isid/ms/2011/10
August 4, 2011
http://www.isid.ac.in/~statmath/eprints

On a conjecture on Ramanujan primes

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ON A CONJECTURE ON RAMANUJAN PRIMES

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Abstract. For $n \geq 1$, the *nth Ramanujan prime* is defined to be the smallest positive integer R_n with the property that if $x \geq R_n$, then $\pi(x) - \pi(\frac{x}{2}) \geq n$ where $\pi(\nu)$ is the number of primes not exceeding ν for any $\nu > 0$ and $\nu \in \mathbb{R}$. In this paper, we prove a conjecture of Sondow on upper bound for Ramanujan primes. An explicit bound of Ramanujan primes is also given. The proof uses explicit bounds of prime π and θ functions due to Dusart.

1. Introduction

In [3], J. Sondow defined Ramanujan primes and gave some conjectures on the behaviour of Ramanujan primes. For $n \geq 1$, the *nth Ramanujan prime* is defined to be the smallest positive integer R_n with the property that if $x \geq R_n$, then $\pi(x) - \pi(\frac{x}{2}) \geq n$ where $\pi(\nu)$ is the number of primes not exceeding ν for any $\nu > 0$ and $\nu \in \mathbb{R}$. It is easy to see that R_n is a prime for each n. The first few Ramanujan primes are given by $R_1 = 2, R_2 = 11, R_3 = 17, R_4 = 29, R_5 = 41, \ldots$. Sondow showed that for every $\epsilon > 0$, there exists $\mathcal{N}_0(\epsilon)$ such that $R_n < (2+\epsilon)n \log n$ for $n \geq \mathcal{N}_0(\epsilon)$. In this note, an explicit value of $\mathcal{N}_0(\epsilon)$ for each $\epsilon > 0$ is given. We prove

Theorem 1. *Let $\epsilon > 0$. For $\epsilon \leq 1.08$, let $\mathcal{N}_0 = \mathcal{N}_0(\epsilon) = \exp(\frac{c}{\epsilon} \log \frac{2}{\epsilon})$ where c is given by the following table.*

$\epsilon \in$	$(0, \frac{2}{11}]$	$(\frac{2}{11}, .4]$	$(.4, .6]$	$(.6, .8]$	$(.8, 1]$	$(1, 1.08]$
c	4	5	6	7	8	9

For $\epsilon > 1.08$, let $\mathcal{N}_0 = \mathcal{N}_0(\epsilon)$ be given by

$\epsilon \in$	$(1.08, 1.1]$	$(1.1, 1.21]$	$(1.21, 1.3]$	$(1.3, 2.5]$	$(2.5, 6]$	$(6, \infty)$
$\mathcal{N}_0$	169	101	74	48	6	2

Then

$$R_n < (2+\epsilon)n \log n \quad \text{for} \quad n \geq \mathcal{N}_0(\epsilon).$$

Sondow also showed that $p_{2n} < R_n < p_{4n}$ for $n > 1$ and he conjectured ([3, Conjecture 1]) that $R_n < p_{3n}$ for all $n \geq 1$, where p_i is the ith prime number. We derive the assertion of conjecture as a consequence of Theorem 1. We have

Theorem 2. *For $n > 1$, we have*

$$p_{2n} < R_n < p_{3n}.$$

1991 *Mathematics Subject Classification.* 11A41.

Key words and phrases. Ramanujan primes; primes.

Appeared in International Journal of Number Theory, Vol. 6, No. 8 (2010) 1869-1873.

We prove Theorems 1 and 2 in Section 3. In Section 2, we give preliminaries and lemmas for the proof which depend on explicit and sharp estimates from prime number theory.

2. Lemmas

We begin with the following estimates from prime number theory. Recall that p_i is the ith prime prime and $\pi(\nu)$ is the number of primes $\leq \nu$. Let $\theta(\nu) = \sum_{p \leq \nu} \log p$ where p is a prime.

Lemma 2.1. *For $\nu \in \mathbb{R}$ and $\nu > 1$, we have*

(*a*) $p_i > i \log i$ *for* $i \geq 1, i \in \mathbb{Z}$.
(*b*) $\nu(1 - \frac{0.006788}{\log \nu}) \leq \theta(\nu) \leq \nu(1 + \frac{0.006788}{\log \nu})$ *for* $\nu \geq 10544111$.
(*c*) $\frac{\nu}{\log \nu - 1} \underset{\nu \geq 5393}{\leq} \pi(\nu) \underset{\nu > 1}{\leq} \frac{\nu}{\log \nu}\left(1 + \frac{1.2762}{\log \nu}\right)$.

The estimate (*a*) is due to Rosser [2] and the estimates (*b*) and (*c*) are due to Dusart [1, p. 54]. □

From Lemma 2.1 (*b*) and (*c*), we obtain

Lemma 2.2. *Hence for $x \geq 2 \cdot 10544111$, we obtain*

$$\pi(x) - \pi(\frac{x}{2}) \geq \frac{x}{2\log x}\left(1 - \frac{0.020364}{\log x}\right) =: F(x) \text{ for } x \geq 2 \cdot 10544111 \tag{1}$$

and

$$\pi(x) - \pi(\frac{x}{2}) \geq \frac{x}{2(\log x - 1)}\left\{1 - \frac{1}{\log \frac{x}{2}}\left(\delta_1 - \frac{\delta_2}{\log \frac{x}{2}}\right)\right\} =: F_1(x) \text{ for } x \geq 5393 \tag{2}$$

where $\delta_1 = .2762 + \log 2$ and $\delta_2 = 1.2762(1 - \log 2)$.

Proof. For $x \geq 2 \cdot 10544111$, we obtain from Lemma 2.1 (*b*) that

$$\begin{aligned} \pi(x) - \pi(\frac{x}{2}) &\geq \frac{\theta(x) - \theta(\frac{x}{2})}{\log x} \\ &\geq \frac{x\left(1 - \frac{0.006788}{\log x}\right) - \frac{x}{2}\left(1 + \frac{0.006788}{\log \frac{x}{2}}\right)}{\log x} \\ &= \frac{x}{2\log x}\left(1 - \frac{0.006788}{\log x}\left(2 + \frac{\log x}{\log \frac{x}{2}}\right)\right) \\ &\geq \frac{x}{2\log x}\left(1 - \frac{0.006788}{\log x}(2+1)\right) \end{aligned}$$

which imply (1). For $x \geq 5393$, we have from Lemma 2.1 (*c*) that

$$\begin{aligned} \pi(x) - \pi(\frac{x}{2}) &\geq \frac{x}{\log x - 1} - \frac{\frac{x}{2}}{\log \frac{x}{2}}\left(1 + \frac{1.2762}{\log \frac{x}{2}}\right) \\ &= \frac{x}{2(\log x - 1)}\left\{2 - \left(1 + \frac{\log 2 - 1}{\log \frac{x}{2}}\right)\left(1 + \frac{1.2762}{\log \frac{x}{2}}\right)\right\} \\ &\geq \frac{x}{2(\log x - 1)}\left\{1 - \frac{1}{\log \frac{x}{2}}\left(\delta_1 - \frac{\delta_2}{\log \frac{x}{2}}\right)\right\} \end{aligned}$$

implying (2). □

For the proof of Theorem 1 for $\epsilon \leq .4$, we shall use the inequality (1). Then we may assume $n \leq \mathcal{N}_0(.4)$ for $\epsilon > .4$ and we use (2) to prove the assertion.

3. Proof of Theorems 1 and 2

For simplicity, we write $\epsilon_1 = \frac{\epsilon}{2}, \log_2 n := \log\log n$ and

$$f_0(n) := \log n + \log_2 n + \log(1+\epsilon_1) \text{ and } f_1(n) := \frac{\log_2 n + \log(2+2\epsilon_1)}{\log n}. \tag{3}$$

Let $x \geq (2+2\epsilon_1)n\log n$ with $n \geq \mathcal{N}_0(\epsilon) = \exp(\frac{c}{2\epsilon_1}\log\frac{1}{\epsilon_1}) =: n_0(\epsilon_1)$. Then $\log x \geq f_0(n) + \log 2$ for $n \geq n_0(\epsilon_1)$.

First we consider $\epsilon_1 \leq .2$. We observe that $F(x)$ is an increasing function of x and $2n_0(.2)\log(n_0(.2)) > 2 \cdot 10544111$. Therefore we have from (1) that

$$\frac{\pi(x) - \pi(\frac{x}{2})}{n} \geq \frac{1+\epsilon_1}{1+f_1(n)}\left(1 - \frac{0.020364}{f_0(n)+\log 2}\right) =: G(n). \tag{4}$$

$G(n)$ is again an increasing function of n. If $G(n_0(\epsilon_1)) > 1$, then $\pi(x) - \pi(\frac{x}{2}) > n$ for all $x \geq (2+2\epsilon_1)n\log n$ when $n \geq n_0(\epsilon_1)$ and hence $R_n < (2+2\epsilon_1)n\log n$ for $n \geq n_0(\epsilon_1)$. Therefore we show that $G(n_0) > 1$. It suffices to show

$$\epsilon_1 - \frac{0.020364(1+\epsilon_1)}{f_0(n)+\log 2} > f_1(n) = \frac{\log_2 n_0 + \log(2+2\epsilon_1)}{\log n_0}$$

for which it is enough to show

$$\epsilon_1 \geq \frac{\log_2 n_0 + \log(2+2\epsilon_1) + 0.020364(1+\epsilon_1)}{\log n_0}.$$

Since $\log n_0 = \frac{c}{2\epsilon_1}\log\frac{1}{\epsilon_1} = \frac{c_1}{\epsilon_1}\log\frac{1}{\epsilon_1}$ with $c_1 = 2, 2.5$ when $\epsilon_1 \leq \frac{1}{11}, \frac{1}{5}$, respectively, we need to show

$$\frac{(c_1 - 1)\log\frac{1}{\epsilon_1}}{\log_2\frac{1}{\epsilon_1} + \log c_1 + \log(2+2\epsilon_1) + 0.020364(1+\epsilon_1)} \geq 1.$$

The left hand side of the above expression is an increasing function of $\frac{1}{\epsilon_1}$ and the inequality is valid at $\frac{1}{\epsilon_1} = 11, 5$ implying the assertion for $\epsilon_1 \leq .2$.

Thus we now take $.2 < \epsilon_1 \leq .49$. We may assume that $n < n_0(.2)$. Since $x \geq (2+2\epsilon_1)n_0\log n_0 > 5393$, we have from (2) that

$$\frac{\pi(x) - \pi(\frac{x}{2})}{n} \geq \frac{1+\epsilon_1}{1+f_1(n) - \frac{1}{\log n}}\left\{1 - \frac{1}{f_0(n)}\left(\delta_1 - \frac{\delta_2}{f_0(n)}\right)\right\}.$$

Note that the right hand side of the above inequality is an increasing function of n since $n < n_0(.2)$. We show that the right hand side of the above inequality is > 1. Since $n \geq n_0(\epsilon_1)$, it suffices to show

$$\begin{aligned}&\log n_0(\epsilon_1 + \frac{1}{\log n_0} - f_1(n_0)) - \frac{1+\epsilon_1}{\frac{f_0(n_0)}{\log n_0}}\left(\delta_1 - \frac{\delta_2}{f_0(n_0)}\right)\\ =&\epsilon_1\log n_0 + 1 - \log_2 n_0 - \log(2+2\epsilon_1) - \frac{1+\epsilon_1}{1+f_1(n_0) - \frac{\log 2}{\log n_0}}\left(\delta_1 - \frac{\delta_2}{f_0(n_0)}\right)\end{aligned}$$

is > 0. Since $n_0(\epsilon_1) = \exp(\frac{c_1}{\epsilon_1}\log\frac{1}{\epsilon_1})$ where $c_1 = 3, 3.5, 4$ if $.2 < \epsilon_1 \leq .3, .3 < \epsilon_1 \leq .4$ and $.4 < \epsilon_1 \leq .49$, respectively, we observe that the right hand side of the above equality is equal to

$$(c_1 - 1)\log\frac{1}{\epsilon_1} + 1 - \log_2\frac{1}{\epsilon_1} - \log(2c_1 + 2c_1\epsilon_1) - \frac{1+\epsilon_1}{1+f_1(n_0) - \frac{\log 2}{\log n_0}}\left(\delta_1 - \frac{\delta_2}{f_0(n_0)}\right)$$

This is an increasing function of $\frac{1}{\epsilon_1}$. We find that the above function is > 0 for $\epsilon_1 \in \{.3, .4, .49\}$ implying $R_n < (2 + 2\epsilon_1)n \log n$ for $n \geq n_0(\epsilon_1)$ when $\epsilon_1 \leq .49$. Further we observe that $n_0(.49) \leq 339$. As a consequence, we have

$$R_n < 2.98n \log n \text{ for } n \geq 339.$$

and

$$\pi(x) - \pi(\frac{x}{2}) \geq 339 \text{ for } x \geq 2.98 \cdot 339 \log 339 > 5885.$$

Let $n < 339$. We now compute R_n by computing $\pi(x) - \pi(\frac{x}{2})$ for $p_{2n} < x \leq 5885$. Recall that $R_n > p_{2n}$ for $n > 1$. We find that $\frac{R_n}{n \log n} < 2.98, 3, 3.05, 3.08$ for $n \geq 220, 219, 171, 169$, respectively. Clearly $\frac{R_n}{n \log n} < 2+\epsilon$ for $n \geq \mathcal{N}_0(\epsilon)$ when $\epsilon \leq 1.08$. Thus $R_n < 3n \log n$ for $n \geq 219$ and $R_n < 3.08n \log n$ for $n \geq 169$. For $\epsilon > 1.08$, we check that the assertion is true by computing R_n for each $n < 169$. This proves Theorem 1.

Now we derive Theorem 2. From the above paragraph, we obtain $R_n < 3n \log n$ for $n \geq 219$. By Lemma 2.1 (a), we have $p_{3n} > 3n \log 3n$ for all $n \geq 1$ implying the assertion of Theorem 2 for $n \geq 219$. For $n < 219$, we check that $R_n < p_{3n}$ and Theorem 2 follows. $\square$

Acknowledgments

I thank Professor Cam Stewart for his encouragement and NSERC for support towards attending CANT 2009 at New York where a part of this work was done. I also thank Professor Jonathan Sondow for sharing his preprint with me and for his comments on an earlier draft of the paper. I would like to thank an anonymous referee for his remarks on an earlier version of the paper.

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