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# On Proportional (Reversed) Hazard Model for Discrete Data

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# ON PROPORTIONAL (REVERSED) HAZARD MODEL FOR DISCRETE DATA

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Abstract. In this paper, we propose a proportional hazards model for discrete data analogous to the version for continuous data and then study its properties. Some ageing properties of the model are discussed. A new definition for reversed hazard rate is introduced. Properties of proportional reversed hazards model are studied.

## 1. Introduction

The literature on reliability theory mainly deals with non-negative absolutely continuous random variables. However, quiet often we come across with situations where the product life can be described through non-negative integer valued random variable. For example (i) a device can be monitored only once per time period and the observation is taken as the number of time periods successfully completed prior to the failure of a device, (ii) a piece of equipment may operate in cycles and we measure the number of cycles completed prior to failure, for example number of copies made by a photocopier before it fails, (iii) number of road accidents in a city in a given month. See Chen and Manatunga [7], Kemp [15] and Yu [26] and the references therein. Therefore we need to develop tools, analogous to the continuous case, for studying the discrete failure data.

Let $X$ be discrete random variable with support $N = \{1, 2, ...\}$ or a subset thereof. Suppose $p(x) = P(X = x)$, $F(x) = P(X \leq x) = \sum_{j=1}^{x} p(j)$ and $R(x) = P(X > x) = \sum_{j=x+1}^{\infty} p(j)$ denote, respectively, the probability mass function, distribution function and reliability/survival function of $X$. Then the hazard rate and the reverse hazard rate functions of $X$ defined by

$k(x)$ and $\lambda(x)$, respectively, are

$$k(x) = \frac{p(x)}{R(x-1)}, \quad x = 1, 2, \ldots \tag{1.1}$$

and

$$\lambda(x) = \frac{p(x)}{F(x)}, \quad x = 1, 2, \ldots. \tag{1.2}$$

It is interesting to note that in the discrete setup both the hazard rate and the reversed hazard rate can be interpreted as a probability which is not the case in the continuous case. If $X$ represents the lifetime of a component then $k(x)$ is the probability that the component will fail at time $X = x$ given that it has survived up to the time before $x$. Similarly, $\lambda(x)$ is the probability that the component will fail at time $X = x$, given it is known to have failed before $x$. Because of this nice interpretation of $k(x)$ and $\lambda(x)$ as a probability of an event in the discrete case, it has received wide spread attention. But many of the properties of the hazard rate and the reversed hazard rate which hold in the continuous case do not hold in the discrete case. Xie, Goudoin and Bracqmond [24] pointed out several problems or anomalies regarding the definition of $k(x)$. They defined a new hazard rate function (see Roy and Gupta [22]) in discrete setup as

$$h(x) = \ln \frac{R(x-1)}{R(x)}, \quad x = 1, 2, \ldots. \tag{1.3}$$

Xie et al. [24] also discussed several advantages of the definition (1.3). And obtained exact expression for hazard rate function $h(x)$ for some well known discrete distributions (see Lai and Xie [17], chapter 6 for more details). It is easy to verify that the functions $k(x)$ and $h(x)$ satisfy the relations of the form

$$k(x) = 1 - \exp(-h(x)), \quad h(x) = -ln(1 - k(x)), \quad \forall x = 1, 2, \ldots. \tag{1.4}$$

Hence both $k(x)$ and $h(x)$ have the same monotonicity property.

The functions $k(x)$ and $h(x)$ uniquely determine the distribution of $X$ through

$$R(x) = \prod_{j=1}^{x} (1 - k(j)) \tag{1.5}$$

and

$$R(x) = exp[-\sum_{j=1}^{x} h(j)], \; x = 1, 2, \ldots. \tag{1.6}$$

These functional forms are extensively used for modelling lifetime data.

Xekalaki [23] pointed out situations where the product life is discrete in nature and gave characterization results concerning the geometric, Waring and negative hyper geometric distributions in terms of hazard rates. Gupta, Gupta and Tripathi [11] characterized certain classes of discrete life distributions based on the convexity properties of $p(x)$. A dedicated study on discrete ageing notion can be found in Bracquemond, Goudoin, Roy and Xie [3]. For a recent survey of discrete reliability concepts and distributions see Bracquemond and Gaudoin [2]. More recently, Kemp [15] did an exhaustive study on the ageing behaviour of discrete life distributions and gave some new insight in this direction.

Modelling discrete lifetime data using the proportional hazards model (PHM) or proportional reversed hazards model (PRHM) is recent interest in statistical literature. See Chen and Manatunga[7], Gupta and Gupta [10] and Yu [26] and references therein for more details. This inspired us to study the properties of the PHM (PRHM) in discrete setup analogous to those in the continuous case.

The rest of the article is organized as follows. In Section 2 we discuss certain ageing properties of the discrete hazard rate. In section 3 we propose a proportional hazards model in discrete set up and then study the preservation of ageing properties under the proposed model. In Section 4 we introduce a new definition for reversed hazard rate for studying the properties of the proportional reversed hazards model. We conclude in Section 5 with indications of direction for future work.

## 2. **Discrete Ageing Notions**

Classification of classes of lifetime distributions based on the notion of ageing helps in identification of the underlying model which generates the data. Here we study the existing notions of discrete ageing. We give some definitions of discrete notion of ageing for completeness.

**Definition 2.1.** Let X be a discrete random variable defined on $N$, then

(i) $X$ is said to be increasing (decreasing) in likelihood ratio (ILR (DLR)) if $f(x)$ is log concave (convex), ie. if $p(x+1)p(x-1) \leq (\geq) p^2(x)$ for all $x$ in $N$.

(ii) $X$ is said to be increasing (decreasing) hazard rate (IHR (DHR)) if $R(x)$ is log concave (convex), ie. if $h(x)$ is increasing (decreasing) in $x$ for all $x$ in $N$.

(iii) $X$ is said to be increasing (decreasing) hazard rate average (IHRA (DHRA)) if $[R(x)]^{1/x}$

is decreasing (increasing) in $x$ for all $x$ in $N$.

(iv) $X$ is said to be decreasing (increasing) mean residual life (DMRL (IMRL)) if $m(x) = E(X - x|X > x)$ is decreasing (increasing) in $x$ for all $x$ in $N$.

(v) $X$ is said to be new better (worse) than used (NBU (NWU)) if $R(x+k) \leq (\geq) R(x)R(k)$ for all $x$, $k$ in $N$.

(vi) $X$ is said to be new better(worse) than used in expectation(NBUE (NWUE)) if $m(x) \leq (\geq)\mu$ for all $x$ in $N$, where $\mu$ is the mean of $X$.

**Remark 2.1.**

Standard text books give two definitions for IHRA (DHRA) and NBU(NWU) notions in discrete setup. Here we stated a unique definition in view of the new hazard rate $h(x)$ given in (1.3). For recent discussion on these aspects see Lai and Xie [17].

Next we discuss, the chain of implications between these ageing concepts.

**Theorem 2.1.** *The random variable $X$ has ILR (DLR) implies it has IHR (DHR).*

**Proof:** Since $X$ has ILR, we have $r(x) = \frac{p(x+1)}{p(x)}$ is decreasing in $x$. Consider

$$\begin{aligned} \frac{1}{k(x+1)} - \frac{1}{k(x)} &= \frac{\sum_{k=x+1}^{\infty} p(k)}{p(x+1)} - \frac{\sum_{k=x}^{\infty} p(k)}{p(x)} \\ &= [r(x+1) - r(x)] \\ &\quad + \sum_{k=1}^{\infty} \Big\{ \prod_{t=1}^{k} r(x+t) \Big( r(x+k+1) - r(x) \Big) \Big\}. \end{aligned}$$

When $< r(x) >$ is decreasing (increasing) each term in the above infinite sum is less than (greater than) zero. In view of the relation (1.4) the results follows.

**Remark 2.2.** The following example illustrates the fact that the converse of the above theorem is not true.

**Example 2.1.** Consider the probability distribution

| $X$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $p(x)$ | 0.10 | 0.1 | 0.40 | 0.30 | 0.10 |
| $k(x)$ | 0.1 | 0.16 | 0.50 | 0.75 | 1 |
| $r(x)$ | $-$ | 1 | 4 | 0.75 | 0.33 |

Here $k(x)$ is increasing, but $r(x)$ is not, that is, X has IHR but not ILR.

**Lemma 2.1.** *The random variable $X$ is IHR (DHR) if and only if $\frac{R(x-1)}{R(x)}$ is increasing (decreasing) for all $x \in N$.*

**Lemma 2.2.** *A sufficient condition that the random variable $X$ has DMRL (IMRL) is that the sequence $\langle s(x)\rangle$ is decreasing(increasing) for all $x \in N$, where $s(x) = R(x)/R(x-1)$.*

**Proof:** By simple algebra we can show that the mean residual life function $m(x)$ takes the form

$$m(x) = \frac{1}{R(x)} \sum_{k=x}^{\infty} R(k).$$

Consider

$$\begin{aligned} m(x+1) - m(x) &= \frac{1}{R(x+1)} \sum_{k=x+1}^{\infty} R(k) - \frac{1}{R(x)} \sum_{k=x}^{\infty} R(k) \\ &= [s(x+2) - s(x+1)] \\ &\quad + \sum_{k=2}^{\infty} \Big\{ \prod_{t=2}^{k} s(x+t) \Big( s(x+k+1) - s(x+1) \Big) \Big\}. \end{aligned}$$

When $\langle s(x)\rangle$ is decreasing (increasing) each term in the above infinite sum is less than (greater than) zero. Hence the result.

It is easy to prove the following theorem.

**Theorem 2.2.** *The random variable $X$ has IHR (DHR) implies it has DMRL (IMRL).*

The following theorem is a partial converse of Theorem 3.2 due to Mi [19].

**Theorem 2.3.** *The random variable $X$ has DMRL (IMRL) and if $\triangle m(x) = m(x+1) - m(x)$ is positive (negative), then $X$ has IHR (DHR).*

The following theorem is proved in Bracqmond et al. [3].

**Theorem 2.4.** *The random variable $X$ has IHR (DHR) implies it has IHRA (DHRA).*

**Theorem 2.5.** *The random variable $X$ has IHRA (DHRA) implies it has NBU (NWU).*

**Proof:** Suppose $X$ has IHRA, then by definition $[R(x)]^{\frac{1}{x}}$ is decreasing, hence we have

$$[R(x+k)]^{\frac{1}{x+k}} \leq [R(x)]^{\frac{1}{x}}.$$

Taking $x^{th}$ power on both sides we have

$$[R(x+k)]^{\frac{x}{x+k}} \leq R(x)$$

or

$$R(x+k) \leq R(x)[R(x+k)]^{\frac{k}{x+k}}.$$

Again using the fact $[R(x+k)]^{\frac{1}{x+k}} \leq [R(k)]^{\frac{1}{k}}$, the above inequality can be written as

$$R(x+k) \leq R(x)R(k).$$

Hence $X$ has NBU and the reversing the inequality we have the proof for the NWU cases.

**Theorem 2.6.** *The random variable X has NBU (NWU) implies it has NBUE (NWUE).*

For the proof see Theorem 3.4 of Kemp [15].

**Theorem 2.7.** *The random variable X has DMRL (IMRL) implies it has NBUE (NWUE).*

**Proof:** Suppose X has DMRL(IMRL). Then by Lemma 2.2 we have

$$\frac{\sum_{k=x-1}^{\infty} R(k)}{R(x-1)} \geq (\leq) \frac{\sum_{k=x}^{\infty} R(k)}{R(x)}.$$

Since $R(0) = 1$, the above inequality can be written as

$$\sum_{k=0}^{\infty} R(k) \geq (\leq) \frac{\sum_{k=x}^{\infty} R(k)}{R(x)}.$$

By definition, $\mu = E(X) = \sum_{k=1}^{\infty} P(X \geq x) = \sum_{k=0}^{\infty} R(k)$, hence X has NBUE (NWUE).

As in the continuous case we have the following chain of implications of the above mentioned classes.

$$IHR \Rightarrow IHRA \Rightarrow NBU \Rightarrow NBUE$$

and

$$IHR \Rightarrow DMRL \Rightarrow NBUE.$$

Next we study the stochastic ordering between two random variables.

**Definition 2.2.** Suppose that $X_1, X_2$ are two random variable with corresponding probability mass functions $p_1(x), p_2(x)$ respectively, then

(i) $X_2$ is said to be smaller than $X_1$ in the likelihood ratio order ($X_1 \geq_{lr} X_2$) if $p_1(x)/p_2(x)$ is increasing for all $x$ in $N$.

(ii) $X_2$ is said to be smaller than $X_1$ in the hazard rate order ($X_1 \geq_{hr} X_2$) if $R_1(x)/R_2(x)$ increases for all $x$ in $N$, where $R_1(x)$, $R_2(x)$ are the reliability function of $X_1$, $X_2$ respectively.

(iii) $X_2$ is said to be smaller than $X_1$ in the reversed hazard rate order ($X_1 \geq_{rhr} X_2$) if $F_1(x)/F_2(x)$ increases for all $x$ in $N$, where $F_1(x)$, $F_2(x)$ are the distribution function of $X_1$, $X_2$ respectively.

(iv) $X_2$ is said to be smaller than $X_1$ in the stochastic order ($X_1 \geq_{st} X_2$) if $R_1(x) \geq R_2(x)$ for all $x$ in $N$.

(v) $X_2$ is said to be smaller than $X_1$ in the mean residual life order ($X_1 \geq_{mrl} X_2$) if $E(X_1 - k|X_1 > k) \geq E(X_2 - k|X_2 > k)$ for all $x$ in $N$.

vi) $X_2$ is said to be smaller than $X_1$ in the mean residual life order ($X_1 \geq_{rmrl} X_2$) if $E(k - X_1|X_1 \leq k) \leq E(k - X_2|X_2 \leq k)$ for all $x$ in $N$.

**Theorem 2.8.** *$X_1 \geq_{lr} X_2$ implies $X_1 \geq_{hr} X_2$.*

**Proof:** Suppose $X_1 \geq_{lr} X_2$. then by definition $p_1(x)/p_2(x)$ is increasing in $x \geq 1$. Hence

$$p_1(x+k)p_2(x+k+1) \leq p_1(x+k+1)p_2(x+k), \forall \quad x, k \in N.$$

Adding $p_1(x+k)p_2(x+k)$ on both sides and rearranging terms we have

$$(p_2(x+k) + p_2(x+k+1))\frac{p_1(x+k)}{p_2(x+k)} \leq (p_1(x+k) + p_1(x+k+1)).$$

Since $\frac{p_1(x+k-1)}{p_2(x+k-1)} \leq \frac{p_1(x+k)}{p_2(x+k)}$ the above inequality can be written as

$$(p_2(x+k) + p_2(x+k+1))\frac{p_1(x+k-2)}{p_2(x+k-2)} \leq (p_2(x+k) + p_2(x+k+1)).$$

By similar argument above, we can arrive at

$$(p_2(x) + p_2(x+1) + ... + p_2(x+k+1))\frac{p_1(x)}{p_2(x)} \leq (p_1(x) + p_1(x+1) + ... + p_1(x+k+1)).$$

As $k \to \infty$, the above inequality reduces to

$$R_2(x)\frac{p_1(x)}{p_2(x)} \leq R_1(x).$$

or

$$h_1(x) \leq h_2(x).$$

The next theorem follows on similar lines.

**Theorem 2.9.** *$X_1 \geq_{lr} X_2$ implies $X_1 \geq_{rhr} X_2$.*

**Theorem 2.10.** *$X_1 \geq_{hr} X_2$ implies $X_1 \geq_{st} X_2$.*

**Proof:** Suppose $X_1 \geq_{hr} X_2$, then by definition, $R_1(x)/R_2(x)$ increases for all $x$ in $N$, if and only

$$\frac{R_1(x-1)}{R_2(x-1)} \leq \frac{R_1(x)}{R_2(x)}$$

or

$$R_1(x-1)R_2(x) \leq R_1(x)R_2(x-1).$$

Putting $x = 0, 1, 2, ...$, we obtain

$$R_2(1) \leq R_1(1), \quad R_2(2) \leq R_2(1), ...$$

Hence

$$R_2(x) \leq R_1(x), \quad x = 1, 2, 3, ...,$$

which implies $X_1 \geq_{st} X_2$.

**Theorem 2.11.** *$X_1 \geq_{rhr} X_2$ implies $X_1 \geq_{st} X_2$.*

The proof is similar to that of the Theorem 2.10.

**Theorem 2.12.** *$X_1 \geq_{hr} X_2$ implies $X_1 \geq_{mrl} X_2$.*

**Proof:** Suppose $X_1 \geq_{hr} X_2$, then by definition, $R_1(x)/R_2(x)$ increases for all $x$ in $N$, if and only

$$\frac{R_1(x)}{R_2(x)} \leq \frac{R_1(x+1)}{R_2(x+1)}$$

or

$$\frac{R_1(x+1)}{R_1(x)} \geq \frac{R_2(x+1)}{R_2(x)}.$$

Changing $x$ to $x+1$ and multiplying with above inequality and simplifying we obtain

$$\frac{R_1(x+2)}{R_1(x)} \geq \frac{R_2(x+2)}{R_2(x)}.$$

Proceeding as above we have the sequence of inequalities,

$$\frac{R_1(x+k)}{R_1(x)} \geq \frac{R_2(x+k)}{R_2(x)}, \quad k = 1, 2, ...$$

Hence

$$\sum_{k=x+1}^{\infty} \frac{R_1(k)}{R_1(x)} \geq \sum_{k=x+1}^{\infty} \frac{R_2(k)}{R_2(x)},$$

which is equivalent to

$$E(X_1 - k|X_1 > k) \geq E(X_2 - k|X_2 > k).$$

**Theorem 2.13.** *$X_1 \geq_{rhr} X_2$ implies $X_1 \geq_{rmrl} X_2$.*

Since we can express reversed mean residual life as

$$E(k - X_1|X_1 \leq k) = \frac{1}{F(x)} \sum_{k=1}^{x-1} F(k),$$

the proof is evident from Theorem 2.12.

Hence we have the following chain of implications in the discrete domain which is well-known in the continuous case.

$$X_1 \geq_{lr} X_2 \quad \text{implies} \quad X_1 \geq_{hr} X_2 \quad \text{implies} \quad X_1 \geq_{st} X_2$$

$$X_1 \geq_{hr} X_2 \quad \text{implies} \quad X_1 \geq_{mrl} X_2.$$

$$X_1 \geq_{lr} X_2 \quad \text{implies} \quad X_1 \geq_{rhr} X_2 \quad \text{implies} \quad X_1 \geq_{st} X_2$$

$$X_1 \geq_{rhr} X_2 \quad \text{implies} \quad X_1 \geq_{rmrl} X_2.$$

## 3. **Proportional Hazards Model**

Prediction of the life of a die is very crucial for the success of the forging industry. This prediction helps the forging industry to estimate the cost of the die and hence supply the die to the customer at a reasonably lower price. Reduced price can escalate the demand from the customer. Thus the study and analysis of failure time of the die is important in this context. One can also estimate the number of components that can be produced using a particular die. Since the life time is computed as the number of metal components produced by the die, discrete ageing concepts are of interest. The performance of the die is effected by the stress applied, environmental factors like temperature and pressure (see Chan and Meeker [4]). Proportional hazards model is a possible model in this context.

Consider a series system consisting of $n$ independent components. Then the lifetime of the system $Z = min(X_1, X_2, ..., X_n)$ has reliability function given by

$$R^*(x) = [R(x)]^n. \tag{3.1}$$

In the continuous case the model (3.1) is known as the PHM where the hazard rate corresponding to the random variable $Z$ is proportional to the hazard rate of individual components. However, the model (3.1) does not yield proportional hazards in the discrete setup when we use $k(x)$ as the hazard rate of $X$.

This inspired us to propose a new PHM in the discrete setup. Let $\overline{G}(y) = P(Y > y)$ be the reliability function of the random variable $Y$. Now we define the proportional hazards model in discrete set up as

$$\overline{G}(x) = [R(x)]^\theta, \quad \theta > 0. \tag{3.2}$$

If $h(x)$ is the hazard rate of $X$ then the hazard rate corresponding to $Y$ is given by

$$h^*(x) = \ln \frac{\overline{G}(x-1)}{\overline{G}(x)} = \ln \frac{[R(x-1)]^\theta}{[R(x)]^\theta} = \theta h(x). \tag{3.3}$$

The hazard rate of $Y$ is proportional to that of the random variable $X$. Hence the model (3.2) is the Proportional Hazards Model for the discrete data. First, we study some structural properties of the model (3.2).

The reliability function of $Y$ in terms of hazard rate is given by

$$\overline{G}(y) = \exp\Big( -\sum_{k=1}^{y} h^*(k)\Big) = \exp\Big( -\theta \sum_{k=1}^{y} h(k)\Big).$$

The probability mass function of $Y$ can be expressed as

$$\begin{aligned} g(y) = \overline{G}(y-1) - \overline{G}(y) &= [R(y-1)]^\theta - [R(y)]^\theta \\ &= [R(y-1)]^\theta \Big(1 - \exp(-\theta h(y))\Big). \end{aligned}$$

The $r^{th}$ moment of $Y$ is given by

$$E(Y^r) = \sum_{k=1}^{\infty} k^r g(k) = \sum_{k=1}^{\infty} k^r \Big( [R(k-1)]^\theta - [R(k)]^\theta \Big).$$

In particular,

$$E(Y) = \sum_{k=0}^{\infty} \overline{G}(k) = \sum_{k=0}^{\infty} [R(k)]^\theta$$

$$E(Y^2) = \sum_{k=0}^{\infty} (2k-1)\overline{G}(k) = \sum_{k=0}^{\infty} (2k-1)[R(k)]^\theta.$$

Hence the variance is given by

$$V(Y) = \sum_{k=0}^{\infty} (2k-1)[R(k)]^\theta - \Big( \sum_{k=0}^{\infty} [R(k)]^\theta \Big)^2.$$

In the following theorems, we prove the preservation of ageing properties under the transformation (3.2).

**Theorem 3.1.** *The random variable $X$ has IHR (DHR) if and only if and $Y$ has IHR (DHR).*

The proof is immediate from the definition by noting that $h^*(x) = \theta h(x)$.

**Theorem 3.2.** *The random variable $X$ has DMRL (IMRL) if and only if $Y$ has DMRL( IMRL).*

**Proof:** Let $X$ has DMRL. To prove $Y$ is DMRL it is enough to show that $\langle s^*(y) \rangle$ is decreasing where $s^*(y) = \overline{G}(y)/\overline{G}(y-1)$. That is $s^*(y+1) \leq s^*(y)$. Suppose

$$s^*(y+1) \geq s^*(y)$$

or

$$\frac{\overline{G}(y+1)}{\overline{G}(y)} \geq \frac{\overline{G}(y)}{\overline{G}(y-1)}.$$

Using (2.1) we obtain

$$\Big[\frac{R(x+1)}{R(x)}\Big]^\theta \geq \Big[\frac{R(x)}{R(x-1)}\Big]^\theta,$$

which gives

$$\Big[\frac{R(x+1)}{R(x)}\Big] \geq \Big[\frac{R(x)}{R(x-1)}\Big].$$

Then by Lemma 2.2 $X$ is IMRL, which is a contradiction. Hence one part of the theorem is proved. By similar argument we can prove the converse.

**Theorem 3.3.** *The random variable $X$ has NBU (NWU) if and only if $Y$ has NBU (NWU).*

**Proof:** Suppose that $X$ has NBU property, then by definition for all $x, k \in N$ we have

$$R(x)R(k) \geq R(x+k).$$

Taking $\theta^{th}$ power on both sides we have

$$[R(x)]^{\theta}[R(k)]^{\theta} \geq [R(x+k)]^{\theta}$$

or

$$\overline{G}(x)\overline{G}(k) \geq \overline{G}(x+k).$$

Now by definition $Y$ has NBU property. Retracing the above steps in the opposite direction we have the proof for the converse part. The proof for the NWU case is similar.
As above we can prove the following theorem.

**Theorem 3.4.** *The random variable $X$ has IHRA (DHRA) if and only if $Y$ has IHRA (DHRA).*

**Example 3.1.** Consider the discrete Pareto distribution with reliability function

$$R(x) = \Big(\frac{d}{x+d}\Big)^{c}, \quad c, d > 0, x \geq 1.$$

The PHM defined by (2.2) has reliability function

$$\overline{G}(x) = \Big(\frac{d}{x+d}\Big)^{c\theta}, \quad c, d > 0, x \geq 1.$$

The hazard rates $h(x)$ and $h^{*}(x)$ are given by

$$h(x) = \ln\frac{(d/x-1+d)^{c}}{(d/x+d)^{c}} = c\ln\Big[\frac{x+d}{x-1+d}\Big]$$

and

$$h^*(x) = \ln \frac{(d/x - 1 + d)^{c\theta}}{(d/x + d)^{c\theta}} = c\theta \ln \Big[\frac{x + d}{x - 1 + d}\Big].$$

Clearly both $h(x)$ and $h^*(x)$ are decreasing in $x$ which is consistent with continuous analogue. The results given in Theorem 3.2, 3.3 and 3.4 can be easily verified.

Next we study the stochastic ordering of two random variable under the transformation (3.2).

Suppose that $Y_1$ and $Y_2$ are two random variable related with $X_1$ and $X_2$ respectively by the transformation (3.2).

**Theorem 3.5.** *If* $X_1 \geq_{lr} X_2$ *then* $Y_1 \geq_{lr} Y_2$.

**Proof:** Let $g_1(x)$ and $g_2(x)$ be the probability mass functions of $Y_1$ and $Y_2$ respectively. Consider

$$g_2(x)/g_1(x) = [R_2(x-1)]^\theta \Big(1 - \exp(-\theta h_1(x))\Big) \Big/ [R_1(x-1)]^\theta \Big(1 - \exp(-\theta h_2(x))\Big)$$

$$= \Big[[R_2(x-1)]^\theta / [R_1(x-1)]^\theta\Big] \Big[\Big(1 - \exp(-\theta h_1(x))\Big) \Big/ \Big(1 - \exp(-\theta h_2(x))\Big)\Big]$$

Suppose $X_1 \geq_{lr} X_2$ holds, then by definition $p_1(x)/p_2(x)$ is increasing in $x$. That is $p_2(x)/p_1(x)$ is decreasing in $x$ so that $R_2(x-1)/R_1(x-1)$ is decreasing in $x$. Hence $h_2(x)/h_1(x)$ is decreasing in $x$ and $[(1-\exp(-\theta h_1(x)))/(1-\exp(-\theta h_2(x)))]$ is decreasing in $x$. Taking all these in account the right hand side of above equation is decreasing in $x$. Hence by definition $Y_1 \geq_{lr} Y_2$.

The proofs of the following two theorems are simple and hence omitted.

**Theorem 3.6.** $X_1 \geq_{hr} X_2$ *if and only if* $Y_1 \geq_{hr} Y_2$.

**Theorem 3.7.** $X_1 \geq_{st} X_2$ *if and only if* $Y_1 \geq_{st} Y_2$.

## 4. **Proportional reversed hazards model**

In lifetime data analysis, the concepts of reversed hazard rate has potential application when the time elapsed since failure is a quantity of interest in order to predict the actual time of failure. The reversed hazard rate is more useful in estimating reliability function when the data are left censored or right truncated. For the discussion on reversed hazard rate in continuous domain see Block, Savits and Singh [1], Chandra and Roy ([5],[6]), Finkelstein [9], Gupta, Gupta and Sankaran [13], Kundu and Gupta [16] and Nanda and Gupta [20] and the reference therein. A little work has been carried out in discrete setup in connection with reversed hazard rate. Recently, Nanda and Sengupta [21] have discussed reversed hazard rate in discrete setup and obtained several interesting results. Gupta, Nair and Asha [14] characterized certain class of discrete life distributions by means of a relationship between reversed hazard rate and right truncated expectation. The results by Gupta et al. [14] enlighten the role of reversed hazard rate and the right truncated expectation in characterizing discrete life distribution which was not covered by the earlier authors.

As parallel to celebrated PHM model the PRHM has become popular in recent times. For survey and some properties of PRHM in continuous domain see Gupta and Gupta [10]. The proportional reversed hazards model in discrete setup has potential application in system reliability. Consider a parallel system with $n$ independent component. If the lifetime of each component has distribution function of the form(discrete logistic)

$$F(x) = \frac{1 - e^{-x/d}}{1 + e^{-(x-c)/d}}.$$

Then the lifetime of the system $Z = max(X_1, X_2, ..., X_n)$ has the distribution function given by

$$F^*(x) = \Big[\frac{1 - e^{-x/d}}{1 + e^{-(x-c)/d}}\Big]^n = [F(x)]^n. \tag{4.1}$$

In continuous domain the model (4.1) is called PRHM (see Gupta, Gupta and Gupta [12]). As pointed out in Section 2 the model (4.1) dose not follow a proportional reversed hazards model in discrete setup when we use $\lambda(x)$ as the reversed hazard rate. Hence we introduce a new definition for the reversed hazard rate in line of continuous case. Besides the model (4.1) ensures proportional reversed hazards rates.

**Definition 4.1.** Let $X$ be discrete random variable with support $N$. The new reversed hazard rate of $X$ is defined as

$$r(x) = \ln \frac{F(x)}{F(x-1)}. \tag{4.2}$$

The rationale behind this definition is as follows. In the continuous case, the reversed hazard rate is defined as

$$\lambda^*(x) = \frac{F'(x)}{F(x)} = \frac{d\ln F(x)}{dx}.$$

Instead of taking $[F(x) - F(x-1)]$ for $F'(x)$ which leads to the expression (1.2), we could use $[\ln F(x) - \ln F(x-1)]$ for $[d\ln F(x)/dx]$ so that (4.2) follows. Note that the $r(x)$ is not bounded by one and is additive for parallel system as in the continuous case. The function $r(x)$ determines the distribution of $X$ uniquely by the relation

$$F(x) = \exp\Big(\sum_{k=x}^{\infty} -r(k)\Big). \tag{4.3}$$

Hence the cumulative reversed hazard rate is given by

$$H(x) = \sum_{k=x}^{\infty} r(k) = -\ln F(x).$$

Similar result holds when the lifetimes are continuous random variables. Since $\lambda(x)$ and $r(x)$ are related through

$$\lambda(x) = 1 - e^{-r(x)}, \tag{4.4}$$

both $\lambda(x)$ and $r(x)$ have same monotonicity properties. That is $\lambda(x)$ is increasing/decreasing if and only if $r(x)$ is increasing/decreasing in $x$. Now consider the reversed hazard rate of $Z$

$$r^*(x) = \ln\Big(\frac{R^*(x)}{R^*(x-1)}\Big) = \ln\Big(\frac{F(x)}{F(x-1)}\Big)^n = n\ln\Big(\frac{F(x)}{F(x-1)}\Big) = nr(x).$$

Clearly $r^*(x)$ is parallel to $r(x)$. Hence we define a PRHM in discrete setup as follows. Let $G(y)$ be the distribution of the random variable $Y$ and suppose that $G(x)$ is related through $F(x)$ as

$$G(x) = [F(x)]^{\theta}, \quad \theta > 0. \tag{4.5}$$

Then the model (4.5) is called the Proportional Reversed Hazards Model as the reversed hazard rate of $Y$ is proportional to that of $X$.

The probability mass function of $Y$ can be expressed as

$$\begin{aligned} g(y) = G(y) - G(y-1) &= [F(y)]^{\theta} - [F(y-1)]^{\theta} \\ &= [F(y)]^{\theta}\Big(1 - \exp(-\theta r(y))\Big). \end{aligned}$$

The $r^{th}$ moment of $Y$ is given by

$$E(Y^r) = \sum_{k=1}^{\infty} k^r g(k) = \sum_{k=1}^{\infty} k^r \Big( [F(k)]^\theta - [F(k-1)]^\theta \Big).$$

In particular,

$$E(Y) = \sum_{k=0}^{\infty} \overline{G}(k) = \sum_{k=0}^{\infty} (1 - [F(k)]^\theta)$$

$$E(Y^2) = \sum_{k=0}^{\infty} (2k-1)\overline{G}(k) = \sum_{k=0}^{\infty} (2k-1)(1 - [F(k)]^\theta).$$

Hence the variance is given by

$$V(Y) = \sum_{k=0}^{\infty} (2k-1)(1 - [F(k)]^\theta) - \Big( \sum_{k=0}^{\infty} (1 - [F(k)]^\theta) \Big)^2.$$

Now we prove an interesting results concerning the compounding sum.

**Definition 4.2.** A random variable defined on $N$ is said to be distributed as Sibuya ($\theta$) if $p(x)$ satisfies (see Devroye [8])

$$p(x) = (-1)^{x+1} \binom{\theta}{x}, \quad x = 1, 2, ..., 0 < \theta < 1.$$

**Theorem 4.1.** *Consider the random variable $Y = min(X_1, X_2, ..., X_N)$, where $N$ is random and independent of $X_i$, $i = 1, 2, ...N$. If the distribution of $N$ is Sibuya ($\theta$), then $Y$ admit a proportional reversed hazards model.*

**Proof:** Consider

$$\begin{aligned}
1 - G(x) &= P(min(X_1, X_2, ..., X_N) > x) \\
&= \sum_{k=1}^{\infty} P(min(X_1, X_2, ..., X_N) > x) P(N = k) \\
&= \sum_{k=1}^{\infty} [R(x)]^k (-1)^{k+1} \binom{\theta}{k}.
\end{aligned}$$

Or

$$\begin{aligned} G(x) &= 1+\sum_{k=1}^{\infty}[R(x)]^k(-1)^{k+1}\binom{\theta}{k} \\ &= (1-[R(x)])^{\theta} \\ &= [F(x)]^{\theta}. \end{aligned}$$

Hence we have the model (4.5).

Next we prove certain aging properties under the transformation (4.5).

**Definition 4.3.** Let $X$ be discrete random variable defined on $N$ (see Li and Xu [18]).
(i) The random variable $X$ is said to be increasing (decreasing) reversed hazard rate (IRHR(DRHR)) if $F(x)$ is log convex (concave), ie. if $r(x)$ is increasing (decreasing) in $x$.
(ii) The random variable $X$ is said to be increasing (decreasing) reversed mean residual life (IRMRL(DMRL)) if $v(x) = E(x - X|X \leq x)$ is increasing (decreasing) for all $x$.
The following theorem is immediate from the definition.

**Theorem 4.2.** *The random variable $X$ has IRHR(DRHR) if and only if $Y$ has IRHR(DRHR).*

As parallel to the results of Gupta et al. [12] we have the following result.

**Theorem 4.3.** *The random variable $X$ has IHR(DHR) and $\theta > 1(\theta < 1)$ then $Y$ has IHR(DHR).*

The proof of the next theorem is greatly facilitated by the following lemma.

**Lemma 4.1.** *Let $\phi$ be a real function defined on an interval $N$ and let $x_1 < y_1 \leq y_2$ and $x_1 \leq x_2 < y_2$*
*(i) if $\phi$ is convex on $N$, then*

$$\frac{\phi(y_1)-\phi(x_1)}{y_1-x_1} \leq \frac{\phi(y_2)-\phi(x_2)}{y_2-x_2}$$

*(ii) if $\phi$ is concave on $N$, then the above inequality is reversed.*

**Theorem 4.4.** *The random variable $X$ has NBU(NWU) and $\theta > 1(\theta < 1)$ then $Y$ has NBU(NWU).*

**Proof :** Suppose that $X$ has NBU then for all $x, t \geq 0$ we have

$$R(x+t-1) \leq R(x-1)R(t-1).$$

After some simple algebra we can show that the above inequality is same as

$$F(x+t-1) - F(x-1) \geq F(t-1) - F(x-1)F(t-1). \tag{4.6}$$

Let $x_1 = F(x-1)F(t-1)$, $x_2 = F(x-1)$, $y_1 = F(t-1)$ and $y_2 = F(x+t-1)$. Hence by the Lemma 4.1, for $\phi(x) = [F(x)]^\theta$ and $\theta > 1$ we have the inequality

$$\begin{aligned}(F(x+t-1) - F(x-1))([F(x-1)]^\theta - [F(x-1)]^\theta[F(t-1)]^\theta) \\ \leq (F(x-1) - F(x-1)F(t-1))([F(x+t-1)]^\theta - [F(x-1)]^\theta).\end{aligned} \tag{4.7}$$

In view of the inequality (4.6), to satisfy (4.7) one must have

$$[F(x-1)]^\theta - [F(x-1)]^\theta[F(t-1)]^\theta \leq [F(x+t-1)]^\theta - [F(x-1)]^\theta.$$

This will reduces to

$$[R(x+t-1)]^\theta \leq [R(x-1)]^\theta[R(t-1)]^\theta,$$

hence the proof for NBU case. Sice for the $\theta < 1$ the inequality in the Lemma 4.1 can be reversed so that the proof for the NWU case can be established in similar line.

**Lemma 4.2.** *A sufficient condition that the random variable $X$ has IRMRL is that the sequence $\langle u(x) \rangle$ is increasing for all $x \in N$, where $u(x) = F(x)/F(x+1)$.*

**Proof:** By simple algebra we can show that $v(x)$ takes the form

$$v(x) = \frac{1}{F(x)} \sum_{k=1}^{x-1} F(k).$$

Consider

$$\begin{aligned} v(x+1)-v(x) &= \frac{1}{F(x+1)}\sum_{k=1}^{x}F(k)-\frac{1}{F(x)}\sum_{k=1}^{x-1}F(k) \\ &= \frac{F(1)}{F(x+1)}+(u(x)-u(x-1)) \\ &\quad +\sum_{k=1}^{x-2}\Big\{\prod_{t=1}^{k}u(x-t))(u(x)-u(x-(k+1)))\Big\}. \end{aligned}$$

When $\langle u(x)\rangle$ is increasing each term in the above finite sum is greater than zero. Hence the results.

As in the same line of Theoem 3.2 we can state the following theorem.

**Theorem 4.5.** *The random variable $X$ has IRMRL if and only if $Y$ has IRMRL.*

**Remark 4.1.** In view of the Definition 4.3 and the Lemma 4.2 it can be easily verified that $X$ has DRHR then $X$ has (IRMRL).

## 5. **Conclusions**

The proportional hazards and reversed hazards model are very common in reliability analysis. Here we introduced these models relevant to situations when lifetimes are discrete and studied their ageing properties.

It would be interesting to consider the estimation and testing of the proportionality parameter $\theta$ introduced in the two models. The concept of frailty model is become popular now if the covariate presented in the model is random. So there is scope for studying the frailty model in discrete setup. Investigating the negative dependence in frailty models in line of the results of Xu and Li [25] is an another open problem.

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