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Positivity and Conditional Positivity of Loewner Matrices

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Abstract We give elementary proofs of the fact that the Loewner matrices $\left[\frac{f(p_i)-f(p_j)}{p_i-p_j}\right]$ corresponding to the function $f(t) = t^r$ on $(0, \infty)$ are positive semidefinite, conditionally negative definite, and conditionally positive definite, for r in $[0, 1], [1, 2]$, and $[2, 3]$, respectively. We show that in contrast to the interval $(0, \infty)$ the Loewner matrices corresponding to an operator convex function on $(-1, 1)$ need not be conditionally negative definite.

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1 Introduction

This is a sequel to our paper [7] and it deals with two issues. Let $p_1, \ldots, p_n$ be distinct positive numbers, and for $r > 0$ let L_r be the $n \times n$ matrix

$$L_r = \left[\frac{p_i^r - p_j^r}{p_i - p_j} \right]. \tag{1.1}$$

Here $[a_{ij}]$ stands for the matrix whose (i, j) entry is a_{ij}, and it is understood that when $i = j$ the quotient in (1.1) means the limiting value rp_i^{r-1}. We call L_r a *Loewner matrix* associated with the function $f(t) = t^r$.

Since Loewner's seminal paper [10] in 1934 it has been known that when $0 < r \leqq 1$ the matrices L_r are positive semidefinite (p.s.d. for short). The proof most often given follows Loewner's arguments. First one proves that a function f from any interval I into $\mathbb{R}$ is *operator monotone* (see [3, Chapter 5] for definitions) if and only if the associated Loewner matrices

$$L_f = \left[\frac{f(p_i) - f(p_j)}{p_i - p_j} \right] \tag{1.2}$$

are p.s.d. for all choices of points $p_1, \ldots, p_n$ in I. Then one proves that the function $f(t) = t^r$ on $(0, \infty)$ is operator monotone. This latter statement is a consequence of another theorem of Loewner: f is operator monotone on I if and only if it has an analytic continuation that maps the upper half plane into itself. A direct and ingenious proof of the operator monotonicity of $f(t) = t^r$ for $0 < r \leqq 1$ was also given by Pedersen [11].

A different approach was adopted by Bhatia and Parthasarathy [6]. If there exists a nonsingular matrix X such that $B = X^*AX$, then B is said to be *congruent* to A. In [6] the authors showed that the matrix L_r is congruent to a matrix of the form

$$\left[\frac{\sinh\ r(x_i - x_j)}{\sinh\ (x_i - x_j)} \right]. \tag{1.3}$$

If $0 < r \leqq 1$ this matrix is positive definite since the function $\sinh(rx)/\sinh x$ is positive definite. Since congruence preserves the p.s.d. property it follows that for $0 < r \leqq 1$ the matrix L_r is p.s.d.

In our paper [7] we began with the well-known representation

$$t^r = \int_0^\infty \frac{t}{\lambda + t}\, d\mu(\lambda), \quad 0 < r < 1, \tag{1.4}$$

where $d\mu(\lambda) = \dfrac{\sin r\pi}{\pi}\lambda^{r-1}\,d\lambda$. Then we observed that a rather simple argument shows that the Loewner matrices associated with the function

$$h_\lambda(t) = \frac{t}{\lambda + t}, \quad \lambda > 0 \tag{1.5}$$

on $(0, \infty)$ are p.s.d. It follows that the matrices L_r for $0 < r \leqq 1$ are p.s.d.

We used a similar argument to show that when $1 \leqq r \leqq 2$, the matrices L_r are *conditionally negative definite* (c.n.d.), and when $2 \leqq r \leqq 3$, they are *conditionally positive definite* (c.p.d.). (The definitions are given in Section 2.)

In the first part of this paper we derive these properties of the matrices L_r by completely elementary arguments. We use just two well-known facts: the *Cauchy matrix*

$$C = \left[\frac{1}{p_i + p_j}\right] \tag{1.6}$$

is p.s.d. for any n positive numbers $p_1, \ldots, p_n$; and the Hadamard (entrywise) product of two p.s.d. matrices is also p.s.d. Our argument consists of combining these two facts with induction and is reminiscent of Pedersen's argument mentioned earlier.

The second part of the paper is concerned with a slightly different issue. From results known earlier and from our work in [7] we know that if f is a function from $(0, \infty)$ into itself, then the Loewner matrices L_f are
(i) p.s.d. if f is operator monotone,
(ii) c.n.d if f is operator convex,
(iii) c.p.d. if $f(t) = tg(t)$ for an operator convex function g.
What happens if f is a real valued function on any open interval I? The statements (i) and (iii) remain true: the first is included in Loewner's original theorem, and the third was proved by R. Horn [8]. However (ii) does not remain true in this case. Our elementary argument is helpful in proving these statements too.

2 The matrices L_r

Let H^n be the subspace of $\mathbb{C}^n$ consisting of all $x = (x_1, \ldots, x_n)$ for which $\sum_{i=1}^{n} x_i = 0$. An $n \times n$ Hermitian matrix A is said to be c.p.d. if $\langle x, Ax\rangle \geqq 0$ for all $x \in H^n$, and it is said to be c.n.d. if $-A$ is c.p.d. Every p.s.d. matrix is obviously c.p.d. We denote by E the matrix with all entries equal to one. Then $Ex = 0$ for all $x \in H^n$. So E is both c.p.d. and c.n.d.

We consider the matrices L_r defined in (1.1) when r is in one of the intervals $[0,1],[1,2]$, and $[2,3]$, and show that these matrices are, respectively, p.s.d., c.n.d., and c.p.d.

To show that L_r is p.s.d. for all r in $[0,1]$ it is enough to prove this for all dyadic rationals $r=(2k+1)/2^m$, $m=1,2,\ldots$, and $0 \leqq k \leqq 2^{m-1}-1$. Replacing p by $p^{1/2^m}$, we see that for such an r, the matrix L_r is of the form

$$L_r = \left[\frac{p_i^{2k+1}-p_j^{2k+1}}{p_i^{2^m}-p_j^{2^m}}\right]. \tag{2.1}$$

For brevity, for any two positive integers k,m let

$$L(k;m) = \left[\frac{p_i^k-p_j^k}{p_i^m-p_j^m}\right].$$

We wish to show that all matrices

$$L(2k+1;2^m), \quad m=1,2,\ldots, \quad 0 \leqq k \leqq 2^{m-1}-1$$

are p.s.d. We prove this by induction on m. When $m=1$, we have to consider only the case $k=0$. In this case

$$L(2k+1;2^m) = L(1;2) = \left[\frac{1}{p_i+p_j}\right].$$

This is a Cauchy matrix and is p.s.d. To see what happens in the induction step it is instructive to consider the cases $m=2$ and $m=3$. When $m=2$ we have two matrices $L(1;4)$ and $L(3;4)$ that are claimed to be p.s.d. The first

$$L(1;4) = \left[\frac{1}{p_i^2+p_j^2}\right] \circ \left[\frac{p_i+p_j}{p_i^2-p_j^2}\right] = \left[\frac{1}{p_i^2+p_j^2}\right] \circ \left[\frac{1}{p_i+p_j}\right]$$

is the Hadamard product of two Cauchy matrices and is p.s.d. One also sees that

$$\begin{aligned} L(3;4) &= \left[\frac{p_i-p_j}{p_i^2-p_j^2}\right] + \left[\frac{p_ip_j}{p_i^2+p_j^2}\right] \circ \left[\frac{p_i-p_j}{p_i^2-p_j^2}\right] \\ &= \left[\frac{1}{p_i+p_j}\right] + \left[\frac{p_ip_j}{(p_i+p_j)(p_i^2+p_j^2)}\right]. \end{aligned}$$

The first matrix on the right hand side is again a Cauchy matrix, and the second is of the form DXD where $D = \operatorname{diag}(p_1,\ldots,p_n)$ and X is the Hadamard product of two Cauchy matrices. Hence $L(3;4)$ is also p.s.d. When $m = 3$, we have to consider four matrices corresponding to $k = 0, 1, 2, 3$. The first two are

$$\begin{aligned} L(1;8) &= \left[\frac{1}{p_i^4 + p_j^4}\right] \circ \left[\frac{p_i - p_j}{p_i^4 - p_j^4}\right] \\ L(3;8) &= \left[\frac{1}{p_i^4 + p_j^4}\right] \circ \left[\frac{p_i^3 - p_j^3}{p_i^4 - p_j^4}\right]. \end{aligned}$$

The first factor of each is a Cauchy matrix, and the second has been shown to be p.s.d. while considering the case $m = 2$. So these products are p.s.d. The remaining two matrices are

$$L(5;8) = \left[\frac{p_i - p_j}{p_i^4 - p_j^4}\right] + \left[\frac{p_i p_j}{p_i^4 + p_j^4}\right] \circ \left[\frac{p_i^3 - p_j^3}{p_i^4 - p_j^4}\right],$$

and

$$L(7;8) = \left[\frac{p_i^3 - p_j^3}{p_i^4 - p_j^4}\right] + \left[\frac{p_i^3 p_j^3}{p_i^4 + p_j^4}\right] \circ \left[\frac{p_i - p_j}{p_i^4 - p_j^4}\right].$$

Both these are p.s.d. by the arguments given earlier.

Now assume that our claim has been proved for the indices $1, 2, \ldots, m-1$. The matrix $L(2k+1; 2^m)$ can be factored as

$$L(2k+1; 2^m) = \left[\frac{1}{p_i^{2^{m-1}} + p_j^{2^{m-1}}}\right] \circ \left[\frac{p_i^{2k+1} - p_j^{2k+1}}{p_i^{2^{m-1}} - p_j^{2^{m-1}}}\right]. \tag{2.2}$$

The first factor is a Cauchy matrix and is p.s.d. If $0 \leqq k \leqq 2^{m-2} - 1$, then the second factor is p.s.d. by the induction hypothesis. Hence the matrix (2.2) is p.s.d. If $2^{m-2} \leqq k \leqq 2^{m-1} - 1$, we use the identity

$$\begin{aligned} \frac{a^{2k+1} - b^{2k+1}}{a^{2^m} - b^{2^m}} &= \frac{a^{2k+1-2^{m-1}} - b^{2k+1-2^{m-1}}}{a^{2^{m-1}} - b^{2^{m-1}}} \\ &\quad + \frac{a^{2k+1-2^{m-1}} b^{2k+1-2^{m-1}}}{a^{2^{m-1}} + b^{2^{m-1}}} \; \frac{a^{2^m-(2k+1)} - b^{2^m-(2k+1)}}{a^{2^{m-1}} - b^{2^{m-1}}}, \end{aligned}$$

to express $L(2k+1; 2^m)$ as a sum of matrices each of which is p.s.d. by the induction hypothesis.

We have shown that the matrix L_r is p.s.d. for $0 \leqq r \leqq 1$. Now we show that this matrix is c.n.d. for $1 \leqq r \leqq 2$. For this it is enough to show that for $m = 1, 2, \ldots,$ and $0 \leqq k \leqq 2^{m-1} - 1$, the matrix

$$L(2^m + 2k + 1; 2^m)$$

is c.n.d. Let us begin with the case $m = 1$ and $k = 0$. Using the identity

$$\frac{a^3 - b^3}{a^2 - b^2} = a + b - ab\frac{a - b}{a^2 - b^2}$$

we see that

$$L(3;2) = DE + ED - DL(1;2)D,$$

where $D = \mathrm{diag}\,(p_1, \ldots, p_n)$. Hence for $x \in H^n$

$$\langle x, L(3;2)x\rangle = -\langle x, DL(1;2)Dx\rangle \leqq 0,$$

and the matrix $L(3;2)$ is c.n.d.

This idea is the basis for the general case. Using the identity

$$\frac{a^{2^m+(2k+1)} - b^{2^m+(2k+1)}}{a^{2^m} - b^{2^m}} = a^{2k+1} + b^{2k+1} - a^{2k+1}b^{2k+1}\,\frac{a^{2^m-(2k+1)} - b^{2^m-(2k+1)}}{a^{2^m} - b^{2^m}},$$

for $0 \leqq k \leqq 2^{m-1} - 1$, we obtain

$$L(2^m + 2k + 1; 2^m) = D^{2k+1}E + ED^{2k+1} - D^{2k+1}L(2^m - (2k+1); 2^m)D^{2k+1}. \qquad (2.3)$$

The matrix $D^{2k+1}E + ED^{2k+1}$ is c.n.d. and the matrix

$$L(2^m - (2k+1); 2^m) = L_{1-(2k+1)/2^m}$$

is p.s.d. Hence the matrix in (2.3) is c.n.d. This shows that the matrix L_r is c.n.d. for $1 \leqq r \leqq 2$.

Finally, we come to the case $2 \leqq r \leqq 3$. To show that L_r is c.p.d. for all such r we do so for $m = 1, 2, \ldots,$ and for k satisfying $0 \leqq k \leqq 2^{m-1} - 1$. Using the identity

$$\frac{a^{2^{m+1}+(2k+1)} - b^{2^{m+1}+(2k+1)}}{a^{2^m} - b^{2^m}} = a^{2^m+2k+1} + b^{2^m+2k+1} + a^{2^m}b^{2^m}\,\frac{a^{2k+1} - b^{2k+1}}{a^{2^m} - b^{2^m}},$$

we obtain

$$L(2^{m+1} + 2k + 1; 2^m) = D^{2^m+2k+1}E + ED^{2^m+2k+1} + D^{2^m}L(2k+1; 2^m)D^{2^m}. \qquad (2.4)$$

We leave it to the reader to check that the sum of the first two terms in (2.4) is a c.p.d. matrix and the third term is a p.s.d matrix. As before, this shows that L_r is c.p.d. for $2 \leqq r \leqq 3$.

In addition to the matrices L_r the matrices

$$K_r = \left[\frac{p_i^r + p_j^r}{p_i + p_j}\right] \tag{2.5}$$

too have been of interest. It was shown by Kwong [9] that for $0 \leqq r \leqq 1$ these matrices are p.s.d. Different proofs of this fact have been given in [6] and [4], and in [5] it was shown that these matrices are not just p.s.d., they are infinitely divisible. (A matrix $[a_{ij}]$ with nonnegative entries is called *infinitely divisible* if for each $\alpha > 0$ the matrix $\left[a_{ij}^{\alpha}\right]$ is p.s.d.) In [7] we showed that for $1 \leqq r \leqq 3$ the matrices K_r are c.n.d. Thus in this respect the behaviour of the matrices L_r and K_r is different in the range $2 \leqq r \leqq 3$. The methods of this paper can be used to derive these results and may provide some further understanding.

We will use a theorem of Bapat [1] saying that if $[a_{ij}]$ is a c.n.d. with positive entries, then the matrix $\left[\frac{1}{a_{ij}}\right]$ is infinitely divisible. We use the notation

$$K(l;m) = \left[\frac{p_i^l + p_j^l}{p_i^m + p_j^m}\right] \tag{2.6}$$

where it is understood that $p_1, \ldots, p_n$ are given positive numbers and l, m are nonnegative integers.

Using the identity

$$\frac{a^{m+2} + b^{m+2}}{a^{m+1} + b^{m+1}} \quad = \quad a + b - ab\, \frac{a^m + b^m}{a^{m+1} + b^{m+1}},$$

one sees that

$$K(m+2; m+1) = DE + ED - DK(m; m+1)D, \tag{2.7}$$

where $D = \text{diag}(p_1, \ldots, p_n)$. So, if we know that $K(m; m+1)$ is p.s.d. then from the identity (2.7) we can conclude that $K(m+2; m+1)$ is c.n.d. Hence by Bapat's theorem $K(m+1; m+2)$ is infinitely divisible.

The matrix $K(0; 1) = 2C$, where C is the Cauchy matrix. Thus $K(0; 1)$ is infinitely divisible and p.s.d. Applying the reasoning in the preceding paragraph we see recursively that all matrices $K(m;m+1)$ are infinitely divisible. This, in turn, implies that for $l < m$ the matrix

$$K(l;m) = K(l;l+1) \circ K(l+1;l+2) \circ \cdots \circ K(m-1;m),$$

being a Hadamard product of infinitely divisible matrices, is infinitely divisible. So, if r is a rational number in $(0,1)$ the matrix K_r is infinitely divisible, and taking limits we see that this is so for all r in $[0,1]$.

Next, we have the identities

$$\begin{aligned}\frac{a^{m+l}+b^{m+l}}{a^m+b^m} &= a^l+b^l-a^lb^l\,\frac{a^{m-l}+b^{m-l}}{a^m+b^m},\\ \frac{a^{2m+l}+b^{2m+l}}{a^m+b^m} &= a^{m+l}+b^{m+l}-a^mb^m\,\frac{a^l+b^l}{a^m+b^m}.\end{aligned}$$

Using these we obtain

$$K(m+l;m) = D^lE+ED^l-D^lK(m-l;m)D^l, \tag{2.8}$$

$$K(2m+l;m) = D^{m+l}E+ED^{m+l}-D^mK(l;m)D^m. \tag{2.9}$$

For $l < m$ we know that the matrices $K(l;m)$ and $K(m-l;m)$ are p.s.d. It follows from (2.8) and (2.9) that the matrices $K(m+l;m)$ and $K(2m+l;m)$ are c.n.d. This, in turn, implies that K_r is c.n.d. for $1 \leqq r \leqq 3$.

3 Operator convex functions on $(-1,1)$

Let $I = (-1,1)$. In this section we consider operator convex functions on I. The best known examples of such functions are

$$g_\lambda(t) := \frac{t^2}{1-\lambda t}, \quad |\lambda| \leqq 1. \tag{3.1}$$

See [3, p. 134] or [2].

Theorem 3.1. Let g_λ be the function in (3.1). Then for $-1 < \lambda < 0$ the Loewner matrices L_{g_λ} are c.n.d., and for $0 < \lambda < 1$ they are c.p.d.

Proof. Let $x_1, \ldots, x_n$ be any points in I. Then

$$L_{g_\lambda}(x_1, \cdots, x_n) = \left[\frac{x_i^2/(1-\lambda x_i) - x_j^2/(1-\lambda x_j)}{x_i - x_j} \right]. \tag{3.2}$$

We have

$$\begin{aligned} \frac{1}{a-b}\left(\frac{a^2}{1-\lambda a} - \frac{b^2}{1-\lambda b}\right) &= \frac{1}{a-b}\,\frac{a^2 - b^2 - \lambda ab(a-b)}{(1-\lambda a)(1-\lambda b)} \\ &= \frac{a+b-\lambda ab}{(1-\lambda a)(1-\lambda b)} \\ &= \frac{-1/\lambda\ \{(1-\lambda a)(1-\lambda b) - 1\}}{(1-\lambda a)(1-\lambda b)} \\ &= -\frac{1}{\lambda} + \frac{1/\lambda}{(1-\lambda a)(1-\lambda b)}. \end{aligned}$$

Thus the matrix (3.2) can be expressed as

$$L_{g_\lambda}(x_1, \ldots, x_n) = -\frac{1}{\lambda}E + \frac{1}{\lambda}D_\lambda E D_\lambda, \tag{3.3}$$

where D_λ is the diagonal matrix with entries $1/(1-\lambda x_i)$ on its diagonal. If $x \in H^n$, then $Ex = 0$. The matrix $D_\lambda E D_\lambda$ is p.s.d. So it follows from (3.3) that L_{g_λ} is c.p.d. if $\lambda > 0$ and c.n.d. if $\lambda < 0$. ■

Remark. By a theorem of Bendat and Sherman [2], [3, Theorem V. 4.6] every operator convex function f on I has a representation

$$f(t) = a + bt + \int_{-1}^{1} \frac{t^2}{1-\lambda t}\, d\mu(\lambda), \tag{3.4}$$

where μ is a probability measure on $[-1, 1]$. Thus the functions $g_\lambda(t)$ are especially important in the theory.

The next theorem is known; see Theorem 10 in [8]. The proof given here is different, and in the spirit of our discussion.

Theorem 3.2. Let $h_\lambda(t) = t g_\lambda(t)$ where g_λ is the function in (3.1). Then every Loewner matrix L_{h_λ} is c.p.d.

Proof. Simple algebraic manipulations show that

$$\begin{aligned}\frac{1}{a-b}\left(\frac{a^3}{1-\lambda a}-\frac{b^3}{1-\lambda b}\right) &= \frac{(a+b)^2-ab-\lambda ab(a+b)}{(1-\lambda a)(1-\lambda b)} \\ &= -\frac{a+b}{\lambda}-\frac{1}{\lambda^2}+\frac{1/\lambda^2}{(1-\lambda a)(1-\lambda b)}.\end{aligned}$$

Using this one can see that

$$L_{h_\lambda}(x_1,\ldots,x_n) = -\frac{1}{\lambda}(DE+ED)-\frac{1}{\lambda^2}E+\frac{1}{\lambda^2}D_\lambda ED_\lambda, \tag{3.5}$$

where $D=\mathrm{diag}(x_1,\ldots,x_n)$ and $D_\lambda=\mathrm{diag}\left(1/(1-\lambda x_1),\ldots,1/(1-\lambda x_n)\right)$. If x is any element of H^n, then $Ex=0$, and $\langle x,(DE+ED)x\rangle=0$. The matrix $D_\lambda ED_\lambda$ is p.s.d. It follows that the matrix L_{h_λ} in (3.5) is c.p.d. ■

From (3.4) it follows that if $f(t)=tg(t)$ where g is operator convex on I, then f has a representation

$$f(t)=at+bt^2+\int_{-1}^{1}\frac{t^3}{1-\lambda t}\,d\mu(\lambda). \tag{3.6}$$

We have shown that for every such f the matrices L_f are c.p.d., a fact proved by Horn [8] using completely different arguments.

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