket* operation $[\ ,\ ] : \mathcal{L} \times \mathcal{L} \rightarrow \mathcal{L}$ which is (i) bilinear, (ii) graded skew-symmetric which means that, for any two homogeneous elements $a, b \in \mathcal{L}$, $[a, b] = -\eta_{ab}[b, a]$ and (iii) satisfies the *super Jacobi identity*

$$[a, [b, c]] = [[a, b], c] + \eta_{ab}[b, [a, c]].$$

A (homogeneous) *superderivation* of a superalgebra $\mathcal{A}$ is a linear map $X : \mathcal{A} \rightarrow \mathcal{A}$ such that $X(AB) = X(A)B + \eta_{XA} AX(B)$. Defining the multiplication operator μ on $\mathcal{A}$ as $\mu(A)B = AB$ and, in the equation defining the superderivation X above, expressing every term as a sequence of mappings acting on the element B, it is easily seen that a necessary and sufficient condition that a linear map $X : \mathcal{A} \rightarrow \mathcal{A}$ is a superderivation is

$$X \circ \mu(A) - \eta_{XA}\ \mu(A) \circ X = \mu(X(A)). \tag{1}$$

The set of all superderivations of $\mathcal{A}$ constitutes a Lie superalgebra SDer($\mathcal{A}$) [$= SDer(\mathcal{A})^{(0)} \oplus SDer(\mathcal{A})^{(1)}$]. The *inner superderivations* D_A defined by $D_A B = [A, B]$ satisfy the relation $[D_A, D_B] = D_{[A,B]}$ and constitute a Lie sub-superalgebra ISDer($\mathcal{A}$) of SDer($\mathcal{A}$).

As in DVNCG, it will be implicitly assumed that the superalgebras being employed have a reasonably rich supply of superderivations so that various constructions involving them have a nontrivial content. Some discussion and useful results relating to the precise characterization of the relevant class of algebras may be found in (Dubois-Violette et al. 2001).

It is easily verified that

(i) If $K \in Z(\mathcal{A})$, then $X(K) \in Z(\mathcal{A})$ for all $X \in SDer(\mathcal{A})$.

(ii) For any $K \in Z(\mathcal{A})$ and $X, Y \in SDer(\mathcal{A})$, we have

$$[X, KY] = X(K)Y + \eta_{XK} K[X, Y]. \tag{2}$$

An involution $*$ on $SDer(\mathcal{A})$ is defined by the relation $X^*(A) = [X(A^*)]^*$. It is easily verified that

(i) $[X, Y]^* = [X^*, Y^*]$; $\quad (ii)(D_A)^* = -D_{A^*}$.

A superalgebra-isomorphism $\Phi : \mathcal{A} \rightarrow \mathcal{B}$ induces a mapping

$$\Phi_* : SDer(\mathcal{A}) \rightarrow SDer(\mathcal{B}) \ \text{ given by } \ (\Phi_* X)(B) = \Phi(X[\Phi^{-1}(B)]) \tag{3}$$

for all $X \in SDer(\mathcal{A})$ and $B \in \mathcal{B}$. It is the analogue (and a generalization) of the push-forward mapping induced by a diffeomorphism between two manifolds on the vector fields and satisfies the expected relations (with $\Psi : \mathcal{B} \to \mathcal{C}$)

$$(\Psi \circ \Phi)_* = \Psi_* \circ \Phi_*; \quad \Phi_*[X, Y] = [\Phi_* X, \Phi_* Y]. \tag{4}$$

Proof : (i) For any $X \in Sder(\mathcal{A})$ and $C \in \mathcal{C}$,

$$\begin{aligned} [(\Psi \circ \Phi)_* X](C) &= (\Psi \circ \Phi)(X[(\Psi \circ \Phi)^{-1}(C)]) = \Psi[\Phi(X[\Phi^{-1}(\Psi^{-1}(C))])] \\ &= \Psi[(\Phi_* X)(\Psi^{-1}(C)] = [\Psi_*(\Phi_* X)](C). \end{aligned}$$

(ii) For any $B \in \mathcal{B}$

$$\begin{aligned} (\Phi_*[X, Y])(B) &= \Phi([X, Y](\Phi^{-1}(B))) \\ &= \Phi[X(Y(\Phi^{-1}(B))) - \eta_{XY} Y(X(\Phi^{-1}(B))]. \end{aligned}$$

Now insert $\Phi^{-1} \circ \Phi$ between X and Y in each of the two terms on the right and follow the obvious steps. $\square$.

Note that Φ_* is a Lie superalgebra isomorphism.

B. Noncommutative differential forms

For the constructions involving the superalgebraic generalization of DVNCG given in this subsection, some relevant background is provided in (Dubois-Violette 1999; Grosse and Reiter 1999; Scheunert 1979a,1979b,1983,1998). Grosse and Reiter (1999) have given a detailed treatment of the differential geometry of graded matrix algebras [generalizing the treatment of differential geometry of matrix algebras in (Dubois-Violette, Kerner and Madore 1990)]. Some related work on supermatrix geometry has also appeared in (Dubois-Violette, Kerner and Madore 1991; Kerner 1993); however, the approach adopted below is closer to (Grosse and Reiter 1999).

The formalism of DVNCG employs Chevalley-Eilenberg cochains (Cartan and Eilenberg 1956; Weibel 1994; Giachetta, Mangiarotti and Sardanshvily 2005). We recall some related basics below.

Let $\mathcal{G}$ be a Lie algebra over the field K [which may be R (real numbers) or C (complex numbers)] and V a $\mathcal{G}$-module which means it is a vector space over K having defined on it a $\mathcal{G}$-action associating a linear mapping $\Psi(\xi)$ on V with every element ξ of $\mathcal{G}$ such that

$$\Psi(0) = id_V \quad \text{and} \quad \Psi([\xi, \eta]) = \Psi(\xi) \circ \Psi(\eta) - \Psi(\eta) \circ \Psi(\xi)$$

where id_V is the identity mapping on V. A V-valued p-cochain $\lambda^{(p)}$ of $\mathcal{G}$ is a skew-symmetric multilinear map from $\mathcal{G}^p$ into V. These cochains constitute a vector space $C^p(\mathcal{G}, V)$. The coboundary operator $d : C^p(\mathcal{G}, V) \to C^{p+1}(\mathcal{G}, V)$ defined by

$$\begin{aligned} (d\lambda^{(p)})(\xi_0, \xi_1, .., \xi_p) = \sum_{i=0}^{p} (-1)^i \Psi(\xi)[\lambda^{(p)}(\xi_0, .., \hat{\xi}_i, .., \xi_p)] + \\ \sum_{0 \leq i < j \leq p} (-1)^j \lambda^{(p)}(\xi_0, .., \xi_{i-1}, [\xi_i, \xi_j], \xi_{i+1}, .., \hat{\xi}_j, .., \xi_p) \end{aligned} \tag{5}$$

for $\xi_0, .., \xi_p \in \mathcal{G}$ satisfies the condition $d^2 = 0$. Defining

$$C(\mathcal{G}, V) = \oplus_{p \geq 0} C^p(\mathcal{G}, V) \text{ [with} C^0(\mathcal{G}, V) = V],$$

the pair $(C(\mathcal{G}, V), d)$ constitutes a cochain complex. The subspaces of $C^p(\mathcal{G}, V)$ consisting of closed cochains (cocycles)[i.e. those λ^p satisfying $d\lambda^p = 0$] and exact cochains (coboundaries) [i.e. those λ^p satisfying $\lambda^p = d\mu^{p-1}$ for some (p-1)-cochain μ^{p-1}] are denoted as $Z^p(\mathcal{G}, V)$ and $B^p(\mathcal{G}, V)$ respectively; the quotient space $H^p(\mathcal{G}, V) \equiv Z^p(\mathcal{G}, V)/B^p(\mathcal{G}, V)$ is called the p-th cohomology group of $\mathcal{G}$ with coefficients in V.

For the special case of the trivial action of $\mathcal{G}$ on V [i.e. $\Psi(\xi) = 0 \ \forall \xi \in \mathcal{G}$], a subscript zero is attached to these spaces [$C_0^p(\mathcal{G}, V)$ etc]. In this case, for p = 1 and p = 2, Eq.(5) takes the form

$$\begin{aligned} d\lambda^{(1)}(\xi_0, \xi_1) &= -\lambda^{(1)}([\xi_0, \xi_1]) \\ d\lambda^{(2)}(\xi_0, \xi_1, \xi_2) &= -[\lambda^{(2)}([\xi_0, \xi_1], \xi_2) + \text{cyclic terms in } \xi_0, \xi_1, \xi_2]. \end{aligned} \tag{6}$$

Recalling that, the classical differential p-forms on a manifold M are defined as skew-symmetric multilinear maps of $\mathcal{X}(M)^p$ into $C^\infty(M)$, the De Rham complex of classical differential forms can be seen as a special case of Chevalley-Eilenberg complex with $\mathcal{G} = \mathcal{X}(M)$, V $= C^\infty(M)$ and $\Psi(X)(f) = X(f)$ in obvious notation. Replacing $C^\infty(M)$ by a superalgebra [complex, associative, unital (i.e. possessing a unit element), not necessarily supercommutative] and $\mathcal{X}(M)$ by $SDer(\mathcal{A})$, a natural choice for the space of (noncommutative) differential p-forms is the space

$$C^p(SDer\mathcal{A}, \mathcal{A}) \ [= C^{p,0}(SDer(\mathcal{A}), \mathcal{A}) \oplus C^{p,1}(SDer(\mathcal{A}), \mathcal{A})]$$

of graded skew-symmetric multilinear maps (for $p \geq 1$) of $[SDer(\mathcal{A})]^p$ into $\mathcal{A}$ [the space of $\mathcal{A}$-valued p-cochains of $SDer(\mathcal{A})$] with $C^0(SDer(\mathcal{A}), \mathcal{A}) = \mathcal{A}$. For $\omega \in C^{p,s}(SDer(\mathcal{A}), \mathcal{A})$, we have

$$\omega(.., X, Y, ..) = -\eta_{XY}\omega(.., Y, X, ..). \tag{7}$$

For a general permutation σ of the arguments of ω, we have

$$\omega(X_{\sigma(1)}, .., X_{\sigma(p)}) = \kappa_\sigma \gamma_p(\sigma; \epsilon_{X_1}, .., \epsilon_{X_p})\omega(X_1, .., X_p)$$

where κ_σ is the parity of the permutation σ and

$$\gamma_p(\sigma; s_1, .., s_p) = \prod_{\substack{j,k = 1, .., p; \\ j < k, \sigma^{-1}(j) > \sigma^{-1}(k)}} (-1)^{s_j s_k}.$$

An involution * on the cochains is defined by the relation $\omega^*(X_1, .., X_p) = [\omega(X_1^*, .., X_p^*)]^*$; ω is said to be real (imaginary) if $\omega^* = \omega(-\omega)$. The *exterior product*

$$\wedge : C^{p,r}(SDer(\mathcal{A}), \mathcal{A}) \times C^{q,s}(SDer(\mathcal{A}), \mathcal{A}) \to C^{p+q,r+s}(SDer(\mathcal{A}), \mathcal{A})$$

is defined as

$$(\alpha \wedge \beta)(X_1, .., X_{p+q}) = \frac{1}{p!q!} \sum_{\sigma \in \mathcal{S}_{p+q}} \kappa_\sigma \gamma_{p+q}(\sigma; \epsilon_{X_1}, .., \epsilon_{X_{p+q}})(-1)^{s \sum_{j=1}^{p} \epsilon_{X_{\sigma(j)}}}$$
$$\alpha(X_{\sigma(1)}, .., X_{\sigma(p)})\beta(X_{\sigma(p+1)}, .., X_{\sigma(p+q)}).$$

With this product, the graded vector space

$$C(SDer(\mathcal{A}), \mathcal{A}) = \bigoplus_{p \geq 0} C^p(SDer(\mathcal{A}), \mathcal{A})$$

becomes an $N_0 \times Z_2$-bigraded complex algebra. (Here N_0 is the set of non-negative integers.)

The Lie superalgebra $SDer(\mathcal{A})$ acts on itself and on C($SDer(\mathcal{A})$,$\mathcal{A}$) through *Lie derivatives*. For each $Y \in SDer(\mathcal{A})^{(r)}$, one defines linear mappings $L_Y : SDer(\mathcal{A})^{(s)} \to SDer(\mathcal{A})^{(r+s)}$ and $L_Y : C^{p,s}(SDer(\mathcal{A}), \mathcal{A}) \to C^{p,r+s}(SDer(\mathcal{A}), \mathcal{A})$ such that the following three conditions hold :

$$L_Y(A) = Y(A) \quad \text{for all } A \in \mathcal{A}$$

$$L_Y[X(A)] = (L_Y X)(A) + \eta_{XY} X[L_Y(A)]$$

$$L_Y[\omega(X_1, .., X_p)] = (L_Y \omega)(X_1, .., X_p) + \sum_{i=1}^{p} (-1)^{\epsilon_Y(\epsilon_\omega + \epsilon_{X_1} + .. + \epsilon_{X_{i-1}})}.$$
$$.\omega(X_1, .., X_{i-1}, L_Y X_i, X_{i+1}, .., X_p).$$

The first two conditions give

$$L_Y X = [Y, X]$$

which, along with the third, gives

$$(L_Y \omega)(X_1, .., X_p) = Y[\omega(X_1, .., X_p)] - \sum_{i=1}^{p} (-1)^{\epsilon_Y(\epsilon_\omega + \epsilon_{X_1} + .. + \epsilon_{X_{i-1}})}.$$
$$.\omega(X_1, .., X_{i-1}, [Y, X_i], X_{i+1}, .., X_p). \tag{8}$$

Two important properties of the Lie derivative are, in obvious notation,

$$[L_X, L_Y] = L_{[X,Y]}$$

$$L_Y(\alpha \wedge \beta) = (L_Y \alpha) \wedge \beta + \eta_{Y\alpha} \; \alpha \wedge (L_Y \beta).$$

For each $X \in SDer(\mathcal{A})^{(r)}$, we define the *interior product* $i_X : C^{p,s}(SDer(\mathcal{A}), \mathcal{A}) \to C^{p-1,r+s}(SDer(\mathcal{A}), \mathcal{A})$ (for $p \geq 1$) by

$$(i_X \omega)(X_1, .., X_{p-1}) = \omega(X, X_1, .., X_{p-1}).$$

One defines $i_X(A) = 0$ for all $A \in \mathcal{A}$. Some important properties of the interior product are :

$$i_X \circ i_Y + \eta_{XY} i_Y \circ i_X = 0$$

$$i_X(\alpha \wedge \beta) = \eta_{X\beta}(i_X\alpha) \wedge \beta + (-1)^p \alpha \wedge (i_X\beta)$$

$$(L_Y \circ i_X - i_X \circ L_Y) = \eta_{X\omega} i_{[X,Y]}\omega. \tag{9}$$

The *exterior derivative* $d : C^{p,r}(SDer(\mathcal{A}), \mathcal{A}) \rightarrow C^{p+1,r}(SDer(\mathcal{A}), \mathcal{A})$ is defined through the relation

$$(i_X \circ d + d \circ i_X)\omega = \eta_{X\omega}\ L_X\omega. \tag{10}$$

For p = 0, it takes the form $(dA)(X) = \eta_{XA}\ X(A)$. Taking, in Eq.(10), ω a homogeneous p-form and contracting both sides with homogeneous derivations $X_1, .., X_p$ gives the quantity $(d\omega)(X, X_1, .., X_p)$ in terms of evaluations of exterior derivatives of lower degree forms. This determines $d\omega$ recursively giving

$$\begin{aligned}(d\omega)(X_0, X_1, .., X_p) = \sum_{i=0}^{p}(-1)^{i+a_i} X_i[\omega(X_0, .., \hat{X}_i, .., X_p)] \\ + \sum_{0 \leq i < j \leq p} (-1)^{j+b_{ij}} \omega(X_0, .., X_{i-1}, [X_i, X_j], X_{i+1}, .., \hat{X}_j, .., X_p)\end{aligned} \tag{11}$$

where the hat indicates omission and

$$a_i = \epsilon_{X_i}(\epsilon_\omega + \sum_{k=0}^{i-1} \epsilon_{X_k});\ b_{ij} = \epsilon_{X_j} \sum_{k=i+1}^{j-1} \epsilon_{X_k};$$

it is clearly a special case of Eq.(5). Some important properties of the exterior derivative are (i) $d^2 (= d \circ d) = 0,\ (ii) d \circ L_Y = L_Y \circ d$ and

$$(iii)\quad d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^p \alpha \wedge (d\beta)$$

where α is a p-cochain. The first of these equations shows that the pair, (C($SDer(\mathcal{A})$, $\mathcal{A}$), d) constitutes a cochain complex.

Taking clue from (Dubois-Violette 1995, 1999) [where the subcomplex of $Z(\mathcal{A})$-linear cochains (Z($\mathcal{A}$) being, in his notation, the center of the algebra $\mathcal{A}$) was adopted as the space of differential forms], we consider the subset $\Omega(\mathcal{A})$ of $C(SDer(\mathcal{A}), \mathcal{A})$ consisting of $Z^{(0)}(\mathcal{A})$-linear cochains. Eq.(2) ensures that this subset is closed under the action of d and, therefore, a subcomplex. We shall take this space to be the space of differential forms in subsequent geometrical work. We have, of course,

$$\Omega(\mathcal{A}) = \oplus_{p \geq 0} \Omega^p(\mathcal{A})$$

with $\Omega^0(\mathcal{A}) = \mathcal{A}$ and $\Omega^p(\mathcal{A}) = \Omega^{p,0}(\mathcal{A}) \oplus \Omega^{p,1}(\mathcal{A})$ for all $p \geq 0$.

C. Induced mappings on differential forms

A superalgebra *-isomorphism $\Phi : \mathcal{A} \to \mathcal{B}$ induces, besides the Lie superalgebra-isomorphism $\Phi_* : SDer(\mathcal{A}) \to SDer(\mathcal{B})$, a mapping

$$\Phi^* : C^{p,s}(SDer(\mathcal{B}), \mathcal{B}) \to C^{p,s}(SDer(\mathcal{A}), \mathcal{A})$$

given, in obvious notation, by

$$(\Phi^*\omega)(X_1, .., X_p) = \Phi^{-1}[\omega(\Phi_* X_1, .., \Phi_* X_p)]. \tag{12}$$

The mapping Φ preserves (bijectively) all the algebraic relations. Eq.(3) shows that Φ_* preserves $Z^{(0)}(\mathcal{A})$-linear combinations of the superderivations. It follows that Φ^* maps differential forms onto differential forms. These mappings are analogues (and generalizations) of the pull-back mappings on differential forms (on manifolds) induced by diffeomorphisms. They satisfy the expected relations [with $\Psi : \mathcal{B} \to \mathcal{C}$]

$$(\Psi \circ \Phi)^* = \Phi^* \circ \Psi^* \tag{13}$$

$$\Phi^*(\alpha \wedge \beta) = (\Phi^*\alpha) \wedge (\Phi^*\beta) \tag{14}$$

$$\Phi^*(d\alpha) = d(\Phi^*\alpha). \tag{15}$$

Outlines of proofs of equations (13-15) :

Eq.(13) : For $\omega \in C^{p,s}(SDer(\mathcal{C}), \mathcal{C})$ and $X_1, .., X_p \in SDer(\mathcal{A})$,

$$\begin{aligned} [(\Psi \circ \Phi)^*\omega](X_1, .., X_p) &= (\Phi^{-1} \circ \Psi^{-1})[\omega(\Psi_*(\Phi_* X_1), .., \Psi_*(\Phi_* X_p))] \\ &= \Phi^{-1}[(\Psi^*\omega)(\Phi_* X_1, .., \Phi_* X_p)] \\ &= [\Phi^*(\Psi^*\omega)](X_1, .., X_p). \ \Box \end{aligned}$$

Eq.(14) : For $\alpha \in C^{p,r}(SDer(\mathcal{B}), \mathcal{B}), \beta \in C^{q,s}(SDer(\mathcal{B}), \mathcal{B})$ and $X_1, .., X_{p+q} \in SDer(\mathcal{A})$,

$$[\Phi^*(\alpha \wedge \beta)](X_1, .., X_{p+q}) = \Phi^{-1}[(\alpha \wedge \beta)(\Phi_* X_1, .., \Phi_* X_{p+q})].$$

Expanding the wedge product and noting that

$$\Phi^{-1}[\alpha(..)\beta(..)] = \Phi^{-1}[\alpha(..)].\Phi^{-1}[\beta(..)],$$

the right hand side is easily seen to be equal to $[(\Phi^*\alpha) \wedge (\Phi^*\beta)](X_1, .., X_{p+q})$. $\Box$
Eq.(15) : We have

$$[\Phi^*(d\alpha)](X_0, .., X_p) = \Phi^{-1}[(d\alpha)(\Phi_* X_0, .., \Phi_* X_p)].$$

Using Eq.(11) for $d\alpha$ and noting that

$$\begin{aligned}\Phi^{-1}[(\Phi_* X_i)(\alpha(\Phi_* X_0, ..)) &= \Phi^{-1}[\Phi(X_i[\Phi^{-1}(\alpha(\Phi_* X_0, ..))]] \\ &= X_i[(\Phi^* \alpha)(X_0, ..)]\end{aligned}$$

and making similar (in fact, simpler) manipulations with the second term in the expression for $d\alpha$, it is easily seen that the left hand side of Eq.(15), evaluated at $(X_0, .., X_p)$, equals $[(d(\Phi^*\alpha)](X_0, .., X_p)$. $\Box$

Now, let $\Phi_t : \mathcal{A} \to \mathcal{A}$ be a one-parameter family of transformations (i.e. superalgebra isomorphisms) given, for small t, by $\Phi_t(A) \simeq A + tg(A)$ where g is some (linear, even) mapping of $\mathcal{A}$ into itself. The condition $\Phi_t(AB) = \Phi_t(A)\Phi_t(B)$ gives $g(AB) = g(A)B + Ag(B)$ implying that $g(A) = Y(A)$ for some even superderivation Y of $\mathcal{A}$ (to be called the *infinitesimal generator* of Φ_t). From Eq.(3), we have, for small t,

$$(\Phi_t)_* X \simeq X + t[Y, X] = X + tL_Y X. \tag{16}$$

Similarly, for any p-form ω, we have

$$\Phi_t^* \omega \simeq \omega - tL_Y \omega. \tag{17}$$

Proof : We have

$$\begin{aligned}(\Phi_t^* \omega)(X_1, .., X_p) &= \Phi_t^{-1}[\omega\,((\Phi_t)_* X_1, ..(\Phi_t)_* X_p)] \\ &\simeq \omega(X_1, .., X_p) - tY\omega(X_1, .., X_p) \\ &\quad + t\sum_{i=1}^{p} \omega(X_1, .., [Y, X_i], .., X_p) \\ &= [\omega - tL_Y \omega]](X_1, .., X_p). \quad \Box\end{aligned}$$

D. A generalization of the DVNCG scheme

In the formula (11) for $d\omega$, the superderivations X_j appear on the right either singly or as supercommutators. It should, therefore, be possible to restrict them to a Lie sub-superalgebra $\mathcal{X}$ of $SDer(\mathcal{A})$ and develop the whole formalism with the pair $(\mathcal{A}, \mathcal{X})$ obtaining thereby a useful generalization of the formalism developed in the previous two subsections. Working with such a pair is the analogue of restricting oneself to a leaf of a foliated manifold as the example below indicates.

Example : $\mathcal{A} = C^\infty(R^3)$; $\mathcal{X}$= the Lie subalgebra of the Lie algebra $\mathcal{X}(R^3)$ of vector fields on R^3 generated by the Lie differential operators $L_j = \epsilon_{jkl} x_k \partial_l$ for the SO(3)-action on R^3. These differential operators, when expressed in terms of the polar coordinates r, θ, ϕ, contain only the partial derivatives with respect to θ and ϕ; they, therefore, act on the 2-dimensional spheres that constitute the leaves of the foliation $R^3 - \{(0, 0, 0)\} \simeq S^2 \times R$. The restriction [of the pair $(\mathcal{A}, \mathcal{X}(R^3))$] to $(\mathcal{A}$,$\mathcal{X})$ amounts to restricting oneself to a leaf (S^2) in the present case.

In the generalized formalism, one obtains the cochains $C^{p,s}(\mathcal{X},\mathcal{A})$ for which the formulas of sections II B and II C are valid (with the $X_j s$ restricted to $\mathcal{X}$). The differential forms $\Omega^{p,s}(\mathcal{A})$ will now be replaced by the objects $\Omega^{p,s}(\mathcal{X},\mathcal{A})$ obtained by restricting the cochains to the $Z_0(\mathcal{A})$-linear ones. [In the new notation, the objects $\Omega^{p,s}(\mathcal{A})$ will be called $\Omega^{p,s}(SDer(\mathcal{A}),\mathcal{A})$.]

To define the induced mappings Φ_* and Φ^* in the present context, one must employ a *pair-isomorphism* $\Phi : (\mathcal{A},\mathcal{X}) \rightarrow (\mathcal{B},\mathcal{Y})$ which consists of a superalgebra *- isomorphism $\Phi : \mathcal{A} \rightarrow \mathcal{B}$ such that the induced mapping $\Phi_* : SDer(\mathcal{A}) \rightarrow SDer(\mathcal{B})$ restricts to an isomorphism of $\mathcal{X}$ onto $\mathcal{Y}$. Various properties of the induced mappings hold as before.

Given a one-parameter family of transformations (i.e. pair isomorphisms) $\Phi_t : (\mathcal{A},\mathcal{X}) \rightarrow (\mathcal{A},\mathcal{X})$, the condition $(\Phi_t)_*\mathcal{X} \subset \mathcal{X}$ implies that the infinitesimal generator Y of Φ_t must satisfy the condition $[Y,X] \in \mathcal{X}$ for all $X \in \mathcal{X}$. In practical applications one will generally have $Y \in \mathcal{X}$ which trivially satisfies this condition.

This generalization will be used in sections III G and in section III of part II.

E. Derivations and differential forms on tensor products of superalgebras

Given two superalgebras $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$, elements of their (skew) tensor product $\mathcal{A} = \mathcal{A}^{(1)} \otimes \mathcal{A}^{(2)}$ are finite sums of tensored pairs :

$$\sum_{j=1}^{m} A_j \otimes B_j \qquad A_j \in \mathcal{A}^{(1)}, \quad B_j \in \mathcal{A}^{(2)}$$

with the multiplication rule

$$(\sum_{j=1}^{m} A_j \otimes B_j)(\sum_{k=1}^{n} A_k \otimes B_k) = \sum_{j,k} \eta_{B_j A_k}(A_j A_k) \otimes (B_j B_k).$$

The superalgebra $\mathcal{A}^{(1)}$ (resp. $\mathcal{A}^{(2)}$) has, in $\mathcal{A}$, an isomorphic copy consisting of the elements $(A \otimes I_2, A \in \mathcal{A}^{(1)})$ (resp. $I_1 \otimes B, B \in \mathcal{A}^{(2)}$) to be denoted as $\tilde{\mathcal{A}}^{(1)}$ (resp. $\tilde{\mathcal{A}}^{(2)}$). Here I_1 and I_2 are the unit elements of $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$ respectively. We shall also use the notations $\tilde{A}^{(1)} = A \otimes I_2$ and $\tilde{B}^{(2)} = I_1 \otimes B$.

Derivations and differential forms on $\mathcal{A}^{(i)}$ and $\tilde{\mathcal{A}}^{(i)}$ (i = 1,2) are formally related through the induced mappings corresponding to the isomorphisms $\Xi^{(i)} : \mathcal{A}^{(i)} \rightarrow \tilde{\mathcal{A}}^{(i)}$ given by $\Xi^{(1)}(A) = A \otimes I_2$ and $\Xi^{(2)}(B) = I_1 \otimes B$. For example, corresponding to $X \in SDer(\mathcal{A}^{(1)})$, we have $\tilde{X}^{(1)} = \Xi^{(1)}_*(X)$ in SDer$(\tilde{\mathcal{A}}^{(1)})$ given by [see Eq.(3)]

$$\tilde{X}^{(1)}(\tilde{A}^{(1)}) = \Xi^{(1)}_*(X)(\tilde{A}^{(1)}) = \Xi^{(1)}[X(A)] = X(A) \otimes I_2. \tag{18}$$

Similarly, corresponding to $Y \in SDer(\mathcal{A}^{(2)})$, we have $\tilde{Y}^{(2)} \in$ SDer$(\tilde{\mathcal{A}}^{(2)})$ given by $\tilde{Y}^{(2)}(\tilde{B}^2) = I_1 \otimes Y(B)$. For the 1-forms $\alpha \in \Omega^1(\mathcal{A}^{(1)})$ and $\beta \in \Omega^1(\mathcal{A}^{(2)})$, we have $\tilde{\alpha}^{(1)} \in \Omega^1(\tilde{\mathcal{A}}^{(1)})$ and $\tilde{\beta}^{(2)} \in \Omega^1(\tilde{\mathcal{A}}^{(2)})$ given by [see Eq.(12)]

$$\tilde{\alpha}^{(1)}(\tilde{X}^{(1)}) = \Xi^{(1)}[\alpha(((\Xi^{(1)})^{-1})_*\tilde{X}^{(1)})] = \Xi^{(1)}[\alpha(X)] = \alpha(X) \otimes I_2 \tag{19}$$

and $\tilde{\beta}^{(2)}(\tilde{Y}^{(2)}) = I_1 \otimes \beta(Y)$. Analogous formulas hold for the higher forms.

We can extend the action of the superderivations $\tilde{X}^{(1)} \in SDer(\tilde{\mathcal{A}}^{(1)})$ and $\tilde{Y}^{(2)} \in SDer(\tilde{\mathcal{A}}^{(2)})$ to $\tilde{\mathcal{A}}^{(2)}$ and $\tilde{\mathcal{A}}^{(1)}$ respectively by defining

$$\tilde{X}^{(1)}(\tilde{B}^{(2)}) = 0, \quad \tilde{Y}^{(2)}(\tilde{A}^{(1)}) = 0 \quad \text{for all} \quad A \in \mathcal{A}^{(1)} \text{ and } \quad B \in \mathcal{A}^{(2)}. \tag{20}$$

Note that an $X \in SDer(\mathcal{A})$ is determined completely by its action on the subalgebras $\tilde{\mathcal{A}}^{(1)}$ and $\tilde{\mathcal{A}}^{(2)}$:

$$X(A \otimes B) = X(\tilde{A}^{(1)}\tilde{B}^{(2)}) = (X\tilde{A}^{(1)})\tilde{B}^{(2)} + \eta_{XA}\tilde{A}^{(1)}X(\tilde{B}^{(2)}).$$

With the extensions described above, we have available to us superderivations belonging to the span of terms of the form [see Eq.(18)]

$$X = X^{(1)} \otimes I_2 + I_1 \otimes X^{(2)}. \tag{21}$$

Replacing I_2 and I_1 in Eq.(21) by elements of $Z(\mathcal{A}^{(2)})$ and $Z(\mathcal{A}^{(1)})$ respectively, we again obtain superderivations of $\mathcal{A}$. We, therefore, have the space of superderivations

$$[SDer(\mathcal{A}^{(1)}) \otimes Z(\mathcal{A}^{(2)})] \oplus [Z(\mathcal{A}^{(1)}) \otimes SDer(\mathcal{A}^{(2)})]. \tag{22}$$

This space, however, is generally only a Lie sub-superalgebra of SDer($\mathcal{A}$). For example, for $\mathcal{A}^{(1)} = M_m(C)$ and $\mathcal{A}^{(2)} = M_n(C)(m, n > 1)$, recalling that all the derivations of these matrix algebras are inner and that their centers consist of scalar multiples of the respective unit matrices , we have the (complex) dimensions of SDer($\mathcal{A}^{(1)}$), and SDer($\mathcal{A}^{(2)}$) respectively, $(m^2 - 1)$ and $(n^2 - 1)$ [so that the dimension of the space (22) is $m^2 + n^2 - 2$] whereas that of $SDer(\mathcal{A})$ is $(m^2 n^2 - 1)$.

We shall need to employ (in section IV) a class of superderivations more general than (22). To this end, it is instructive to obtain explicit representation(s) for a general derivation of the matrix algebra $\mathcal{A} = M_m(C) \otimes M_n(C)$. We have

$$[A \otimes B, C \otimes D]_{ir,js} = A_{ik}B_{rt}C_{kj}D_{ts} - C_{ik}D_{rt}A_{kj}B_{ts}$$

which gives

$$\begin{aligned} [A \otimes B, C \otimes D]_- &= AC \otimes BD - CA \otimes DB && (23) \\ &= [A, C]_- \otimes \frac{1}{2}[B, D]_+ + \frac{1}{2}[A, C]_+ \otimes [B, D]_-. && (24) \end{aligned}$$

This gives, in obvious notation,

$$\begin{aligned} D_{A \otimes B} \equiv [A \otimes B, .]_- &= A.(.) \otimes B.(.) - (.).A \otimes (.).B && (25) \\ &= D_A \otimes J_B + J_A \otimes D_B && (26) \end{aligned}$$

where J_B is the linear mapping on $\mathcal{A}^{(2)}$ given by $J_B(D) = \frac{1}{2}[B, D]_+$ and a similar expression for J_A as a linear mapping on $\mathcal{A}^{(1)}$. Eq.(25) shows that a derivation of the algebra $\mathcal{A} = \mathcal{A}^{(1)} \otimes \mathcal{A}^{(2)}$

need not explicitly contain those of $\mathcal{A}^{(i)}$. We shall, however, not get involved in the search for the most general expression for a derivation of the tensor product algebra $\mathcal{A}$(although such an expression would be very useful). The expression (26) is more useful for us; it is a special case of the more general form

$$X = X_1 \otimes \Psi_2 + \Psi_1 \otimes X_2 \tag{27}$$

where $X_i \in SDer(\mathcal{A}^{(i)})$ (i=1,2) and $\Psi_i : \mathcal{A}^{(i)} \to \mathcal{A}^{(i)}$ (i =1,2) are linear mappings. Our constructions in section IV A will lead us to structures of the form (27). It is important to note, however, that an expression of the form (27) (which represents a linear mapping of $\mathcal{A}$ into itself) need not always be a derivation as can be easily checked. One should impose the condition (1) on such an expression to obtain a derivation.

A straightforward procedure to obtain general differential forms and the exterior derivative on $\mathcal{A}$ is to obtain the graded differential space $(\Omega(\mathcal{A}), d)$ as the tensor product (Greub 1978) of the graded differential spaces $(\Omega(\mathcal{A}^{(1)}), d_1)$ and $(\Omega(\mathcal{A}^{(2)}), d_2)$. A (homogeneous) differential k-form on $\mathcal{A}$ is of the form (in obvious notation)

$$\alpha_{kt} = \sum_{\substack{i+j=k \\ r+s=t\ mod(2)}} \alpha^{(1)}_{ir} \otimes \alpha^{(2)}_{js}.$$

The d operation on $\Omega(\mathcal{A})$ is given by [here $\alpha \in \Omega^p(\mathcal{A}^{(1)})$ and $\beta \in \Omega(\mathcal{A}^{(2)})$]

$$d(\alpha \otimes \beta) = (d_1\alpha) \otimes \beta + (-1)^p \alpha \otimes d_2\beta. \tag{28}$$

III. NONCOMMUTATIVE SYMPLECTIC GEOMETRY AND HAMILTONIAN MECHANICS

We shall now present a treatment of noncommutative symplectic geometry [extending the treatment of noncommutative symplectic structures by Dubois-Violette (1991, 1995, 1999)] and Hamiltonian mechanics along lines parallel to the classical symplectic geometry and hamiltonian mechanics.

A. Symplectic structures

Note. The sign conventions about various quantities adopted below are parallel to those of Woodhouse (1980). This results in a (super-) Poisson bracket which, when applied to classical Hamiltonian mechanics, gives a Poisson bracket differing from the one in most current books on mechanics by a minus sign. [See Eq.(56).] The main virtue of the adopted conventions is that Eq.(33) below has no unpleasant minus sign.

A *symplectic structure* on a superalgebra $\mathcal{A}$ is a 2- form ω (the *symplectic form*) which is even, closed and *non-degenerate* in the sense that, for every $A \in \mathcal{A}$, there exists a unique superderivation Y_A in $SDer(\mathcal{A})$ [the *(globally) Hamiltonian superderivation* corresponding to A] such that

$$i_{Y_A}\omega = -dA. \tag{29}$$

The pair $(\mathcal{A}, \omega)$ will be called a *symplectic superalgebra.* A symplectic structure is said to be *exact* if the symplectic form is exact ($\omega = d\theta$ for some 1-form θ called the *symplectic potential*).

A *symplectic mapping* from a symplectic superalgebra $(\mathcal{A}, \alpha)$ to another one $(\mathcal{B}, \beta)$ is a superalgebra isomorphism $\Phi : \mathcal{A} \to \mathcal{B}$ such that $\Phi^*\beta = \alpha$. [If the symplectic structures involved are exact, one requires a symplectic mapping to preserve the symplectic potential under the pull-back action; Eq.(15) then guarantees the preservation of the symplectic form.] A symplectic mapping from a symplectic superalgebra onto itself will be called a *canonical/symplectic transformation.* The symplectic form and its exterior powers are invariant under canonical transformations.

If Φ_t is a one-parameter family of canonical transformations generated by $X \in SDer(\mathcal{A})$, the condition $\Phi_t^*\omega = \omega$ implies, with Eq.(17),

$$L_X\omega = 0. \tag{30}$$

The argument just presented gives Eq.(30) with X an even superderivation. More generally, a superderivation X (even or odd or inhomogeneous) satisfying Eq.(30) will be called a *locally Hamiltonian* superderivation. Eq.(10) and the condition $d\omega = 0$ imply that Eq.(30) is equivalent to the condition

$$d(i_X\omega) = 0. \tag{31}$$

The (globally) Hamiltonian superderivations defined by Eq(29) constitute a subclass of locally Hamiltonian superderivations for which $i_X\omega$ is exact. Note from Eq(29) that $\epsilon(Y_A) = \epsilon(A)$. In analogy with the commutative case, the supercommutator of two locally Hamiltonian superderivations is a globally Hamiltonian superderivation. Indeed, given two locally Hamiltonian superderivations X and Y, we have

$$\begin{aligned} \eta_{X\omega}\; i_{[X,Y]}\omega &= (L_Y \circ i_X - i_X \circ L_Y)\omega \\ &= \eta_{Y\omega}\; (i_Y \circ d + d \circ i_Y)(i_X\omega) \\ &= \eta_{Y\omega}\; d(i_Y i_X\omega) \end{aligned}$$

which is exact. It follows that the locally Hamiltonian superderivations constitute a Lie superalgebra in which the globally Hamiltonian superderivations constitute an ideal.

The *Poisson bracket* (PB) of two elements A and B of $\mathcal{A}$ is defined as

$$\{A, B\} = \omega(Y_A, Y_B) = Y_A(B) = -\eta_{AB}Y_B(A). \tag{32}$$

It obeys the superanalogue of the Leibnitz rule :

$$\begin{aligned} \{A, BC\} = Y_A(BC) &= Y_A(B)C + \eta_{AB}BY_A(C) \\ &= \{A, B\}C + \eta_{AB}B\{A, C\}. \end{aligned}$$

As in the classical case, we have the relation

$$[Y_A, Y_B] = Y_{\{A,B\}}. \tag{33}$$

Eqn.(33) follows by using the equation for $i_{[X,Y]}\omega$ above with $X = Y_A$ and $Y = Y_B$ and equations (32) and (29), remembering that Eq.(29) determines Y_A uniquely. The super-Jacobi identity

$$\begin{aligned} 0 &= \frac{1}{2}(d\omega)(Y_A, Y_B, Y_C) \\ &= \{A, \{B, C\}\} + (-1)^{\epsilon_A(\epsilon_B+\epsilon_C)}\{B, \{C, A\}\} \\ &\quad +(-1)^{\epsilon_C(\epsilon_A+\epsilon_B)}\{C, \{A, B\}\} \end{aligned} \tag{34}$$

is obtained by using Eq.(11) and noting that

$$\begin{aligned} Y_A[\omega(Y_B, Y_C)] &= \{A, \{B, C\}\} \\ \omega([Y_A, Y_B], Y_C) &= \omega(Y_{\{A,B\}}, Y_C) = \{\{A, B\}, C\}. \end{aligned}$$

Clearly, the pair $(\mathcal{A}, \{\ ,\ \})$ is a Lie superalgebra. Eq.(33) shows that the linear mapping $A \mapsto Y_A$ is a Lie-superalgebra homomorphism.

An element A of $\mathcal{A}$ can act, via Y_A, as the infinitesimal generator of a one-parameter family of canonical transformations. The change in $B \in \mathcal{A}$ due to such an infinitesimal transformation is

$$\delta B = \epsilon Y_A(B) = \epsilon\{A, B\}. \tag{35}$$

In particular, if $\delta B = \epsilon I$ (infinitesimal 'translation' in B), we have

$$\{A, B\} = I \tag{36}$$

which is the noncommutative analogue of the classical PB relation $\{p, q\} = 1$. A pair (A,B) of elements of $\mathcal{A}$ satisfying the condition (36) will be called a *canonical pair*.

B. Reality properties of the symplectic form and the Poisson bracket

For classical superdynamical systems, conventions about reality properties of the symplectic form are based on the fact that the Lagrangian is a real, even object (Berezin and marinov 1977; Dass 1993). The matrix of the symplectic form is then real- antisymmetric in the 'bosonic sector' and imaginary-symmetric in the 'fermionic sector' (which means anti-Hermitian in both sectors). Keeping this in view, it appears appropriate to impose, in noncommutative Hamiltonian mechanics, the following reality condition on the symplectic form ω:

$$\omega^*(X, Y) = -\eta_{XY}\omega(Y, X) \quad \text{for all } X, Y \in SDer(\mathcal{A}); \tag{37}$$

but this means, by Eq.(7), that $\omega^* = \omega$ (i.e. ω is real) which is the most natural assumption to make about ω. Eq.(37) is equivalent to the condition

$$\omega(X^*, Y^*) = -\eta_{XY}[\omega(Y, X)]^*.$$

Now, for arbitrary $A, B \in \mathcal{A}$, we have

$$\begin{aligned}\{A,B\}^* &= Y_A(B)^* = Y_A^*(B^*) = \eta_{AB} dB^*(Y_A^*) \\ &= -\eta_{AB}\omega(Y_{B^*}, Y_A^*) = [\omega(Y_A, Y_{B^*}^*)]^* \\ &= -[dA(Y_{B^*}^*)]^* = -\eta_{AB}[Y_{B^*}^*(A)]^* \\ &= -\eta_{AB} Y_{B^*}(A^*)\end{aligned}$$

giving finally

$$\{A,B\}^* = -\eta_{AB}\{B^*, A^*\}. \tag{38}$$

Eq.(38) is consistent with the reality properties of the classical and quantum Poisson brackets.[See equations (56) and (43) below.]

C. Special algebras; the canonical symplectic form

In this subsection, we shall consider a distinguished class of superalgebras (TD93, TD02; Dubois-Violette 1995, 1999) which have a canonical symplectic structure associated with them. As we shall see below and in section III in part II, these superalgebras play an important role in Quantum mechanics.

A complex, associative, non-supercommutative *-superalgebra will be called *special* if all its superderivations are inner. The differential 2-form ω_c defined on such a superalgebra $\mathcal{A}$ by

$$\omega_c(D_A, D_B) = [A, B] \tag{39}$$

is said to be the *canonical form* on $\mathcal{A}$. It is easily seen to be closed [the equation $(d\omega_c)(D_A, D_B, D_C) = 0$ is nothing but the Jacobi identity for the supercommutator], imaginary (i.e. $\omega_c^* = -\omega_c$) and dimensionless. For any $A \in \mathcal{A}$, the equation

$$\omega_c(Y_A, D_B) = -(dA)(D_B) = [A, B]$$

has the unique solution $Y_A = D_A$. (To see this, note that, since all derivations are inner, $Y_A = D_C$ for some $C \in \mathcal{A}$; the condition $\omega_c(D_C, D_B) = [C, B] = [A, B]$ for all $B \in \mathcal{A}$ implies that $(C - A) \in Z(\mathcal{A})$. But then $D_C = D_A$.$\square$) This gives

$$i_{D_A}\omega_c = -dA. \tag{40}$$

The closed and non-degenerate form ω_c defines, on $\mathcal{A}$, the *canonical symplectic structure.* It gives, as Poisson bracket, the supercommutator :

$$\{A, B\} = Y_A(B) = D_A(B) = [A, B]. \tag{41}$$

Using equations (40) and (10), it is easily seen that the form ω_c is *invariant* in the sense that $L_X\omega_c = 0$ for all $X \in SDer(\mathcal{A})$. The invariant symplectic structure on the algebra $M_n(C)$ of complex $n \times n$ matrices obtained by Dubois-Violette and coworkers (1994) is a special case of the canonical symplectic structure on special algebras described above.

If, on a special superalgebra $\mathcal{A}$, instead of ω_c, we take $\omega = b\ \omega_c$ as the symplectic form (where b is a nonzero complex number), we have

$$Y_A = b^{-1} D_A, \qquad \{A, B\} = b^{-1}[A, B]. \tag{42}$$

The *quantum Poisson bracket*

$$\{A, B\}_Q = (-i\hbar)^{-1}[A, B] \tag{43}$$

is a special case of this with $b = -i\hbar$. (Note that b must be imaginary to make ω real.) In the case of the Schrödinger representation for a nonrelativistic spinless particle, the algebra $\mathcal{A}$ generated by the position and momentum operators X_j, P_j (j= 1,2,3) defined on an invariant domain in the Hilbert space $\mathcal{H} = L^2(R^3, dx)$ is special (Dubois-Violette 1995,1999); one has, therefore, a canonical form ω_c and the *quantum symplectic structure* on $\mathcal{A}$ given by the *quantum symplectic form*

$$\omega_Q = -i\hbar\omega_c \tag{44}$$

which gives the PB (43).

We shall refer to a symplectic structure of the above sort with a general nonzero b as the *quantum symplectic structure with parameter b.*

D. Noncommutative Hamiltonian mechanics

We shall now present the formalism of noncommutative Hamiltonian mechanics (NHM) combining elements of noncommutative symplectic geometry and noncommutative probability as mentioned earlier.

The system algebra and states

In NHM, one associates, with every physical system, a symplectic superalgebra $(\mathcal{A}, \omega)$. Here we shall treat the term 'physical system' informally as is traditionally done; some formalities in this connection will be taken care of in section V in part II where the axioms are stated. The even Hermitian elements of $\mathcal{A}$ represent *observables* of the system. The collection of all observables in $\mathcal{A}$ will be denoted as $\mathcal{O}(\mathcal{A})$.

To take care of limit processes and continuity of mappings, we must employ topological algebras. The choice of the admissible class of topological algebras must meet the following reasonable requirements:

(i) It should be closed under the formation of (a) topological completions and (b) tensor products. [Both are nontrivial requirements (Dubin and Hennings 1990).]
(ii) It should include

(a) the Op*-algebras (Horuzhy 1990) based on the pairs $(\mathcal{H}, \mathcal{D})$ where $\mathcal{H}$ is a separable Hilbert space and $\mathcal{D}$ a dense linear subset of $\mathcal{H}$. [Such an algebra is an algebra of operators which, along with their adjoints, map $\mathcal{D}$ into itself. The *-operation on the algebra is defined as the restriction of the Hilbert space adjoint on $\mathcal{D}$. These are the algebras of operators (not necessarily bounded) appearing in the traditional Hilbert space QM; for example, the (Heisenberg-Schrödinger) operator algebra $\mathcal{A}$ in subsection C above belongs to this class.];
(b) Algebras of smooth functions on manifolds (to accommodate classical dynamics).
(iii) The GNS representations of the system algebra (in the non-supercommutative case) induced by various states must have *separable* Hilbert spaces as the representation spaces.

The right choice appears to be the $\hat{\otimes}$-(star-)algebras of Helemskii (1989) (i.e. locally convex *-algebras which are complete and Hausdorff with a jointly continuous product) satisfying the additional condition of being separable. [*Note.* The condition of separability may have to be dropped in applications to quantum field theory.] Henceforth all (super-)algebras employed will be assumed to belong to this class. For easy reference, unital *-algebras of this class will be called *NHM-admissible*.

A *state* on a (unital) *-algebra $\mathcal{A}$ is a linear functional ϕ on $\mathcal{A}$ which is (i) positive [which means $\phi(A^*A) \geq 0$ for all $A \in \mathcal{A}$) and (ii) normalized [i.e. $\phi(I) = 1$]. Given a state ϕ, the quantity $\phi(A)$ for any observable A is real (this can be seen by considering, for example, the quantity$\phi[(I+A)^*(I+A)]$) and is to be interpreted as the expectation value of A in the state ϕ. Following general usage in literature, we shall call observables of the form A^*A or sum of such terms *positive* (strictly speaking, the term 'non-negative' would be more appropriate); states assign non-negative expectation values to such observables. The family of all states on $\mathcal{A}$ will be denoted as $\mathcal{S}(\mathcal{A})$. It is easily seen to be closed under convex combinations: given $\phi_i \in \mathcal{S}(\mathcal{A})$, i = 1,..,n and $p_i \geq 0$ with $p_1 + .. + p_n = 1$, we have $\phi = \sum_{i=1}^n p_i \phi_i$ also in $\mathcal{S}(\mathcal{A})$. States which cannot be expressed as nontrivial convex combinations of other states will be called *pure* states and others *mixed* states or *mixtures*. The family of pure states of $\mathcal{A}$ will be denoted as $\mathcal{S}_1(\mathcal{A})$. The triple $(\mathcal{A}, \mathcal{S}_1(\mathcal{A}), \omega)$ will be referred to as a *symplectic triple*.

Expectation values of all even elements of $\mathcal{A}$ can be expressed in terms of those of the observables (by considering the breakup of such an element into its Hermitian and anti-Hermitian part). This leaves out the odd elements of $\mathcal{A}$. It appears reasonable to demand that the expectation values $\phi(A)$ of all odd elements $A \in \mathcal{A}$ must vanish for all pure states (and, therefore, for all states).

Denoting the algebraic dual of the superalgebra $\mathcal{A}$ by $\mathcal{A}^*$, an automorphism $\Phi : \mathcal{A} \rightarrow \mathcal{A}$ induces the transpose mapping $\tilde{\Phi} : \mathcal{A}^* \rightarrow \mathcal{A}^*$ such that

$$\tilde{\Phi}(\phi)(A) = \phi(\Phi(A)) \text{ or } < \tilde{\Phi}(\phi), A > = < \phi, \Phi(A) > \tag{45}$$

where the second alternative has employed the dual space pairing $<,>$. The mapping $\tilde{\Phi}$ (which is easily seen to be linear and bijective) maps states (which form a subset of $\mathcal{A}^*$) onto states. To see this, note that

(i) $\tilde{\Phi}(\phi)(A^*A) = \phi(\Phi(A^*A)) = \phi(\Phi(A)^*\Phi(A)) \geq 0$;

(ii)$[\tilde{\Phi}(\phi)](I) = \phi(\Phi(I)) = \phi(I) = 1$.

The linearity of $\tilde{\Phi}$ (as a mapping on $\mathcal{A}^*$) ensures that it preserves convex combinations of states. In particular, it maps pure states onto pure states. We have, therefore, a bijective mapping $\tilde{\Phi} : \mathcal{S}_1(\mathcal{A}) \rightarrow \mathcal{S}_1(\mathcal{A})$.

When Φ is a canonical transformation, the condition $\Phi^*\omega = \omega$ gives, for $X, Y \in SDer(\mathcal{A})$,

$$\omega(X, Y) = (\Phi^*\omega)(X, Y) = \Phi^{-1}[\omega(\Phi_* X, \Phi_* Y)]$$

which gives

$$\Phi[\omega(X, Y)] = \omega(\Phi_* X, \Phi_* Y). \tag{46}$$

Taking expectation value of both sides of this equation in a state ϕ, we get

$$(\tilde{\Phi}\phi)[\omega(X, Y)] = \phi[\omega(\Phi_* X, \Phi_* Y)]. \tag{47}$$

When Φ is an infinitesimal canonical transformation generated by $G \in \mathcal{A}$, we have

$$\tilde{\Phi}(\phi)(A) = \phi(\Phi(A)) \simeq \phi(A + \epsilon\{G, A\}).$$

Putting $\tilde{\Phi}(\phi) = \phi + \delta\phi$, we have

$$(\delta\phi)(A) = \epsilon\phi(\{G, A\}). \tag{48}$$

Dynamics

Dynamics of the system is described in terms of the one-parameter family Φ_t of canonical transformations generated by an observable H, called the *Hamiltonian*. (The parameter t is supposed to be an evolution parameter which need not always be the conventional time.) Writing $\Phi_t(A) = A(t)$ and recalling Eq.(35), we have the *Hamilton's equation* of NHM :

$$\frac{dA(t)}{dt} = Y_H[A(t)] = \{H, A(t)\}. \tag{49}$$

The triple $(\mathcal{A}, \omega, H)$ [or, more appropriately, the quadruple $(\mathcal{A}, \mathcal{S}_1(\mathcal{A}), \omega, H)$] will be called an *NHM Hamiltonian system*. It is the analogue of a classical Hamiltonian system (M, ω_{cl}, H_{cl}) [where (M, ω_{cl}) is a symplectic manifold and H_{cl}, the classical Hamiltonian (a smooth real-valued function on M); note that the specification of the symplectic manifold M serves to define both observables and pure states in classical mechanics]. As far as the evolution is concerned, the Hamiltonian is, as in the classical case, arbitrary up to the addition of a constant multiple of the identity element. We shall assume that H is bounded below in the sense that its expectation values in all pure states (hence in all states) are bounded below.

This is the analogue of the Heisenberg picture in traditional QM. An equivalent description, the analogue of the Schrödinger picture, is obtained by transferring the time dependence to states through the relation [see Eq.(45)]

$$< \phi(t), A >=< \phi, A(t) > \tag{50}$$

where $\phi(t) = \tilde{\Phi}_t(\phi)$. The mapping $\tilde{\Phi}_t$ satisfies the condition (47) which [with $\Phi = \Phi_t$] may be said to represent the canonicality of the evolution of states. With $\Phi = \Phi_t$ and G = H, Eq.(48) gives the *Liouville equation* of NHM:

$$\frac{d\phi(t)}{dt}(A) = \phi(t)(\{H, A\}) \quad \text{or} \quad \frac{d\phi(t)}{dt}(.) = \phi(t)(\{H, .\}). \tag{51}$$

Equivalent descriptions; Symmetries and conservation laws

By a 'description' of a system, we shall mean specification of its triple $(\mathcal{A}, \mathcal{S}(\mathcal{A}), \omega)$. Two descriptions are said to be *equivalent* if they are related through a pair of isomorphisms $\Phi_1 : \mathcal{A} \rightarrow \mathcal{A}$ and $\Phi_2 : \mathcal{S}(\mathcal{A}) \rightarrow \mathcal{S}(\mathcal{A})$ such that the symplectic form and the expectation values are preserved :

$$\Phi_1^* \omega = \omega; \quad \Phi_2(\phi)[\Phi_1(A)] = \phi(A) \tag{52}$$

for all $A \in \mathcal{A}$ and $\phi \in \mathcal{S}(\mathcal{A})$. The second equation above and Eq(45) imply that we must have $\Phi_2 = (\tilde{\Phi}_1)^{-1}$. Two equivalent descriptions are, therefore, related through a canonical transformation on $\mathcal{A}$ and the corresponding inverse transpose transformation on the states. An infinitesimal transformation of this type generated by $G \in \mathcal{A}$ takes the form

$$\delta A = \epsilon\{G, A\}, \qquad (\delta\phi)(A) = -\epsilon\phi(\{G, A\}) \tag{53}$$

for all $A \in \mathcal{A}$ and $\phi \in \mathcal{S}(\mathcal{A})$.

These transformations may be called symmetries of the formalism; they are the analogues of simultaneous unitary transformations on operators and state vectors in a Hilbert space preserving expectation values of operators. Symmetries of dynamics are the subclass of these which leave the Hamiltonian invariant:

$$\Phi_1(H) = H. \tag{54}$$

For an infinitesimal transformation generated by $G \in \mathcal{A}$, this equation gives

$$\{G, H\} = 0. \tag{55}$$

It now follows from the Hamilton's equation (49) that (in the 'Heisenberg picture' evolution) G is a constant of motion. This is the situation familiar from classical and quantum mechanics: generators of symmetries of the Hamiltonian are conserved quantities and vice-versa.
Note. Noting that a symmetry operation is uniquely defined by any one of the two mappings Φ_1 and Φ_2, we can be flexible in the implementation of symmetry operations. It is often

useful to implement them such that the symmetry operations act, in a single implementation, *either* on states *or* on observables, and the two actions are related as the Heisenberg and Schrödinger picture evolutions above [see equations (50) and (45)]; we shall refer to this type of implementation as *unimodal*. In such an implementation, the second equation in (53) will not have a minus sign on the right.

For future reference, we define equivalence of NHM Hamiltonian systems. Two NHM Hamiltonian systems

$$(\mathcal{A}, \mathcal{S}_1(\mathcal{A}), \omega, H) \quad \text{and} \quad (\mathcal{A}', \mathcal{S}_1(\mathcal{A}'), \omega', H')$$

are said to be equivalent if they are related through a pair $\Phi = (\Phi_1, \Phi_2)$ of bijective mappings such that $\Phi_1 : (\mathcal{A}, \omega) \rightarrow (\mathcal{A}', \omega')$ is a symplectic mapping connecting the Hamiltonians [i.e. $\Phi_1^* \omega' = \omega$ and $\Phi_1(H) = H'$] and $\Phi_2 : \mathcal{S}_1(\mathcal{A}) \rightarrow \mathcal{S}_1(\mathcal{A}')$ such that $< \Phi_2(\phi), \Phi_1(A) > = < \phi, A >$.

Classical Hamiltonian mechanics and traditional Hilbert space QM as subdisciplines of NHM

A classical hamiltonian system (M, ω_{cl}, H_{cl}) is a special case of an NHM Hamiltonian system $(\mathcal{A}, \omega, H)$ with $\mathcal{A} = \mathcal{A}_{cl} \equiv C^\infty(M), \omega = \omega_{cl} \equiv \sum dp_j \wedge dq^j$ and H $= H_{cl}$; the NHM PBs are now the classical PBs

$$\{f, g\}_{cl} = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial p_j} \frac{\partial g}{\partial q_j} - \frac{\partial f}{\partial q_j} \frac{\partial g}{\partial p_j} \right). \tag{56}$$

Eq.(49) now becomes the classical Hamilton's equation. Representing states by probability densities in phase space, Eq.(51) goes over, in appropriate cases (for $M = R^{2n}$, for example, after the obvious partial integrations) to the classical Liouville equation for the density function.

To see the traditional Hilbert space QM as a subdiscipline of NHM, it is useful to introduce the concept of a *quantum triple* $(\mathcal{H}, \mathcal{D}, \mathcal{A})$ where $\mathcal{H}$ is a complex separable Hilbert space, $\mathcal{D}$ a dense linear subset of $\mathcal{H}$ and $\mathcal{A}$ an Op-*-algebra of operators based on $(\mathcal{H}, \mathcal{D})$. When $\mathcal{A}$ is special in the sense of section III C (this is the case for the Schrödinger dynamics of a finite number of nonrelativistic particles and is trivially so for spin dynamics), choosing an appropriate self adjoint element H of $\mathcal{A}$ as the Hamiltonian operator, we have a quantum hamiltonian system $(\mathcal{A}, \omega_Q, H)$ [with ω_Q as in Eq.(44)] as a special case of an NHM Hamiltonian system. With the quantum PBs of Eq.(43), Eq.(49) goes over to the Heisenberg equation. The states are represented by density operators ρ satisfying the condition $|Tr(\rho A)| < \infty$ for all observables A in $\mathcal{A}$; Eq.(51) then goes over to the von Neumann equation

$$\frac{d\rho(t)}{dt} = (-i\hbar)^{-1}[\rho(t), H]. \tag{57}$$

E. Symplectic actions of Lie groups

The study of symplectic actions of Lie groups in NHM proceeds generally parallel to the classical case (Sudarshan and Mukunda 1974; Arnold 1978; Guillemin and Sternberg 1984;

Woodhouse 1980) and promises to be quite rich and rewarding. Here we shall present the essential developments mainly to provide background material for the following sections.

Let G be a connected Lie group with Lie algebra $\mathcal{G}$. Elements of G, $\mathcal{G}$ and $\mathcal{G}^*$ (the dual space of $\mathcal{G}$) will be denoted, respectively, as g,h,.., $\xi, \eta, ..$ and $\lambda, \mu, ...$ The pairing between $\mathcal{G}^*$ and $\mathcal{G}$ will be denoted as $< .,. >$. Choosing a basis $\{\xi_a; a = 1,..,r\}$ in $\mathcal{G}$, we have the commutation relations $[\xi_a, \xi_b] = C^c_{ab}\xi_c$. The dual basis in $\mathcal{G}^*$ is denoted as $\{\lambda^a\}$ (so that $< \lambda^a, \xi_b >= \delta^a_b$). The action of G on $\mathcal{G}$ (adjoint representation) will be denoted as $Ad_g : \mathcal{G} \rightarrow \mathcal{G}$ and that on $\mathcal{G}^*$ (the coadjoint representation) by $Cad_g : \mathcal{G}^* \rightarrow \mathcal{G}^*$; the two are related as $< Cad_g\lambda, \xi >=< \lambda, Ad_{g^{-1}}\xi >$. With the bases chosen as above, the matrices in the two representations are related as $(Cad_g)_{ab} = (Ad_{g^{-1}})_{ba}$.

Recalling the mappings Φ_1 and Φ_2 of the previous subsection, a *symplectic action* of of G on a symplectic superalgebra $(\mathcal{A}, \omega)$ is given by the assignment, to each $g \in G$, a symplectic mapping (canonical transformation) $\Phi_1(g) : \mathcal{A} \rightarrow \mathcal{A}$ which is a group action [which means that $\Phi_1(g)\Phi_1(h) = \Phi_1(gh)$ and $\Phi_1(e) = id_{\mathcal{A}}$ in obvious notation]. The action on the states is given by the mappings $\Phi_2(g) = [\tilde{\Phi}_1(g)]^{-1}$.

A one-parameter subgroup g(t) of G generated by $\xi \in \mathcal{G}$ induces a locally Hamiltonian derivation $Z_\xi \in SDer(\mathcal{A})$ as the generator of the one-parameter family $\Phi_1(g(t)^{-1})$ of canonical transformations of $\mathcal{A}$: For small t

$$\Phi_1(g(t)^{-1})(A) \simeq A + tZ_\xi(A).$$

[*Note.* We employed $\Phi_1(g(t)^{-1})$ (and not $\Phi_1(g(t))$) for defining Z_ξ above because the former correspond to a right action of G on $\mathcal{A}$.] The correspondence $\xi \rightarrow Z_\xi$ is a Lie algebra homomorphism :

$$Z_{[\xi,\eta]} = [Z_\xi, Z_\eta]. \tag{58}$$

[A proof of (58), whose steps are parallel to those for Lie group actions on manifolds, was given in TD08 (it is an instructive application of the mathematical techniques of section II); we shall skip the details here.]

The G-action is said to be *hamiltonian* if the derivations Z_ξ are Hamiltonian, i.e. for each $\xi \in \mathcal{G}, Z_\xi = Y_{h_\xi}$ for some $h_\xi \in \mathcal{A}$ (called the *hamiltonian* corresponding to ξ). These hamiltonians are arbitrary up to addition of multiples of the unit element. This arbitrariness can be somewhat reduced by insisting that h_ξ be linear in ξ.(This can be achieved by first defining the hamiltonians for the members of a basis in $\mathcal{G}$ and then for general elements as appropriate linear combinations of these.) We shall always assume this linearity.

A hamiltonian G-action satisfying the additional condition

$$\{h_\xi, h_\eta\} = h_{[\xi,\eta]} \text{ for all } \xi, \eta \in \mathcal{G} \tag{59}$$

is called a *Poisson action*. The hamiltonians of a Poisson action have the following equivariance property :

$$\Phi_1(g)(h_\xi) = h_{Ad_g(\xi)}. \tag{60}$$

Since G is connected, it is adequate to verify this relation for infinitesimal group actions. Denoting by g(t) the one-parameter group generated by $\eta \in \mathcal{G}$, we have, for small t,

$$\Phi_1(g(t))(h_\xi) \quad \simeq \quad h_\xi + t\{h_\eta, h_\xi\} \quad = \quad h_\xi + th_{[\eta,\xi]} \quad = \quad h_{\xi+t[\eta,\xi]} \quad \simeq \quad h_{Ad_{g(t)}\xi}$$

completing the verification.

A Poisson action is not always admissible. The obstruction to such an action is determined by the objects

$$\alpha(\xi, \eta) = \{h_\xi, h_\eta\} - h_{[\xi,\eta]} \tag{61}$$

which are easily seen to have vanishing Hamiltonian derivations :

$$Y_{\alpha(\xi,\eta)} \quad = \quad [Y_{h_\xi}, Y_{h_\eta}] - Y_{h_{[\xi,\eta]}} = [Z_\xi, Z_\eta] - Z_{[\xi,\eta]} = 0$$

and hence vanishing Poisson brackets with all elements of $\mathcal{A}$. [This last condition defines the so-called *neutral elements* (Sudarshan and Mukunda 1974) of the Lie algebra$(\mathcal{A}, \{,\})$. They clearly form a complex vector space which will be denoted as $\mathcal{N}$.] We also have

$$\alpha([\xi, \eta], \zeta) + \alpha([\eta, \zeta], \xi) + \alpha([\zeta, \xi], \eta) = 0.$$

The derivation (Woodhouse 1980) of this result in classical mechanics employs only the standard properties of PBs and remains valid in NHM. Comparing this equation with Eq.(6), we see that $\alpha(.,.) \in Z_0^2(\mathcal{G}, \mathcal{N})$. A redefinition of the hamiltonians $h_\xi \to h'_\xi = h_\xi + k_\xi I$ (where the scalars k_ξ are linear in ξ) changes α by a coboundary term:

$$\alpha'(\xi, \eta) = \alpha(\xi, \eta) - k_{[\xi,\eta]}I$$

showing that the obstruction is characterized by a cohomology class of $\mathcal{G}$ [i.e. an element of $H_0^2(\mathcal{G}, \mathcal{N})$]. A necessary and sufficient condition for the admissibility of Poisson action of G on $\mathcal{A}$ is that it should be possible to transform away all the obstruction 2-cocycles by redefining the hamiltonians, or, equivalently, $H_0^2(\mathcal{G}, \mathcal{N}) = 0$.

We restrict ourselves to the special case, relevant for application in section II in part II, in which the cocycles α are multiples of the unit element :

$$\alpha(\xi, \eta) = \underline{\alpha}(\xi, \eta)I; \tag{62}$$

here the quantities $\underline{\alpha}(\xi, \eta)$ must be real numbers because the set of observables is closed under Poisson brackets. This implies $\mathcal{N} = R$, the set of real numbers. In this case, the relevant cohomology group $H_0^2(\mathcal{G}, R)$ is a real finite dimensional vector space; we shall take it to be R^m. In this case, as in classical symplectic mechanics (Sudarshan and Mukunda 1974; Cariñena and Santander 1975; Alonso 1979), Hamiltonian group actions (more generally, Lie algebra actions) with nontrivial neutral elements can be treated as Poisson actions of a (Lie group with a) larger Lie algebra $\hat{\mathcal{G}}$ obtained as follows: Let $\eta_r(.,.)(r = 1, .., m)$ be a set of representatives in $Z_0^2(\mathcal{G}, R)$ of a basis in $H_0^2(\mathcal{G}, R)$. We add extra generators M_r to the basis $\{\xi_a\}$ of $\mathcal{G}$ and take the commutation relations of the larger Lie algebra $\hat{\mathcal{G}}$ as

$$[\xi_a, \xi_b] = C_{ab}^c \xi_c + \sum_{r=1}^{m} \eta_r(\xi_a, \xi_b) M_r; \quad [\xi_a, M_r] \ = \ 0 \ = \ [M_r, M_s]. \tag{63}$$

The simply connected Lie group $\hat{G}$ with the Lie algebra $\hat{\mathcal{G}}$ is called the *projective group* of G(called 'projective covering group' of G by Cariñena and Santander; we follow the terminology of Alonso 1979); it is generally a central extension of the universal covering group $\tilde{G}$ of G.

The hamiltonian action of G with the cocycle α now becomes a Poisson action of $\hat{G}$ with the Poisson bracket relations (writing $h_{M_r} = h_r$)

$$\{h_a, h_b\} = C_{ab}^c h_c + \sum_{r=1}^{m} \eta_r(\xi_a, \xi_b) h_r; \ \{h_a, h_r\} = 0 = \{h_r, h_s\}. \tag{64}$$

F. The momentum map.

In classical mechanics, given a Poisson action of a connected Lie group G on a symplectic manifold (M, ω_{cl}) [with hamiltonians/comoments $h_\xi^{(cl)} \in C^\infty(M)$], a useful construction is the so-called *momentum map*(Souriau 1997; Arnold 1978; Guillemin and Sternberg 1984) $P : M \to \mathcal{G}^*$ given by

$$< P(x), \xi > \ = \ h_\xi^{(cl)}(x) \ \ \forall x \in M \text{ and } \ \xi \in \mathcal{G}. \tag{65}$$

This map relates the symplectic action Φ_g of G on M ($\Phi_g : M \to M, \Phi_g^* \omega_{cl} = \omega_{cl} \ \forall g \in G$) and the transposed adjoint action on $\mathcal{G}^*$ through the equivariance property

$$P(\Phi_g(x)) = Ad_g^*(P(x)) \ \forall x \in M \text{ and } \ g \in G. \tag{66}$$

Noting that points of M are pure states of the algebra $\mathcal{A}_{cl} = C^\infty(M)$, the map P may be considered as the restriction to M of the dual/transpose $\tilde{h}^{(cl)} : \mathcal{A}_{cl}^* \to \mathcal{G}^*$ of the linear map $h^{(cl)} : \mathcal{G} \to \mathcal{A}_{cl}$ [given by $h^{(cl)}(\xi) = h_\xi^{(cl)}$]:

$$< \tilde{h}^{(cl)}(u), \xi >=< u, h^{(cl)}(\xi) > \ \forall \ u \in \mathcal{A}_{cl}^* \text{ and } \ \xi \in \mathcal{G}.$$

The analogue of M in NHM is $\mathcal{S}_1 = \mathcal{S}_1(\mathcal{A})$. Defining $h : \mathcal{G} \to \mathcal{A}$ by $h(\xi) = h_\xi$, the analogue of the momentum map in NHM is the mapping $\tilde{h} : \mathcal{S}_1 \to \mathcal{G}^*$ (considered as the restriction to $\mathcal{S}_1$ of the mapping $\tilde{h} : \mathcal{A}^* \to \mathcal{G}^*$) given by

$$< \tilde{h}(\phi), \xi >=< \phi, h(\xi) >=< \phi, h_\xi > . \tag{67}$$

Now

$$\begin{aligned} < \tilde{h}(\Phi_2(g)\phi), \xi > \ &= \ < \Phi_2(g)\phi, h_\xi >=< \phi, \Phi_1(g^{-1})(h_\xi) >=< \phi, h_{Ad_{g^{-1}}(\xi)} > \\ &= \ < \phi, h(Ad_{g^{-1}}(\xi)) >=< Cad_g(\tilde{h}(\phi)), \xi > \end{aligned}$$

giving finally

$$\tilde{h}(\Phi_2(g)\phi) = Cad_g(\tilde{h}(\phi)) \tag{68}$$

which is the noncommutative analogue of Eq.(66). [*Note.* In Eq.(68),the co-adjoint (and not the transposed adjoint) action appears on the right because $\Phi_2(g)$ is inverse transpose (and not transpose) of $\Phi_1(g)$. With this understanding, (66) is obviously a special case of (68).]

G. Generalized symplectic structures and Hamiltonian systems

The generalization of the DVNCG scheme introduced in section II D can be employed to obtain the corresponding generalization of the NHM formalism. One picks up a distinguished Lie sub-superalgebra $\mathcal{X}$ of $SDer(\mathcal{A})$ and restricts the superderivations of $\mathcal{A}$ in all definitions and constructions to those in $\mathcal{X}$. Thus, a symplectic superalgebra $(\mathcal{A},\omega)$ is now replaced by a *generalized symplectic superalgebra* $(\mathcal{A},\mathcal{X},\omega)$ and a symplectic mappings $\Phi : (\mathcal{A},\mathcal{X},\alpha) \rightarrow (\mathcal{B},\mathcal{Y},\beta)$ is restricted to a superalgebra-isomorphism $\Phi : \mathcal{A} \rightarrow \mathcal{B}$ such that $\Phi_* : \mathcal{X} \rightarrow \mathcal{Y}$ is a Lie-superalgebra- isomorphism and $\Phi^*\beta = \alpha$. An NHM Hamiltonian system $(\mathcal{A},\mathcal{S}_1(\mathcal{A}),\omega,H)$ is now replace by a *generalized NHM Hamiltonian sytem* $(\mathcal{A},\mathcal{S}_1(\mathcal{A}),\mathcal{X},\omega,H)$. In section III of part II, we shall employ the pairs ($\mathcal{A}$, $\mathcal{X}$) with $\mathcal{X}$= ISDer($\mathcal{A}$) to define quantum symplectic structure on superalgebras admitting outer as well as inner superderivations.

H. Augmented symplectics including time; the noncommutative analogue of Poincaré-Cartan form

We shall now augment the kinematic framework of NHM by including time and obtain the non-commutative analogues of the Poincaré-Cartan form and the symplectic version of Noether's theorem (Souriau 1997).

For a system S with associated symplectic superalgebra $(\mathcal{A},\omega)$ we construct the *extended system algebra* $\mathcal{A}^e = C^\infty(R) \otimes \mathcal{A}$ (where the real line R is the carrier space of the 'time' t) whose elements are finite sums $\sum_i f_i \otimes A_i$ (with $f_i \in C^\infty(R) \equiv \mathcal{A}_0$) which may be written as $\sum_i f_i A_i$. This algebra is the analogue of the algebra of functions on the evolution space of Souriau (the Cartesian product of the time axis and the phase space — often referred to as the phase bundle).The superscript e in $\mathcal{A}^e$, may, therefore, also be taken to refer to 'evolution'.

Derivations on $\mathcal{A}_0$ are of the form $g(t)\frac{d}{dt}$ and one-forms of the form h(t)dt where g and h are smooth functions; there are no nonzero higher order forms. We have, of course, $dt(\frac{d}{dt}) = 1$.

A (super-)derivation D_1 on $\mathcal{A}_0$ and D_2 on $\mathcal{A}$ extend trivially to (super-)derivations on $\mathcal{A}^e$ as $D_1 \otimes id_{\mathcal{A}}$ and $id_{\mathcal{A}_0} \otimes D_2$ respectively; these trivial extensions may be informally denoted as D_1 and D_2. With $f \otimes A$ written as fA, we can write $D_1(fA) = (D_1 f)A$ and $D_2(fA) = f(D_2 A)$.

The mapping $\Xi : \mathcal{A} \rightarrow \mathcal{A}^e$ given by $\Xi(A) = 1 \otimes A (= A)$ is an isomorphism of the algebra $\mathcal{A}$ onto the subalgebra $\tilde{\mathcal{A}} \equiv 1 \otimes \mathcal{A}$ of $\mathcal{A}^e$ and can be employed to pull back the differential forms on $\mathcal{A}$ to those on $\tilde{\mathcal{A}}$. We write, for a p-form α on $\mathcal{A}$, $(\Xi^{-1})^*(\alpha) = \tilde{\alpha}$ and extend this form on $\tilde{\mathcal{A}}$ to one on $\mathcal{A}^e$ by defining $\tilde{\alpha}(\frac{d}{dt},...) = 0$. We shall generally skip the tilde. Similarly, we may extend the one-form dt on $\mathcal{A}_0$ to one on $\mathcal{A}^e$ by defining (dt)(X) =0 for all $X \in SDer(\mathcal{A})$.

The symplectic structure ω on $\mathcal{A}$ induces, on $\mathcal{A}^e$, a generalized symplectic structure (of the type introduced in subsection G above) with the distinguished Lie sub-superalgebra $\mathcal{X}$ of $SDer(\mathcal{A}^e)$ taken to be the one consisting of the objects $\{id_{\mathcal{A}_0} \otimes D; D \in SDer(\mathcal{A})\}$ which constitute a Lie sub-superalgebra of $SDer(\mathcal{A}^e)$ isomorphic to $SDer(\mathcal{A})$, thus giving a generalized symplectic superalgebra $(\mathcal{A}^e, \mathcal{X}, \tilde{\omega})$. The corresponding PBs on $\mathcal{A}^e$ are trivial extensions of those on $\mathcal{A}$ obtained by treating the 'time' t as an external parameter; this amounts to extending the C-linearity of PBs on $\mathcal{A}$ to what is essentially $\mathcal{A}_0$-linearity :

$$\{fA + gB, hC\}_{\mathcal{A}^e} = fh\{A, C\}_{\mathcal{A}} + gh\{B, C\}_{\mathcal{A}}$$

where, for clarity, we have put subscripts on the PBs to indicate the underlying superalgebras. We shall often drop these subscripts; the underlying (super-)algebra will be clear from the context.

To describe dynamics in $\mathcal{A}^e$, we extend the one-parameter family Φ_t of canonical transformations on $\mathcal{A}$ generated by a Hamiltonian $H \in \mathcal{A}$ to a one-parameter family Φ_t^e of transformations on $\mathcal{A}^e$ (which are 'canonical' in a certain sense, as we shall see below) given by

$$\Phi_t^e(fA) \equiv (fA)(t) = f(t)A(t) \equiv (\Phi_t^{(0)} f)\Phi_t(A)$$

where $\Phi_t^{(0)}$ is the one-parameter group of translations on $\mathcal{A}_0$ generated by the derivation $\frac{d}{dt}$. An infinitesimal transformation under the evolution Φ_t^e is given by

$$\begin{aligned} \delta(fA)(t) &\equiv (fA)(t + \delta t) - (fA)(t) \\ &= [\frac{df}{dt}A + f\{H, A\}_{\mathcal{A}}]\delta t \equiv \hat{Y}_H(fA)\delta t \end{aligned}$$

where

$$\hat{Y}_H = \frac{\partial}{\partial t} + \tilde{Y}_H. \tag{69}$$

Here $\frac{\partial}{\partial t}$ is the derivation on $\mathcal{A}^e$ corresponding to the derivation $\frac{d}{dt}$ on $\mathcal{A}_0$ and

$$\tilde{Y}_H = \{H, .\}_{\mathcal{A}^e}. \tag{70}$$

Note that
(i) $dt(\hat{Y}_H) = 1$;
(ii) the right hand side of Eq.(70) remains well defined if $H \in \mathcal{A}^e$ ('time dependent' Hamiltonian). Henceforth, in various formulas in this subsection, H will be understood to be an element of $\mathcal{A}^e$.

The obvious generalization of the NHM Hamilton's equation (49) to $\mathcal{A}^e$ is the equation

$$\frac{dF(t)}{dt} = \hat{Y}_H F(t) = \frac{\partial F(t)}{\partial t} + \{H(t), F(t)\}. \tag{71}$$

We next consider an object in $\mathcal{A}^e$ which contains complete information about the symplectic structure *and* dynamics [i.e. about $\tilde{\omega}$ and H (up to an additive constant multiple of I)] and is canonically determined by these objects. It is the 2-form

$$\Omega = \tilde{\omega} + dt \wedge dH \tag{72}$$

which is 'obviously' closed. [To have a formal proof, apply Eq.(28) with $\Omega = 1 \otimes \omega + d_1 t \otimes d_2 H$.] If the symplectic structure on $\mathcal{A}$ is exact (with $\omega = d\theta$), we have ('obviously') $\Omega = d\Theta$ where

$$\Theta = \tilde{\theta} - H dt \tag{73}$$

is the NHM avatar of the Poincaré-Cartan form in classical mechanics. [Again, for a formal derivation, use Eq.(28) with $\Theta = 1 \otimes \theta - dt \otimes H$.]

The closed form Ω is generally not non-degenerate. It defines what may be called a *presymplectic structure* (Souriau 1997) on $\mathcal{A}^e$. In fact, we have here the noncommutative analogue of a special type of presymplectic structure called *contact structure* (Abraham and Marsden 1978; Berndt 2001); it may be called the Poincaré- Cartan contact structure. We shall, however, not attempt a formal development of noncommutative contact structures here.

A *symplectic action* of a Lie group G on the presymplectic space $(\mathcal{A}^e, \Omega)$ is the assignment, to every $g \in G$, an automorphism $\Phi(g)$ of the superalgebra $\mathcal{A}^e$ having the usual group action properties and such that $\Phi(g)^* \Omega = \Omega$. This implies, as in section III E, that, to every element ξ of the Lie algebra $\mathcal{G}$ of G, corresponds a derivation $\hat{Z}_\xi$ such that $L_{\hat{Z}_\xi} \Omega = 0$ which, in view of the condition $d\Omega = 0$, is equivalent to the condition

$$d(i_{\hat{Z}_\xi} \Omega) = 0. \tag{74}$$

We shall now verify that the one-parameter family $\Phi_t^{(e)}$ of transformations on $\mathcal{A}^e$ is symplectic/canonical. For this, it is adequate to verify that Eq.(74) holds with $\hat{Z}_\xi = \hat{Y}_H$. We have, in fact, the stronger relation

$$i_{\hat{Y}_H} \Omega = 0. \tag{75}$$

Indeed

$$\begin{aligned} i_{\hat{Y}_H} \Omega &= i_{\partial/\partial t} \Omega + i_{\tilde{Y}_H} \Omega \\ &= i_{\partial/\partial t}(dt \wedge dH) + i_{\tilde{Y}_H} \tilde{\omega} + i_{\tilde{Y}_H}(dt \wedge dH) \\ &= dH - dH - i_{\tilde{Y}_H}(dH) dt \\ &= [i_{\tilde{Y}_H}(i_{\tilde{Y}_H} \tilde{\omega})] dt = 0. \end{aligned}$$

The equation in note (i) above and Eq.(75) are analogous to the properties of the 'characteristic vector field' of a contact structure. The derivation $\hat{Y}_H$ may, therefore, be called the *characteristic derivation* of the Poincaré-Cartan contact structure.

A symplectic G-action (in the present context) is said to be *hamiltonian* if the 1-forms $i_{\hat{Z}_\xi} \Omega$ are exact, i.e. to each $\xi \in \mathcal{G}$, corresponds a 'hamiltonian' $\hat{h}_\xi \in \mathcal{A}^e$ (unique up to an additive constant multiple of the unit element) such that

$$i_{\hat{Z}_\xi} \Omega = -d\hat{h}_\xi. \tag{76}$$

These 'hamiltonians' (*Noether invariants*) are constants of motion :

$$\begin{aligned} \frac{d\hat{h}_\xi(t)}{dt} &= \hat{Y}_H(\hat{h}_\xi(t)) = (d\hat{h}_\xi)(\hat{Y}_H)(t) \\ &= -(i_{\hat{Z}_\xi}\Omega)(\hat{Y}_H)(t) = 0 \end{aligned} \tag{77}$$

where, in the last step, Eq.(75) has been used. This is the NHM analogue of the symplectic version of Noether's theorem. Some concrete examples of Noether invariants will be given in section II F in part II.

IV. COUPLED SYSTEMS IN NONCOMMUTATIVE HAMILTONIAN MECHANICS

We shall now consider the interaction of two systems S_1 and S_2 described individually as the NHM Hamiltonian systems $(\mathcal{A}^{(i)}, \omega^{(i)}, H^{(i)})$ (i=1,2) and treat the coupled system $S_1 + S_2$ also as an NHM Hamiltonian system. To facilitate this, we must obtain the relevant mathematical objects for the coupled system.

A. The symplectic form and Poisson bracket on the tensor product of two superalgebras

We shall freely use the notations and constructions in section II E.

Given the symplectic forms $\omega^{(i)}$ on $\mathcal{A}^{(i)}$ [with PBs $\{,\}_i$(i=1,2)] we shall construct the 'canonically induced' symplectic form ω on $\mathcal{A}$ satisfying the following conditions :
(a) It should not depend on anything other than the objects $\omega^{(i)}$ and $I_{(i)}$ (i=1,2) [the 'naturality'/'canonicality' assumption for ω. (Note that the unit elements are the only distinguished elements of the algebras being considered)].
(b) The restrictions of ω to $\tilde{\mathcal{A}}^{(1)}$ and $\tilde{\mathcal{A}}^{(2)}$ be, respectively, $\omega^{(1)} \otimes I_2$ and $I_1 \otimes \omega^{(2)}$.
This determines ω uniquely :

$$\omega = \omega^{(1)} \otimes I_2 + I_1 \otimes \omega^{(2)}. \tag{78}$$

To verify that it is a symplectic form, we must show that it is (i) closed and (ii) nondegenerate. Eq.(28) gives

$$d\omega = (d_1\omega^{(1)}) \otimes I_2 + \omega^{(1)} \otimes d_2(I_2) + d_1(I_1) \otimes \omega^{(2)} + I_1 \otimes d_2\omega^{(2)} = 0$$

showing that ω is closed. To show the nondegeneracy of ω, it is necessary and sufficient to show that, given $A \otimes B \in \mathcal{A}$, there exists a unique superderivation $Y = Y_{A\otimes B}$ in $SDer(\mathcal{A})$ such that

$$\begin{aligned} i_Y\omega = -d(A \otimes B) &= -(d_1A) \otimes B - A \otimes d_2B \\ &= i_{Y_A^{(1)}}\omega^{(1)} \otimes B + A \otimes i_{Y_B^{(2)}}\omega^{(2)}. \end{aligned} \tag{79}$$

where $Y_A^{(1)}$ and $Y_B^{(2)}$ are the Hamiltonian superderivations associated with $A \in \mathcal{A}^{(1)}$ and $B \in \mathcal{A}^{(2)}$. The structure of Eq.(79) suggests that Y must be of the form [see Eq.(27)]

$$Y = Y_A^{(1)} \otimes \Psi_B^{(2)} + \Psi_A^{(1)} \otimes Y_B^{(2)} \tag{80}$$

where the linear mappings $\Psi_A^{(1)}$ and $\Psi_B^{(2)}$ satisfy the conditions $\Psi_A^{(1)}(I_1) = A$ and $\Psi_B^{(2)}(I_2) = B$. Recalling the discussion after Eq.(27) and Eq.(1) [and denoting the multiplication operators in $\mathcal{A}^{(1)}, \mathcal{A}^{(2)}$ and $\mathcal{A}$ by μ_1, μ_2 and μ respectively], the (necessary and sufficient) condition for Y to be a superderivation may be written as

$$Y \circ \mu(C \otimes D) - \eta_{Y,C\otimes D}\ \mu(C \otimes D) \circ Y = \mu(Y(C \otimes D)). \tag{81}$$

Noting that $\mu(C \otimes D) = \mu_1(C) \otimes \mu_2(D)$ (the skew tensor product causes no problems here), Eq.(81) with Y of Eq.(80) gives

$$\begin{aligned}
&\eta_{BC}\{[Y_A^{(1)} \circ \mu_1(C)] \otimes [\Psi_B^{(2)} \circ \mu_2(D)] + [\Psi_A^{(1)} \circ \mu_1(C)] \otimes Y_B^{(2)} \circ \mu_2(D)]\} \\
&-(-1)^{\epsilon}\{[\mu_1(C) \circ Y_A^{(1)}] \otimes [\mu_2(D) \circ \Psi_B^{(2)}] + [\mu_1(C) \circ \Psi_A^{(1)}] \otimes [\mu_2(D) \circ Y_B^{(2)}]\} \\
&= \eta_{BC}[\mu_1(\{A, C\}_1) \otimes \mu_2(\Psi_B^{(2)}(D)) + \mu_1(\Psi_A^{(1)}(C)) \otimes \mu_2(\{B, D\}_2)]
\end{aligned} \tag{82}$$

where $\epsilon \equiv \epsilon_A\epsilon_C + \epsilon_B\epsilon_D + \epsilon_B\epsilon_C$ and we have used the relations $Y_A^{(1)}(C) = \{A, C\}_1$ and $Y_B^{(2)}(D) = \{B, D\}_2$.

The objects $Y_A^{(1)}$ and $Y_B^{(2)}$, being superderivations, satisfy relations of the form (1) :

$$\begin{aligned}
Y_A^{(1)} \circ \mu_1(C) - \eta_{AC}\mu_1(C) \circ Y_A^{(1)} &= \mu_1(Y_A^{(1)}(C)) = \mu_1(\{A, C\}_1) \\
Y_B^{(2)} \circ \mu_2(D) - \eta_{BD}\mu_2(D) \circ Y_B^{(2)} &= \mu_2(\{B, D\}_2).
\end{aligned} \tag{83}$$

Putting $D = I_2$ in Eq.(82), we have [noting that $\mu_2(D) = \mu_2(I_2) = id_{\mathcal{A}^{(2)}}$, and $\{B, I_2\}_2 = Y_B^{(2)}(I_2) = 0$]

$$\begin{aligned}
[Y_A^{(1)} \circ \mu_1(C)] \otimes \Psi_B^{(2)} \quad + \quad & [\Psi_A^{(1)} \circ \mu_1(C)] \otimes Y_B^{(2)} \\
& -\eta_{AC}\{[\mu_1(C) \circ Y_A^{(1)}] \otimes \Psi_B^{(2)} + [\mu_1(C) \circ \Psi_A^{(1)}] \otimes Y_B^{(2)}\} \\
& = \mu_1(\{A, C\}_1) \otimes \mu_2(B)
\end{aligned} \tag{84}$$

which, along with equations (83), gives

$$\begin{aligned}
\mu_1(\{A, C\}_1) \otimes [\Psi_B^{(2)} - \mu_2(B)] \quad &= \\
-[\Psi_A^{(1)} \circ \mu_1(C) \quad &- \quad \eta_{AC}\mu_1(C) \circ \Psi_A^{(1)}] \otimes Y_B^{(2)}.
\end{aligned} \tag{85}$$

Similarly, putting $C = I_1$ in Eq.(82), we get

$$\begin{aligned}
[\Psi_A^{(1)} - \mu_1(A)] \otimes \mu_2(\{B, D\}_2) \quad &= \\
-Y_A^{(1)} \otimes [\Psi_B^{(2)} \circ \mu_2(D) \quad &- \quad \eta_{BD}\mu_2(D) \circ \Psi_B^{(2)}].
\end{aligned} \tag{86}$$

Now, equations (86) and (85) give the relations

$$\Psi_A^{(1)} - \mu_1(A) = \lambda_1 Y_A^{(1)} \tag{87}$$

$$\Psi_B^{(2)} \circ \mu_2(D) - \eta_{BD}\ \mu_2(D) \circ \Psi_B^{(2)} = -\lambda_1\ \mu_2(\{B, D\}_2) \tag{88}$$

$$\Psi_B^{(2)} - \mu_2(B) = \lambda_2 Y_B^{(2)} \tag{89}$$

$$\Psi_A^{(1)} \circ \mu_1(C) - \eta_{AC}\ \mu_1(C) \circ \Psi_A^{(1)} = -\lambda_2\ \mu_1(\{A, C\}_1) \tag{90}$$

where λ_1 and λ_2 are complex numbers.

Equations (80), (87) and (89) now give

$$\begin{aligned} Y &= Y_A^{(1)} \otimes [\mu_2(B) + \lambda_2 Y_B^{(2)}] + [\mu_1(A) + \lambda_1 Y_A^{(1)}] \otimes Y_B^{(2)} \\ &= Y_A^{(1)} \otimes \mu_2(B) + \mu_1(A) \otimes Y_B^{(2)} + (\lambda_1 + \lambda_2) Y_A^{(1)} \otimes Y_B^{(2)}. \end{aligned} \tag{91}$$

Note that only the combination $(\lambda_1 + \lambda_2) \equiv \lambda$ appears in Eq.(91). To have a unique Y, we must obtain an equation fixing λ in terms of given quantities.

Substituting for $\Psi_A^{(1)}$ and $\Psi_B^{(2)}$ from equations (87) and (89) into equations (88) and (90) and using equations (83), we obtain the equations

$$\lambda\mu_1(\{A, C\}_1) = -\mu_1([A, C]) \quad \text{for all} \quad A, C \in \mathcal{A}^{(1)} \tag{92}$$

$$\lambda\mu_2(\{B, D\}_2) = -\mu_2([B, D]) \quad \text{for all} \quad B, D \in \mathcal{A}^{(2)}. \tag{93}$$

We have not one but two equations of the type we have been looking for. This is a signal for the emergence of nontrivial conditions (for the desired symplectic structure on the tensor product superalgebra to exist).

Let us consider the equations (92,93) for the various possible situations (corresponding to whether or not one or both the superalgebras are super-commutative) :

(i) Let $\mathcal{A}^{(1)}$ be supercommutative. Assuming that the PB $\{,\}_1$ is nontrivial, Eq.(92) implies that $\lambda = 0$. Eq.(93) then implies that $\mathcal{A}^{(2)}$ must also be super-commutative. It follows that
(a) when both the superalgebras $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$ are super-commutative, the unique Y is given by Eq.(91) with $\lambda = 0$;
(b) when one of them is supercommutative and the other is not, the ω of Eq.(78) does not define a symplectic structure on $\mathcal{A}$; hence a 'canonically induced' symplectic structure does not exist on the tensor product of a super-commutative and a non-supercommutative superalgebra.

(ii) Let the superalgebra $\mathcal{A}^{(1)}$ be non-supercommutative. Eq.(92) then implies that $\lambda \neq 0$, which, along with Eq.(93) implies that the superalgebra $\mathcal{A}^{(2)}$ is also non-supercommutative [which is also expected from (b) above]. These equations now give

$$\{A, C\}_1 = -\lambda^{-1}[A, C], \quad \{B, D\}_2 = -\lambda^{-1}[B, D] \tag{94}$$

which shows that, when both the superalgebras are non-supercommutative, a 'canonically induced' symplectic structure on their (skew) tensor product exists if and only if each superalgebra has a quantum symplectic structure with the *same* parameter $(-\lambda)$, i.e.

$$\omega^{(1)} = -\lambda\omega_c^{(1)}, \quad \omega^{(2)} = -\lambda\omega_c^{(2)} \tag{95}$$

where $\omega_c^{(i)}$ (i=1,2) are the canonical symplectic forms on the two superalgebras. The traditional quantum symplectic structure is obtained with $\lambda = i\hbar$ [see Eq.(44)].

Note. The two forms $\omega^{(i)}$(i=1,2) of Eq.(95) represent genuine symplectic structures only if the superalgebras $\mathcal{A}^{(i)}$ (i=1,2) have only inner superderivations (see section III C). In general, we have generalized symplectic superalgebras $(\mathcal{A}^{(i)}, \mathcal{X}^{(i)}, \omega^{(i)})$ (i=1,2) where $\mathcal{X}^{(i)} = ISDer(\mathcal{A}^{(i)})$.

In all the permitted cases, the PB on the superalgebra $\mathcal{A} = \mathcal{A}^{(1)} \otimes \mathcal{A}^{(2)}$ is given by

$$\begin{aligned} \{A \otimes B, C \otimes D\} = Y_{A\otimes B}(C \otimes D) &= \eta_{BC}[\{A, C\}_1 \otimes BD + AC \otimes \{B, D\}_2 \\ &\quad + \lambda\{A, C\}_1 \otimes \{B, D\}_2] \end{aligned} \tag{96}$$

where the parameter λ vanishes in the super-commutative case; in the non-supercommutative case, it is the universal parameter appearing in the symplectic forms (95).

Noting that, in the non-supercommutative case,

$$\begin{aligned} \lambda\{A, C\}_1 \otimes \{B, D\}_2 &= -[A, C] \otimes \{B, D\}_2 = -\{A, C\}_1 \otimes [B, D] \\ &= -\frac{1}{2}[A, C] \otimes \{B, D\}_2 - \frac{1}{2}\{A, C\}_1 \otimes [B, D], \end{aligned}$$

the PB of Eq.(96) can be written in the more symmetric form

$$\begin{aligned} &\{A \otimes B, C \otimes D\} \\ &= \eta_{BC}[\{A, C\}_1 \otimes \tfrac{BD+\eta_{BD}DB}{2} + \tfrac{AC+\eta_{AC}CA}{2} \times \{B, D\}_2]. \end{aligned} \tag{97}$$

Note. The non-super version of Eq.(97) was [wrongly, not realizing that Y of Eq.(80) is not always a (super-)derivation] put forward by the author as the PB for a tensor product of algebras in the general case [in arXiv : quant-ph/0612224, henceforth referred to as TD06]. M.J.W. Hall pointed out to the author (private communication) that it does not satisfy Jacobi identity in some cases, as shown, for example, in (Caro and Salcedo, 1999). Revised calculations by the author then led to the results presented above.

B. Dynamics of coupled systems

Given the individual systems S_1 and S_2 as the NHM Hamiltonian systems $(\mathcal{A}^{(i)}, \omega^{(i)}, H^{(i)})$ (i = 1,2) where the two superalgebras are either both supercommutative or both non-supercommutative, the coupled system $(S_1 + S_2)$ is an NHM Hamiltonian system with the system algebra and symplectic form as discussed above and the Hamiltonian H given by

$$H = H^{(1)} \otimes I_2 + I_1 \otimes H^{(2)} + H_{int} \tag{98}$$

where the interaction Hamiltonian is generally of the form

$$H_{int} = \sum_{i=1}^{n} F_i \otimes G_i.$$

The evolution (in the Heisenberg type picture) of a typical obsevable $A(t) \otimes B(t)$ is governed by the supmech Hamilton's equation

$$\begin{aligned} \frac{d}{dt}[A(t) \otimes B(t)] &= \{H, A(t) \otimes B(t)\} \\ &= \{H^{(1)}, A(t)\}_1 \otimes B(t) + A(t) \otimes \{H^{(2)}, B(t)\}_2 \\ &\quad + \{H_{int}, A(t) \otimes B(t)\}. \end{aligned} \quad (99)$$

The last Poisson bracket in this equation can be evaluated using Eq.(96) or (97). In the Schrӧdinger type picture, the time evolution of states of the coupled system is given by the NHM Liouville equation (51) with the Hamiltonian of Eq.(98). In favorable situations, the NHM Heisenberg or Liouville equations may be written for finite time intervals by using appropriate exponentiations of operators.

The main lesson from this section is that *all* systems in nature whose interaction with other systems can be talked about must belong to one of the two 'worlds' : the 'commutative world' in which all system superalgebras are super-commutative and the 'noncommutative world' in which all system superalgebras are non-supercommutative with a *universal* quantum symplectic structure. In view of the familiar inadequacy of the commutative world, the 'real' world must clearly be the noncommutative (hence quantum) world; its systems will be called quantum systems. (This is formalized as axiom **A7** in section V in part II.) The classical systems with commutative system algebras and traditional symplectic structures will appear only in the appropriately defined classical limit (or, more generally, in the classical approximation) of quantum systems.

Concluding remarks and acknowledgements will appear in part II.

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