Inverse semigroups and the Cuntz-Li algebras

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Abstract. In this paper, we apply the theory of inverse semigroups to the $C^*$-algebra $U[Z]$ considered in [Cun08]. We show that the $C^*$-algebra $U[Z]$ is generated by an inverse semigroup of partial isometries. We explicitly identify the groupoid $G_{tight}$ associated to the inverse semigroup and show that $G_{tight}$ is exactly the same groupoid obtained in [CL10].

1. Introduction

Ever since the appearance of the Cuntz algebras $O_n$ and the Cuntz-Krieger algebras $O_A$ there has been a great deal of interest in understanding the structure of $C^*$-algebras generated by partial isometries. The theory of graph $C^*$-algebras owes much to these examples. It has now been well known that these algebras admit a groupoid realisation and the groupoid turns out to be r-discrete. Another object that is closely related with an r-discrete groupoid is that of an inverse semigroup. The relationship between r-discrete groupoids and inverse semigroups was already clear from [Ren80].

An inverse semigroup $S$ is a semigroup such that for every $s \in S$, there exists a unique $s^* \in S$ for which $s^*ss^* = s^*$ and $ss^*s = s$. The universal example of an inverse semigroup is the semigroup of partial bijections on a set. Just like one can associate a $C^*$-algebra to a group, one can associate a universal $C^*$-algebra related with an inverse semigroup $S$ and is denoted $C^*(S)$. This universal $C^*$-algebra captures the representations of the inverse semigroup (as partial isometries on a Hilbert space). One can canonically associate an r-discrete groupoid $G_S$ to an inverse semigroup $S$ such that the $C^*$-algebra of the groupoid $G_S$ coincides with $C^*(S)$. For a more detailed account of inverse semigroups and r-discrete groupoids, we refer to [Pat99] and [Exe08].

Recently, Cuntz and Li in [CL10] has introduced a $C^*$-algebra associated to every integral domain with only finite quotients. Earlier in [Cun08], Cuntz considered the integral domain $Z$. Let $R$ be an integral domain with only finite quotients. Then the universal algebra $U[R]$ is the universal $C^*$-algebra generated by a set of unitaries $\{u^n : n \in R\}$ and a set of partial isometries $\{s_m : m \in R^\times\}$ satisfying certain relations. In [CL10], it was proved that $U[R]$ is simple and purely infinite. A concrete realisation of $U[R]$ can be obtained by representing $s_m$ and $u^n$ on...
Then $U[R]$ is isomorphic to the $C^*$-algebra generated by $S_m$ and $U^n$ (by the simplicity of $U[R]$). The operator $S_m$ is implemented by the multiplication by $m$ (an injection) and $U^n$ is implemented by the addition by $n$ (a bijection). Thus it is immediately clear that $U[R]$ is generated by an inverse semigroup of partial isometries. Thus the theory of inverse semigroups should explain some of the results obtained by Cuntz and Li in \cite{CL10}. The purpose of this paper is to obtain the groupoid realisation (obtained in \cite{CL10}) by using the theory of inverse semigroups. We spell out the details only for the case $R = \mathbb{Z}$ as the analysis for general integral domains with finite quotients is similar. We should also remark that alternate approaches to the Cuntz-Li algebras were considered in \cite{BE10} and in \cite{KLQ10}. The main point we want to stress is if one uses the language of inverse semigroups one can obtain a groupoid realisation systematically without having to guess anything about the structure of the Cuntz-Li algebras.

Now we indicate the organisation of this paper. In Section 2, the definition of $U[\mathbb{Z}]$ is recalled and we show that $U[\mathbb{Z}]$ is generated by an inverse semigroup of partial isometries which we denote by $T$. In Section 3, we recall the notion of tight representations of an inverse semigroup, a notion introduced by Exel in \cite{Exe08}. We show that the identity representation of $T$ in $U[\mathbb{Z}]$ is in fact tight, and show that $U[\mathbb{Z}]$ is isomorphic to the $C^*$-algebra of the groupoid $\mathcal{G}_{\text{tight}}$ (considered in \cite{Exe08}) associated to $T$. In Sections 4 and 5, we explicity identify the groupoid $\mathcal{G}_{\text{tight}}$ which turns out to be exactly the groupoid considered in \cite{CL10}. In Section 6, we show that $U[\mathbb{Z}]$ is simple. In Section 7, we digress a bit to explain the connection between Crisp and Laca’s boundary relations and Exel’s tight representations of Nica’s inverse semigroup. In the final Section, we give a few remarks of how to adapt the analysis carried out in Sections 1 – 6 for a general integral domain. A bit of notation: For non-zero integers $m$ and $n$, we let $[m,n]$ to denote the lcm of $m$ and $n$ and $(m,n)$ to denote the gcd of $m$ and $n$. For a ring $R$, $R^\times$ denotes the set of non-zero elements in $R$.

\section{The Regular $C^*$-algebra associated to $\mathbb{Z}$}

**Definition 2.1** (\cite{Cun08}). Let $U[\mathbb{Z}]$ be the universal $C^*$-algebra generated by a set of unitaries $\{u^n : n \in \mathbb{Z}\}$ and a set of isometries $\{s_m : m \in \mathbb{Z}^\times\}$ satisfying the following relations.

\begin{align*}
    s_m s_n &= s_{mn} \\
    u^n u^m &= u^{n+m} \\
    s_m u^n &= u^{mn} s_m \\
    \sum_{n \in \mathbb{Z}/(m)} u^n e_m u^{-n} &= 1
\end{align*}
where \( e_m \) denotes the final projection of \( s_m \).

**Remark 2.2.** Let \( \chi \) be a character of the discrete multiplicative group \( \mathbb{Q}^\times \). Then the universal property of the \( C^* \)-algebra \( U[\mathbb{Z}] \) ensures that there exists an automorphism \( \alpha_\chi \) of the algebra \( U[\mathbb{Z}] \) such that \( \alpha_\chi(s_m) = \chi(m) s_m \) and \( \alpha_\chi(u^n) = u^n \). This action of the character group of the multiplicative group \( \mathbb{Q}^\times \) was considered in [CL10].

For \( m \neq 0 \) and \( n \in \mathbb{Z} \), consider the operators \( S_m \) and \( U^n \) defined on \( l^2(\mathbb{Z}) \) as follows:

\[
S_m(\delta_r) = \delta_{rm} \\
U^n(\delta_r) = \delta_{r+n}
\]

Then \( s_m \to S_m \) and \( u^n \to U^n \) gives a representation of the universal \( C^* \)-algebra \( U[\mathbb{Z}] \) called the regular representation and its image is denoted by \( U_r[\mathbb{Z}] \). We begin with a series of Lemmas (highly inspired and adapted from [Cun08] and from [CL10]) which will be helpful in proving that \( U[\mathbb{Z}] \) is generated by an inverse semigroup of partial isometries.

**Lemma 2.3.** For every \( m, n \neq 0 \), one has \( e_m = \sum_{k \in \mathbb{Z}/(n)} u^{mk} e_{mn} u^{-mk} \).

**Proof.** One has

\[
e_m = s_m s_m^* \\
= s_m \left( \sum_{k \in \mathbb{Z}/(n)} u^k e_n u^{-k} \right) s_m^* \\
= \sum_{k \in \mathbb{Z}/(n)} s_m u^k s_n^* s_m^* u^{-k} s_m^* \\
= \sum_{k \in \mathbb{Z}/(n)} u^{km} s_m s_n^* s_m^* u^{-km} \\
= \sum_{k \in \mathbb{Z}/(n)} u^{km} s_m s_m^* s_m u^{-km} \\
= \sum_{k \in \mathbb{Z}/(n)} u^{km} e_{mn} u^{-km}.
\]

This completes the proof. \( \square \)

**Lemma 2.4.** For every \( m, n \neq 0 \), one has \( e_m e_n = e_{[m,n]} \) where \([m,n]\) denotes the least common multiple of \( m \) and \( n \).

**Proof.** Let \( c := [m,n] \) be the lcm of \( m \) and \( n \). Then \( c = am = bn \) for some \( a, b \). Now from Lemma 2.3, it follows that

\[
e_m e_n = \sum_{r \in \mathbb{Z}/(a), s \in \mathbb{Z}/(b)} u^{mr} c_r u^{-mr} u^{ns} c_s u^{-ns} \]
The product $u^{mr} e_r u^{mr} u^{ns} e_r u^{-ns}$ survives if and only if $mr \equiv ns \mod c$. But the only choice for such an $r$ and an $s$ is when $r \equiv 0 \mod a$ and $s \equiv 0 \mod b$. [Reason: Suppose there exists $r$ and $s$ such that $mr \equiv ns \mod c$. Then $\frac{mr-ns}{c}$ is an integer. That is $\frac{r}{a} - \frac{s}{b}$ is an integer. Multiplying by $b$, one has that $\frac{br}{a} - s$ and hence $\frac{br}{a}$ is an integer. But $a$ and $b$ are relatively prime. Hence $a$ divides $r$. Similarly $b$ divides $s$. Thus $e_m e_n = e_c$. This completes the proof.]

**Lemma 2.5.** Suppose $r \neq s$ in $\mathbb{Z}/(d)$ then the projections $u^r e_m u^{-r}$ and $u^s e_n u^{-s}$ are orthogonal where $d$ is the gcd of $m$ and $n$.

**Proof.** First note that $e_d u^{-r} u^s e_d u^{-s} u^r = 0$. Hence $e_d u^{-r} u^s e_d = 0$. Now note that

$$u^r e_m u^{-r} u^s e_n u^{-s} = u^r e_m (e_d u^{-r} u^s e_d) e_n u^{-s} \quad \text{[by Lemma 2.4]}$$

$$= u^r e_m (e_d u^{-r} u^s e_d) e_n u^{-s} = 0$$

This completes the proof. □

**Lemma 2.6.** Let $m, n \neq 0$ be given. Let $d = (m, n)$ and $c = [m, n]$. Suppose $r \equiv s \mod d$. Let $k$ be such that $k \equiv r \mod m$ and $k \equiv s \mod n$. Then $u^r e_m u^{-r} u^s e_n u^{-s} = u^k e_c u^{-k}$.

**Proof.** First note that $u^r e_m u^{-r} = u^k e_m u^{-k}$ and $u^s e_n u^{-s} = u^k e_n u^{-k}$. The result follows from Lemma 2.4 □

**Lemma 2.7.** For $m, n \neq 0$, one has $s_m^* e_n s_m = e_n'$ where $n' := \frac{n}{(m, n)}$.

**Proof.** First note that without loss of generality, we can assume that $m$ and $n$ are relatively prime. Otherwise write $m := m_1 d$ and $n := n_1 d$ where $d$ is the gcd of $m$ and $n$. Then $(m_1, n_1) = 1$ and

$$s_m^* e_n s_m = s_{m_1}^* s_{d}^* s_n s_{m_1}^* s_d s_{m_1}$$

$$= s_{m_1}^* e_{n_1} s_{m_1}$$

So now assume $m$ and $n$ are relatively prime. Observe that $s_m^* e_n s_m$ is a selfadjoint projection. For $s_m^* e_n s_m s_m e_n s_n = s_m^* e_n e_m s_m = s_m^* e_m e_n s_m = s_m^* e_m s_m$. Again,

$$(s_m^* e_n s_m)^2 = s_m^* e_n e_m s_m$$

$$= s_m^* e_{mn} s_m \quad \text{[by Lemma 2.4]}$$

$$= s_m^* s_m s_n s_n^* s_m$$

$$= e_n$$

This completes the proof. □

**Lemma 2.8.** Let $m, n \neq 0$ and $k \in \mathbb{Z}$ be given. If $(m, n)$ does not divide $k$ then one has $s_m^* u^k e_n u^{-k} s_m = 0$. 
Proof. It is enough to show that $x := e_n u^{-k} s_m$ vanishes. Thus it is enough to show that $x x^* = e_n u^{-k} e_m u^k e_n$. Now Lemma 2.5 implies that $x x^* = 0$. This completes the proof. 

Lemma 2.9. Let $m, n \neq 0$ and $k \in \mathbb{Z}$ be given. Suppose that $d := (m, n)$ divides $k$. Choose an integer $r$ such that $mr \equiv k \mod n$. Then $s_m u^k e_n u^{-k} s_m = u e_n u^{-r}$ where $n_1 = \frac{n}{d}$.

Proof. Now observe that $u^k e_n u^{-k} = u^m e_n u^{-mr}$. Hence one has

$$s_m u^k e_n u^{-k} s_m = s_m u^m e_n u^{-mr} s_m$$

$$= u^r s_m e_n s_m u^{-r}$$

$$= u^r e_n u^{-r} \quad \text{[by Lemma 2.7]}$$

This completes the proof.

Remark 2.10. Let $P := \{ u^n e_m u^{-n} : m \neq 0, n \in \mathbb{Z} \} \cup \{0\}$. Then the above observations show that $P$ is a commutative semigroup of projections which is invariant under the map $x \rightarrow s_m x s_m$.

The proof of the following proposition is adapted from [CL10].

Proposition 2.11. Let $T := \{ s_m u^n e_k u^{-n'} s_m' : m, m', k \neq 0, n, n' \in \mathbb{Z} \} \cup \{0\}$. Then $T$ is an inverse semigroup of partial isometries. Let $P := \{ u^n e_m u^{-n} : m \neq 0, n \in \mathbb{Z} \} \cup \{0\}$. Then the set of projections in $T$ coincide with $P$. Also the linear span of $T$ is dense in $U[\mathbb{Z}]$.

Proof. The fact that $T$ is closed under multiplication follows from the following calculation.

$s_m u^n e_r u^{-n'} s_m' s_k' u^\ell u^{-\ell'} s_{k'} = s_m u^n e_r u^{-n'} s_m' s_k' s_m u^\ell u^{-\ell'} s_{k'}$

$$= s_m u^n e_r u^{-n'} e_m' s_k' s_m' u^\ell u^{-\ell'} s_{k'}$$

$$= s_m u^n e_r e_m' s_k' s_m' e_s u^\ell u^{-\ell'} s_{k'} \quad \text{[where } \tilde{e} = u^n e_r u^{-n'} \text{ and } \tilde{f} = u^\ell u^{-\ell'}]$$

$$= s_m u^n s_k' e_k s_k' (s_k e_m' s_k') (s_k' \tilde{f}^* s_k' s_m' u^\ell u^{-\ell'} s_{k'})$$

$$= s_m u^n s_k' e_k s_k' (s_k e_m' s_k') (s_k' \tilde{f} s_k s_m') \quad \text{[where } p := (s_k \tilde{e} s_k')(s_k e_m' s_k')(s_k' \tilde{f} s_k s_m') \in P \]}

Thus we have shown that $T$ is closed under multiplication. Clearly $T$ is closed under the involution *. Thus the linear span of $T$ is a * algebra containing $s_m$ and $u^n$ for every $m \neq 0$ and $n \in \mathbb{Z}$. Hence the linear span of $T$ is dense in $U[\mathbb{Z}]$.

Now we show that every element of $T$ is a partial isometry. Let $v := s_m u^n e_k u^{-n'} s_m'$ be given. Now,

$v v^* = s_m u^n e_k u^{-n'} s_m' s_m u^{-n'} e_k u^{-n} s_m$

$$= s_m u^n (e_k u^{-n'} e_m u^{-n'} e_k) u^{-n} s_m$$

$$= s_m u^n e u^{-n} s_m \quad \text{[where } e := (e_k u^{-n'} e_m u^{-n'} e_k) \in P \]
Now it follows from Remark 2.10 that $vv^* \in P$. It also shows that the set of projections in $T$ coincides with $P$. This completes the proof. □

The following equality will be used later. Let us isolate it now.

\[(2.1) s_{m_1}^* u_1^k s_{n_1}^* s_{m_2}^* u_2^k s_{n_2} = s_{m_1} m_2 u m_2 n_1 u k_2 n_1 s_{n_1} n_2\]

**Remark 2.12.** We also need the following fact. If $v \in T$, let us denote its image in the regular representation by $V$. Observe that $v \neq 0$ if and only if $V \neq 0$. This is clear for projections in $T$. Now let $v \in T$ be a non-zero element. Then $vv^* \in P$ is non-zero. Thus $VV^* \neq 0$ which implies $V \neq 0$.

3. Tight representations of an inverse semigroup

Let us recall the notion of tight characters and tight representations from [Exe08].

**Definition 3.1.** Let $S$ be an inverse semigroup with $0$. Denote the set of projections in $S$ by $E$. A character for $E$ is a map $x : E \to \{0, 1\}$ such that

1. the map $x$ is a semigroup homomorphism, and
2. $x(0) = 0$.

We denote the set of characters of $E$ by $\hat{E}_0$. We consider $\hat{E}_0$ as a locally compact Hausdorff topological space where the topology on $\hat{E}_0$ is the subspace topology induced from the product topology on $\{0, 1\}^E$.

For a character $x$ of $E$, let $A_x := \{ e \in E : x(e) = 1 \}$. Then $A_x$ is a nonempty set satisfying the following properties.

1. The element $0 \notin A_x$.
2. If $e \in A_x$ and $f \geq e$ then $f \in A_x$.
3. If $e, f \in A_x$ then $ef \in A_x$.

Any nonempty subset $A$ of $E$ for which (1), (2) and (3) are satisfied is called a filter. Moreover if $A$ is a filter then the indicator function $1_A$ is a character. Thus there is a bijective correspondence between the set of characters and filters. A filter is called an ultrafilter if it is maximal. We also call a character $x$ maximal or an ultrafilter if its support $A_x$ is maximal.

The set of maximal characters is denoted by $\hat{E}_\infty$ and its closure in $\hat{E}_0$ is denoted by $\hat{E}_\text{tight}$.

The following characterization of maximal characters will be extremely useful for us and we refer to [Exe09] for a proof. Let $E$ be an inverse semigroup of projections. Let $e, f \in E$. We say that $f$ intersects $e$ if $fe \neq 0$.

**Lemma 3.2.** Let $E$ be an inverse semigroup of projections with $0$ and $x$ be a character of $E$. Then the following are equivalent.

1. The character $x$ is maximal.
2. The support $A_x$ contains every element of $E$ which intersects every element of $A_x$. 

Corollary 3.3. Let $A$ be a unital $C^*$-algebra and $E \subset A$ be an inverse semigroup of projections containing $\{0, 1\}$. Suppose that $E$ contains a finite set $\{e_1, e_2, \cdots, e_n\}$ of mutually orthogonal projections such that $\sum_{i=1}^{n} e_i = 1$. Then for every maximal character $x$ of $E$, there exists a unique $e_i$ for which $x(e_i) = 1$.

Proof. The uniqueness of $e_i$ is clear as the projections $e_1, e_2, \cdots, e_n$ are orthogonal. Now to show the existence of an $e_i$ in $A_x$, we prove by contradiction. Assume that $e_i \notin A_x$ for every $i$. Then by Lemma 3.2, we have that for every $i$, there exists an $f_i \in A_x$ such that $e_if_i = 0$. Let $f = \prod f_i$. Then $f \in A_x$ and thus nonzero and also $fe_i = 0$ for every $i$. As $\sum_i e_i = 1$, this forces $f = 0$. Thus we have a contradiction. \qed

Let us recall the notion of tight representations of semilattices from [Exe08] and from [Exe09]. The only semilattice we consider is that of an inverse semigroup of projections or in otherwords the idempotent semilattice of an inverse semigroup. Also our semilattice contains a maximal element 1. First let us recall the notion of a cover from [Exe08].

Definition 3.4. Let $E$ be an inverse semigroup of projections containing $\{0, 1\}$ and $Z$ be a subset of $E$. A subset $F$ of $Z$ is called a cover for $Z$ if given a non-zero element $z \in Z$ there exists an $f \in F$ such that $fz \neq 0$. A cover $F$ of $Z$ is called a finite cover if $F$ is finite.

The following definition is actually Proposition 11.8 in [Exe08].

Definition 3.5. Let $E$ be an inverse semigroup of projections containing $\{0, 1\}$. A representation $\sigma : E \to B$ of the semilattice $E$ in a Boolean algebra $B$ is said to be tight if given $e \neq 0$ in $E$ and for every finite cover $F$ of the interval $[0, e] := \{x \in E : x \leq e\}$, one has $\sup_{f \in F} \sigma(f) = \sigma(e)$.

Let $A$ be a unital $C^*$ algebra and $S$ be an inverse semigroup containing $\{0, 1\}$. Let $\sigma : S \to A$ be a unital representation of $S$ as partial isometries in $A$. Let $\sigma(C^*(E))$ be the $C^*$-subalgebra in $A$ generated by $\sigma(E)$. Then $\sigma(C^*(E))$ is a unital, commutative $C^*$-algebra and hence the set of projections in it is a Boolean algebra which we denote by $B_{\sigma(C^*(E))}$. We say the representation $\sigma$ is tight if the representation $\sigma : E \to B_{\sigma(C^*(E))}$ is tight.

Lemma 3.6. Let $X$ be a compact metric space and $E \subset C(X)$ be an inverse semigroup of projections containing $\{0, 1\}$. Suppose that for every finite set of projections $\{f_1, f_2, \cdots, f_m\}$ in $E$, there exists a finite set of mutually orthogonal non-zero projections $\{e_1, e_2, \cdots, e_n\}$ in $E$ and a matrix $(a_{ij})$ such that

$$\sum_{i=1}^{n} e_i = 1$$

$$f_i = \sum_{j} a_{ij} e_j.$$

Then the identity representation of $E$ in $C(X)$ is tight.
Proof. Let \( e \in E\setminus\{0\} \) be given and let \( F \) be a finite cover for the interval \([0,e]\). Without loss of generality, we can assume that \( e = 1 \) (Just cut everything down by \( e \)). Let 
\( F := \{f_1,f_2,\cdots ,f_m\} \). Then by the hypothesis there exists a finite set of mutually orthogonal projections \( \{e_1,e_2,\cdots ,e_n\} \) and a matrix \( (a_{ij}) \) such that 
\[ f_i = \sum_j a_{ij} e_j \quad \text{and} \quad \sum_i e_i = 1. \]
For a given \( j \), let \( A_j := \{i : a_{ij} \neq 0\} \). Since \( F \) covers \( C(X) \), it follows that for every \( j \), \( A_j \) is nonempty. In otherwords, given \( j \), there exists an \( i \) such that \( f_i \geq e_j \). Thus \( f := \sup_i f_i \geq e_j \) for every \( j \). Hence \( f \geq \sup_j e_j = 1 \). This completes the proof. \( \square \)

In the next proposition, \( T \) denotes the inverse semigroup associated to \( U[Z] \) in Proposition 2.11

**Proposition 3.7.** The identity representation of \( T \) in \( U[Z] \) is tight.

**Proof.** We apply Lemma 2.6. Let \( \{u^{r_1}e_{m_1}u^{-r_1},u^{r_2}e_{m_2}u^{-r_2},\cdots ,u^{r_k}e_{m_k}u^{-r_k}\} \) be a finite set of non-zero projections in \( P \). By Lemma 2.3 it follows that each \( f_i := u^{r_i}e_{m_i}u^{-r_i} \) is a linear combination of \( \{u^c e_c u^{-c} : s \in \mathbb{Z}/(c)\} \) where \( c \) is the lcm of \( m_1,m_2,\cdots ,m_k \). Then Lemma 3.6 implies that the identity representation of \( T \) in \( U[Z] \) is tight. This completes the proof. \( \square \).

Now we will show that the \( C^* \)-algebra of the groupoid \( G_{tight} \) of the inverse semigroup \( T \) is isomorphic to the algebra \( U[Z] \). First let us recall the construction of the groupoid \( G_{tight} \) considered in [Exe08]. Let \( S \) be an inverse semigroup with 0 and let \( E \) denote its set of projections. Note that \( S \) acts on \( \hat{E}_0 \) partially. For \( x \in \hat{E}_0 \) and \( s \in S \), define \( (x.s)(e) = x( ses^*) \).

Then
\[ \begin{align*}
\bullet & \quad \text{The map } x.s \text{ is a semigroup homomorphism, and} \\
\bullet & \quad (x.s)(0) = 0.
\end{align*} \]
But \( x.s \) is non-zero if and only if \( x(ss^*) = 1 \). For \( s \in S \), define the domain and range of \( s \) as Let \( S \) be an inverse semigroup with 0 and let \( E \) denote its set of projections. Note that \( S \) acts on \( \hat{E}_0 \) partially. For \( x \in \hat{E}_0 \) and \( s \in S \), define \( (x.s)(e) = x( ses^*) \). Then
\[ \begin{align*}
\bullet & \quad \text{The map } x.s \text{ is a semigroup homomorphism, and} \\
\bullet & \quad (x.s)(0) = 0.
\end{align*} \]
But \( x.s \) is non-zero if and only if \( x(ss^*) = 1 \). For \( s \in S \), define the domain and range of \( s \) as
\[ \begin{align*}
D_s : & = \{x \in \hat{E}_0 : x(ss^*) = 1\} \\
R_s : & = \{x \in \hat{E}_0 : x(s^*s) = 1\}.
\end{align*} \]
Note that both \( D_s \) and \( R_s \) are compact and open. Moreover \( s \) defines a homeomorphism from \( D_s \) to \( R_s \) with \( s^* \) as its inverse. Also observe that \( \hat{E}_{tight} \) is invariant under the action of \( S \).

Consider the transformation groupoid \( \Sigma := \{(x,s) : x \in D_s\} \) with the composition and the inversion being given by:
\[ \begin{align*}
(x,s)(y,t) : & = (x,st) \text{ if } y = x.s \\
(x,s)^{-1} : & = (x.s,s^*)
\end{align*} \]
Define an equivalence relation \( \sim \) on \( \Sigma \) as \((x, s) \sim (y, t)\) if \( x = y \) and if there exists an \( e \in E \) such that \( x \in D_e \) for which \( es = et \). Let \( \mathcal{G} = \Sigma / \sim \). Then \( \mathcal{G} \) is a groupoid as the product and the inversion respects the equivalence relation \( \sim \). Now we describe a topology on \( \mathcal{G} \) which makes \( \mathcal{G} \) into a topological groupoid.

For \( s \in S \) and \( U \) an open subset of \( D_s \), let \( \theta(s, U) := \{[x, s] : x \in U\} \). We refer to [Exe08] for the proof of the following two propositions. We denote \( \theta(s, D_s) \) by \( \theta_s \). Then \( \theta_s \) is homeomorphic to \( D_s \) and hence is compact, open and Hausdorff.

**Proposition 3.8.** The collection \( \{\theta(s, U) : s \in S, U \text{ open in } D_s\} \) forms a basis for a topology on \( \mathcal{G} \). The groupoid \( \mathcal{G} \) with this topology is a topological groupoid whose unit space can be identified with \( \hat{E}_0 \). Also one has the following.

1. For \( s, t \in S \), \( \theta_s \theta_t = \theta_{st} \).
2. For \( s \in S \), \( \theta_s^{-1} = \theta_s^* \), and
3. The set \( \{1_{\theta_s} : s \in T\} \) generates the \( C^* \)-algebra \( C^*(\mathcal{G}) \).

We define the groupoid \( \mathcal{G}_{\text{tight}} \) to be the reduction of the groupoid \( \mathcal{G} \) to \( E_{\text{tight}} \). In [Exe08], it is shown that the representation \( s \mapsto 1_{\theta_s} \in C^*(\mathcal{G}_{\text{tight}}) \) is tight and any tight representation factors through this universal one.

**Proposition 3.9.** Let \( T \) be the inverse semigroup associated to \( U[\mathbb{Z}] \) in Proposition 2.11. Let \( \mathcal{G}_{\text{tight}} \) be the tight groupoid associated to \( T \). Then \( U[\mathbb{Z}] \) is isomorphic to \( C^*(\mathcal{G}_{\text{tight}}) \).

**Proof.** Let \( t_m, v^n \) denote the images of \( s_m, u^n \) in \( C^*(\mathcal{G}_{\text{tight}}) \). The universality of the \( C^* \)-algebra \( C^*(\mathcal{G}_{\text{tight}}) \) together with Proposition 3.7 implies that there exists a homomorphism \( \rho : C^*(\mathcal{G}_{\text{tight}}) \to U[\mathbb{Z}] \) such that \( \rho(t_m) = s_m \) and \( \rho(v^n) = u^n \).

Note that the mutually orthogonal set of projections \( \{u^r e_m u^{-r} : r \in \mathbb{Z}/(m)\} \) cover \( T \). Since the representation of \( T \) in \( C^*(\mathcal{G}_{\text{tight}}) \) is tight, it follows that \( \sum_r v^r t_m t^*_m v^{-r} = 1 \). Now the universal property of \( U[\mathbb{Z}] \) implies that there exists a homomorphism \( \sigma : U[\mathbb{Z}] \to C^*(\mathcal{G}_{\text{tight}}) \) such that \( \sigma(s_m) = t_m \) and \( \sigma(u^n) = v^n \). Now it is clear that \( \rho \) and \( \sigma \) are inverses of each other. This completes the proof.

In the next two sections, we identify the groupoid \( \mathcal{G}_{\text{tight}} \) explicitly.

4. **Tight characters of the inverse semigroup \( T \)**

In this section, we determine the tight characters of the inverse semigroup \( T \) defined in Proposition 2.11. Let us recall a few ring theoretical notions. We denote the set of strictly positive integers by \( \mathbb{N}^+ \). Consider the directed set \( (\mathbb{N}^+, \leq) \) where we say \( m \leq n \) if \( m | n \). If \( m | n \) then there exists a natural map from \( \mathbb{Z}/(n) \) to \( \mathbb{Z}/(m) \). The inverse limit of this system is called the profinite completion of \( \mathbb{Z} \) and is denoted \( \hat{\mathbb{Z}} \). In other words,

\[
\hat{\mathbb{Z}} := \{(r_m) \in \prod_{m \in \mathbb{N}^+} \mathbb{Z}/(m) : r_{mk} \equiv r_m \mod m \}
\]
Also $\hat{Z}$ is a compact ring with the subspace topology induced by the product topology on $\prod Z/(m)$. Also $Z$ embeds naturally in $\hat{Z}$. We also need the easily verifiable fact that the kernel of the $m^{th}$ projection $r = (r_m) \to r_m$ is in fact $m\hat{Z}$.

For $r \in \hat{Z}$, define a character $\xi_r : P \to \{0, 1\}$ by the following formula:

$$\xi_r(u^n e_m u^{-n}) = \delta_{r_m, n}$$
$$\xi_r(0) = 0$$

In the above formula, the Dirac-delta function is over the set $Z/(m)$. Thus $\delta_{r_m, n} = 1$ if and only if $r_m \equiv n \mod m$.

**Proposition 4.1.** The map $r \to \xi_r$ is a topological isomorphism from $\hat{Z}$ to $\hat{P}_{\text{tight}}$

**Proof.** First let us check that for $r \in \hat{Z}$, $\xi_r$ is in fact a character and is maximal. Consider an element $r \in \hat{Z}$. Let $e := u^{n_1} e_m u^{-n_1}$ and $f := u^{n_2} e_m u^{-n_2}$ be given. Let $d := (m_1, m_2)$ and $c := [m_1, m_2]$. Suppose $\xi_r(e) = \xi_r(f) = 1$. Then $r_{m_1} \equiv n_1 \mod m_1$ and $r_{m_2} \equiv n_2 \mod m_2$. Moreover, $r_c \equiv r_m \mod m_i$ for $i = 1, 2$. Thus $ef = u^{e_c} u^{-e_c}$ by Lemma 2.6. Hence by definition $\xi_r(ef) = 1$. Now suppose $\xi_r(e) = 1$ and $e \leq f$. Then by Lemma 2.5 and Lemma 2.6, it follows that $m_2$ divides $m_1$ and $r_{m_2} \equiv r_{m_1} \equiv n_1 \equiv n_2 \mod m_2$. Hence $\xi_r(f) = 1$. By definition 0 is not in the support of $\xi_r$. Thus we have shown that the support of $\xi_r$ is a filter or in other words $\xi_r$ is a character.

Now we claim $\xi_r$ is maximal. This follows from the observation that for every $m \in \mathbb{N}^+$, the set of projections $\{u^n e_m u^{-n} : n \in Z/(m)\}$ are mutually orthogonal. Thus if $\xi$ is a character then for every $m$ there exists at most one $r_m$ for which $\xi(u^{r_m} e_m u^{-r_m}) = 1$. This implies that if $\xi$ is a character which contains the support of $\xi_r$, then $\xi = \xi_r$.

Now let $\xi$ be a maximal character of $P$. Then by Corollary 3.3 and by the observation in the previous paragraph, it follows that for every $m$ there exists a unique $r_m$ such that $\xi(u^{r_m} e_m u^{-r_m}) = 1$. Now let $k$ be given. Since both $u^{r_m} e_m u^{-r_m}$ and $u^{r_m} e_{mk} u^{-r_m}$ belong to the support of $\xi$, it follows that the product $u^{r_m} e_m u^{-r_m} u^{r_m} e_{mk} u^{-r_m}$ does not vanish. Then by Lemma 2.5, it follows that $r_{mk} \equiv r_m \mod m$. Thus $r = (r_m) \in \hat{Z}$ and the support of $\xi_r$ is contained in the support of $\xi$. Thus again by the observation in the preceeding paragraph, it follows that $\xi = \xi_r$.

It is clear from the definition that the map $r \to \xi_r$ is one-one and continuous. As $\hat{Z}$ is compact, it follows that the range of the map $r \to \xi_r$ which is $\hat{P}_\infty$ is also compact. Hence $\hat{P}_\infty = \hat{P}_{\text{tight}}$. Thus we have shown that $r \to \xi_r$ is a one-one and onto continuous map from $\hat{Z}$ to $\hat{P}_{\text{tight}}$. Since $\hat{Z}$ is compact, it follows that the above map is in fact a homeomorphism. This completes the proof.

From now on we will simply write $r(e)$ in place of $\xi_r(e)$ if $r \in \hat{Z}$ and $e \in P$. 

\[\square\]
5. The Groupoid $G_{tight}$ of the Inverse Semigroup $T$

Let us recall a few ring theoretical constructions. Consider the directed set $(\mathbb{N}^+, \leq)$ where the partial order $\leq$ is defined by $m \leq n$ if $m$ divides $n$. For $m \in \mathbb{N}^+$, let $\mathcal{R}_m := \hat{\mathbb{Z}}$. Let $\phi_{mt,m} : \mathcal{R}_m \to \mathcal{R}_{tm}$ be the map defined by multiplication by $t$. Then $\phi_{mt,m}$ is only an additive homomorphism and it does not preserve the multiplication. We let $\mathcal{R}$ be the inductive limit of $(\mathcal{R}_m, \phi_{mt,m})$. Then $\mathcal{R}$ is an abelian group and $\hat{\mathbb{Z}}$ is a subgroup of $\mathcal{R}$ via the inclusion $\mathcal{R}_1 \subset \mathcal{R}$.

Note that $\mathcal{R}$ is a locally compact Hausdorff space. Moreover the group $\mathcal{P} := \left\{ \begin{bmatrix} 1 & 0 \\ b & a \end{bmatrix} : a \in Q^*, b \in Q \right\}$ acts on $\mathcal{R}$ by affine transformations. The action is described explicitly by the following formula. For $x \in \mathcal{R}$,

$$\begin{bmatrix} 1 & 0 \\ \frac{n}{m} & \frac{m}{m'} \end{bmatrix} x = mx + np \in \mathcal{R}_{m'p}$$

One can check that the above formula defines an action of $\mathcal{P}$ on $\mathcal{R}$. We need the following lemma.

**Lemma 5.1.** Let $a := \frac{n}{m}$ and $b := \frac{m}{m'}$. Then $s^*, u^n s_m$ depends only on $a$ and $b$.

**Proof.** Suppose $\frac{n_1}{m_1} = \frac{n_2}{m_2}$ and $\frac{m_1}{m_2} = \frac{m_2}{m_2}$. Then $n_1 m_2 = n_2 m_1'$ and $m_1 m_1' = m_1' m_2$. Now, we have

$$s^*, u^n s_m = s^*, s_{m_2} s_{m_2} u^n s_m = s^*, s_{m_2} s_{m_1} s_{m_1'} s_{m_1} = s^*, s_{m_2} s_{m_1} s_{m_1'} u^{n_1 m_2} s_{m_1} s_m = s^*, s_{m_2} s_{m_1} u^{n_1 m_2} s_m = s^*, u^{n_1 m_2} s_{m_1} s_{m_1} = s^*, u^{n_1 m_2} s_{m_1} s_{m_1} = s^*, u^{n_1 m_2} s_m = s^*, u^{n_1 m_2} s_m$$

This completes the proof. \qed

**Remark 5.2.** The above lemma has also been used in [BE10].

Now we explicitly identify the groupoid $G_{tight}$ associated to the inverse semigroup $T$. When we consider transformation groupoids, we consider only right actions. Thus we let $P_Q$ act on $\mathcal{R}$ on the right by defining $x.g = g^{-1}x$ for $x \in \mathcal{R}$ and $g \in P_Q$. We show that that groupoid $G_{tight}$ of the inverse semigroup $T$ is isomorphic to the restriction of the transformation groupoid...
\( \mathcal{R} \times P_Q \) to the closed subset \( \hat{\mathbb{Z}} \). (Here we consider \( P_Q \) as a discrete group.) Let us begin with a lemma which will be useful in the proof.

**Lemma 5.3.** In \( \mathcal{G}_{\text{tight}} \) one has \([ (r, s_m^* s_m^e_k u^n s_m) ] = [(r, s_m^* u^{n+s} s_m)] \)

*Proof.* First observe that \([(r, s_m^* s_m^e_k u^n s_m)] = [(r, s_m^* u^n e_k u^n s_m)]\). Thus it is enough to consider the case \( m' = 1 \). Now let \( s := u^n e_k u^n s_m \), \( t := u'^n e_k u'^n s_m \) and \( e := u^n e_k u^n - e' \). Now observe that \( ss^* := ett^* \). Hence if \( r(s) = 1 \) then \( r(tt^*) = 1 \) and \( r(e) = 1 \). Moreover \( es = et \). Thus \([ (r, s) ] = [(r, t)] \). This completes the proof. \( \square \).

**Theorem 5.4.** Let \( \phi : \mathcal{R} \times P_Q|_\mathbb{Z} \rightarrow \mathcal{G}_{\text{tight}} \) be the map defined by

\[
\phi \left( \left( r, \begin{bmatrix} 1 & 0 \\ \frac{k}{m} & \frac{n}{m} \end{bmatrix} \right) \right) = [(r, s_m^* u^k s_n)]
\]

Then \( \phi \) is a topological groupoid isomorphism.

*Proof.*

The map \( \phi \) is well defined.

Let \( \left( r, \begin{bmatrix} 1 & 0 \\ \frac{k}{m} & \frac{n}{m} \end{bmatrix} \right) \) be an element in \( \mathcal{R} \times P_Q|_\mathbb{Z} \). Then we have \( mr - k = ns \) for some \( s \in \mathbb{Z} \). Now we need to show that \( r(s_m^* u^k e_n u^{-k} s_m) = 1 \). By Lemma 2.9 it follows that \( s_m^* u^k e_n u^{-k} s_m = u^r e_{n_1} u^{-r} \) where \( n_1 := n \pmod{(m, n)} \). Thus

\[
r(s_m^* u^k e_n u^{-k} s_m) = r(u^r e_{n_1} u^{-r})
\]

\[
= \delta_{r n_1} ^{r n_1}
\]

\[
= 1 \quad \text{[Since } r_n = r n_1 \text{ in } \mathbb{Z}/(n_1)]\]

**Surjectivity of \( \phi \):**

First let us show that if \( [(r, s_m^* u^k s_n)] \in \mathcal{G}_{\text{tight}} \) then \( \left( r, \begin{bmatrix} 1 & 0 \\ \frac{k}{m} & \frac{n}{m} \end{bmatrix} \right) \in \mathcal{R} \times P_Q|_\mathbb{Z} \). Consider an element \( [(r, v := s_m^* u^k s_n)] \) in \( \mathcal{G}_{\text{tight}} \). Then \( rvv^* = 1 \) and \( vv^* := s_m^* u^k e_n u^{-k} s_m \). Now Lemma 2.8 and 2.9 implies that \( (m, n) \mid k \). Let \( s \) be an integer such that \( ms \equiv k \pmod n \). Again Lemma 2.9 implies that \( vv^* = u^s e_{n_1} u^{-s} \) where \( n_1 := \frac{n}{(m, n)} \). Now \( r(vv^*) = 1 \) implies that \( r_n \equiv s \pmod{n_1} \). But \( r_n \equiv r_n \pmod{n_1} \). This in turn implies that \( mr_n \equiv ms \equiv k \pmod{n} \). Hence \( \left( r, \begin{bmatrix} 1 & 0 \\ \frac{k}{m} & \frac{n}{m} \end{bmatrix} \right) \in \mathcal{R} \times P_Q|_\mathbb{Z} \).

Now the surjectivity of \( \phi \) follows from Lemma 5.3.

**Injectivity of \( \phi \):**

Now suppose \( [(r, s_m^* u^k s_n)] = [(r, s_m^* u^k s_n)] \). Then by definition there exists a projection of the form \( e := u^r e_p u^{-r} \) such that \( e(s_m^* u^k s_n) = e(s_m^* u^k s_n) \neq 0 \). Consider a character \( \chi \)
of the discrete group $\mathbb{Q}^*$. Let $\alpha_\chi$ be the automorphism of the algebra $U[\mathbb{Z}]$ such that $\alpha_\chi(u^n) = u^n$ and $\alpha_\chi(s_m) = \chi(m)s_m$.

$$\chi\left(\frac{n_1}{m_1}\right)e(s_{m_1}^*u^{k_1}s_{n_1}) = \alpha_\chi\left(e(s_{m_1}^*u^{k_1}s_{n_1})\right)$$

$$= \alpha_\chi\left(e(s_{m_2}^*u^{k_2}s_{n_2})\right)$$

$$= \chi\left(\frac{n_2}{m_2}\right)e(s_{m_2}^*u^{k_2}s_{n_2})$$

$$= \chi\left(\frac{n_2}{m_2}\right)e(s_{m_1}^*u^{k_1}s_{n_1})$$

Since $e(s_{m_1}^*u^{k_1}s_{n_1}) \neq 0$, it follows that $\chi\left(\frac{n_1}{m_1}\right) = \chi\left(\frac{n_2}{m_2}\right)$ for every character $\chi$ of the discrete, multiplicative group $\mathbb{Q}^*$. Thus $\frac{n_1}{m_1} = \frac{n_2}{m_2}$.

From remark 2.12 it follows that $e(s_{m_1}^*u^{k_1}s_{n_1}) = e(s_{m_2}^*u^{k_2}s_{n_2}) \neq 0$ in $U_r[\mathbb{Z}]$. Since $\frac{n_1}{m_1} = \frac{n_2}{m_2}$, it follows immediately that $\frac{k_1}{m_1} = \frac{k_2}{m_2}$. Thus we have shown that $\phi$ is injective.

The map $\phi$ is a groupoid morphism.

First we show that $\phi$ is continuous. Let $(r_n, g_n)$ be a sequence in $\mathcal{R} \times P_\mathbb{Q}[\hat{\mathbb{Z}}]$ converging to $(r, g)$. Since we are considering $P_\mathbb{Q}$ as a discrete group, we can without loss of generality assume that $g_n = g$ for every $n$. Then, from Lemma 4.1 it follows that $\phi(r_n, g_n)$ converges to $\phi(r, g)$.

For an open subset $U$ of $\hat{\mathbb{Z}}$ and $g := \begin{bmatrix} 1 & 0 \\ k & m \end{bmatrix}$, consider the open set

$$\theta(U, g) := \{(r, g) : r \in U \text{ and } r.g \in \hat{\mathbb{Z}}\}.$$

Then the collection $\{\theta(U, g) : U \subset \hat{\mathbb{Z}}, g \in P_\mathbb{Q}\}$ forms a basis for $\mathcal{R} \times P_\mathbb{Q}[\hat{\mathbb{Z}}]$. Moreover $\phi(\theta(U, g)) = \theta(U, s_m^*u^n s_n)$. Hence $\phi$ is an open map. Thus we have shown that $\phi$ is a homeomorphism.

$\phi$ is a groupoid morphism.

First we show that $\phi$ preserves the source and range. By definition $\phi$ preserves the range. Let $r, g := \begin{bmatrix} 1 & 0 \\ k & m \end{bmatrix} \in \mathcal{R} \times P_\mathbb{Q}[\hat{\mathbb{Z}}]$ be given. Let $v := s_m^*u^n s_n$. Since $r.g \in \hat{\mathbb{Z}}$, it follows that there exists $t \in \hat{\mathbb{Z}}$ such that $mr - k = nt$. We need to show that $\xi_r, v = \xi_t$. (Just to keep things clear we write $\xi_r$ for the character determined by $r$). It is enough to show that the support of $\xi_t$ and that of $\xi_r, v$ coincide. But then both the characters are maximal and thus it is enough to show that the support of $\xi_t$ is contained in the support of $\xi_r, v$. Thus, suppose that $\xi_t(u^n e_s u^{-t}) = 1$. Then $t_{ns} \equiv t \equiv \ell \mod s$. This implies $mr_{ns} - k \equiv nt_{ns} \equiv n\ell \mod ns$. 
Thus $mr_{ns} \equiv k + n\ell \mod ns$. Let $n_1 := \frac{ns}{(ns, m)}$. Now observe that

\[(\xi_r \cdot v)(u^\ell e_s u^{-\ell}) = \xi_r (vu^\ell e_s u^{-\ell} v^*)
= \xi_r (s^*_m u^k s_n u^\ell s^*_n u^{-k} s_m)
= \xi_r (s^*_m u^{k + n\ell} e_{ns} u^{-(k + n\ell)} s_m)
= \xi_r (u^{ns} e_{n_1} u^{-ns}) \text{ [By Lemma 2.9]}
= \delta_{r_{ns}, r_{n_1}}
= 1 \text{ [Since } r_{ns} = r_{n_1} \text{ in } \mathbb{Z}/(n_1)]
\]

Thus we have shown that the support of $\xi_t$ is contained in the support of $\xi_r \cdot v$ which in turn implies that $\xi_t = \xi_r \cdot v$. Hence $\phi$ preserves the source.

Now we show that $\phi$ preserves multiplication. Let $\gamma_i := (r_i, \begin{bmatrix} 1 & 0 \\ k_i & n_i \end{bmatrix})$ for $i = 1, 2$. Since $\phi$ preserves the range and source, it follows that if $\gamma_1$ and $\gamma_2$ are composable, so do $\phi(\gamma_1)$ and $\phi(\gamma_2)$. Observe that

\[
\phi(\gamma_1) \phi(\gamma_2) = [(r_1, s^*_m u^{k_1} s_n u^{k_2} s_{n_2})]
= [r_1, s^*_m m_{m_2} u^{m_2 k_1} e_{m_2 n_1} u^{m_2 k_1} s_{n_1} s_{n_2}] \text{ (Eq. 2.1)}
= [r_1, s^*_m m_{m_2} u^{m_2 k_1 + n_1 k_2} s_{n_1} s_{n_2}] \text{ (Lemma 5.3)}
= \phi(\gamma_1 \gamma_2)
\]

It is easily verifiable that $\phi$ preserves inversion. This completes the proof. \qed

**Remark 5.5.** Combining Proposition 3.9 and Theorem 8.3, we obtain that $U[\mathbb{Z}]$ is isomorphic to $C^*(\mathcal{R} \times P_Q|\mathbb{Z})$ which is Remark 2 in page 17 of [CL10].

### 6. Simplicity of $U[\mathbb{Z}]$

First we recall a few definitions from [Ren09]. Let $\mathcal{G}$ be an r-discrete, Hausdorff and locally compact topological groupoid. Let $\mathcal{G}^0$ be its unit space. We denote the source and range maps by $s$ and $r$ respectively. The arrows of $\mathcal{G}$ define an equivalence relation on $\mathcal{G}^0$ as follows:

$x \sim y$ if there exists $\gamma \in \mathcal{G}$ such that $s(\gamma) = x$ and $r(\gamma) = y$

A subset $E$ of $\mathcal{G}^0$ is said to be invariant if the orbit of $x$ is contained in $E$ whenever $x \in E$. For $x \in \mathcal{G}^0$, define the isotropy group at $x$ denoted $\mathcal{G}(x)$ by $\mathcal{G}(x) := \{\gamma \in \mathcal{G} : s(\gamma) = r(\gamma) = x\}$.

A groupoid $\mathcal{G}$ is said to be

- topologically principal if the set of $x \in \mathcal{G}^0$ for which $\mathcal{G}(x) = \{x\}$ is dense in $\mathcal{G}^0$.
- minimal if the only non-empty open invariant subset of $\mathcal{G}^0$ is $\mathcal{G}^0$.

We need the following theorem. We refer to [Ren09] for a proof.
Theorem 6.1. Let $G$ be an $r$-discrete, Hausdorff and locally compact topological groupoid. If $G$ is topologically principal and minimal then $C^*_{red}(G)$ is simple.

Proposition 6.2. The $C^*$-algebra $U[\mathbb{Z}]$ is simple.

Proof. Let $G$ denote the groupoid $R \times P \big| \hat{\mathbb{Z}}$. Since the group $P \big| \hat{\mathbb{Z}}$ is solvable, it is amenable and thus by Proposition 2.15 of [MR82], it follows that the full groupoid $C^*(G)$ is isomorphic to the reduced algebra $C^*_{red}(G)$. Now we apply Theorem 6.1 to complete the proof.

First let us show $G$ is minimal. Let $U$ be a non-empty open invariant subset of $G^0$. For $m = (m_1, m_2, \ldots, m_n) \in (\mathbb{Z}\setminus\{0\})^n$ and $k \in \mathbb{Z}$, let

$$U_{m,k} := \{r \in \hat{\mathbb{Z}} : r_{m_i} \equiv k \mod m_i\}$$

Then the collection $\{U_{m,k}\}$ (where $m$ varies over $(\mathbb{Z}\setminus\{0\})^n$ (we let $n$ vary too) and $k \in \mathbb{Z}$) is a basis for the topology on $\hat{\mathbb{Z}}$. Also observe that for a given $m$, $\bigcup_{k \in \mathbb{Z}} U_{m,k} = \hat{\mathbb{Z}}$. Moreover the translation matrix $egin{bmatrix} 1 & 0 \\ k_1 - k_2 & 1 \end{bmatrix}$ maps $U_{m,k_1}$ onto $U_{m,k_2}$. Now since $U$ is non-empty and open, there exists an $m$ and a $k_0$ such that $U_{m,k_0} \subset U$. But since $U$ is invariant, it follows that $U_{m,k} \subset U$ for every $k \in \mathbb{Z}$. Thus $\bigcup_{k \in \mathbb{Z}} U_{m,k} \subset U$. This forces $U = \hat{\mathbb{Z}}$. This completes the proof.

Now we show $G$ is topologically principal. Let

$$E := \{r \in \hat{\mathbb{Z}} : r \neq 0, r_p = 0 \forall i, \text{ except for finitely many primes } p\}$$

If one identifies $\hat{\mathbb{Z}}$ with $\prod_{p \text{ prime}} \hat{\mathbb{Z}}_p$ then it is clear that $E$ is dense in $\hat{\mathbb{Z}}$. Now let $r \in E$ be given.

We claim that $G(r) = \{r\}$. Suppose $r. \begin{bmatrix} 1 & 0 \\ k & m \end{bmatrix} = r$. Then $mr - k = nr$. But $r_p = 0$ except for finitely many primes. Thus it follows that $k$ is divisible by infinitely many primes which forces $k = 0$. Now $mr = nr$ and $r \neq 0$ implies $m = n$. Thus $G(r) = \{r\}$. This proves that $G$ is topologically principal. This completes the proof.

7. Nica-covariance, tightness and boundary relations

In this section, we digress a bit to understand some of the results in [Nic92], [CL07] and in [LR10] from the point of view of inverse semigroups. Let us recall the notion of quasi-lattice ordered groups considered by Nica in [Nic92]. Let $G$ be a discrete group and $P$ a subsemigroup of $G$ containing the identity $e$. Also assume that $P \cap P^{-1} = \{e\}$. Then $P$ induces a left-invariant partial order $\leq$ on $G$ defined by $x \leq y$ if and only if $x^{-1}y \in P$. The pair $(G, P)$ is said to be quasi-lattice ordered if the following conditions are satisfied.

(1) Any $x \in PP^{-1}$ has a least upper bound in $P$, and

(2) If $s, t \in P$ have a common upper bound in $P$ then $s, t$ have a least upper bound.
If $s, t \in P$ have a common upper bound in $P$ then we denote the least upper bound in $P$ by $\sigma(s, t)$. It is easy to show that $s, t \in P$ have a common upper bound if and only if $s^{-1}t \in PP^{-1}$. Let us recall the Wiener-Hopf representation from \cite{Nic92}. Consider the representation $W : P \to B(\ell^2(P))$ defined by

$$W(p)(\delta_a) := \delta_{pa}$$

where $\{\delta_a : a \in P\}$ denotes the canonical orthonormal basis of $\ell^2(P)$. Note that for $s \in P$, $W(s)$ is an isometry and $W(s)W(t) = W(st)$ for $s, t \in P$. For $s \in P$, let $M(s) = W(s)W(s)^*$ then

$$(7.2) \quad M(s)M(t) = \begin{cases} M(\sigma(s, t)) & \text{if } s \text{ and } t \text{ have a common upper bound in } P \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathcal{N} := \{W(s)W(t)^* : s, t \in P\} \cup \{0\}$. Then Equation (5) of Proposition 3.2 in \cite{Nic92} implies that $\mathcal{N}$ is an inverse semigroup of partial isometries. The following definition is due to Nica.

**Definition 7.1** \cite{Nic92}. Let $(G, P)$ be a quasi-lattice ordered group. An isometric representation $V : P \to B(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ (i.e. $V(t)^*V(t) = 1$ for $t \in P$, $V(e) = 1$ and $V(s)V(t) = V(st)$ for every $s, t \in P$) is said to be Nica-covariant if the following holds

$$(7.3) \quad L(s)L(t) = \begin{cases} L(\sigma(s, t)) & \text{if } s \text{ and } t \text{ have common upper bound in } P \\ 0 & \text{otherwise.} \end{cases}$$

where we set $L(t) = V(t)V(t)^*$. In other words a Nica-covariant representation of $(G, P)$ is nothing but a unital representation of the inverse semigroup $\mathcal{N}$ which sends $0$ to $0$.

Let us say a Nica-covariant representation is tight if the corresponding representation on $\mathcal{N}$ is tight. Now one might ask what are the tight representations of the inverse semigroup $\mathcal{N}$? We prove that tight representations are nothing but Nica-covariant representations satisfying the boundary relations considered by Laca and Crisp in \cite{CL07}. This fact is implicit in \cite{CL07} and it is in fact explicit if one applies Theorem 13.2 of \cite{Exe09}. The author believes that it is worth recording this connection and we do this in the next proposition.

First let us fix a few notations. A finite subset $F$ of $P$ is said to cover $P$ if given $x \in P$ there exists $y \in F$ such that $x$ and $y$ have a common upper bound in $P$. Let

$$\mathcal{F} := \{F \subset P : F \text{ is finite and covers } P\}$$

**Proposition 7.2.** Let $(G, P)$ be a quasi-lattice ordered group. Consider a Nica-covariant representation $V : P \to B(\mathcal{H})$. Then $V$ is tight if and only if for every $F \in \mathcal{F}$, one has $\prod_{t \in F}(1 - V(t)V(t)^*) = 0$.

**Proof.** Consider a Nica-covariant representation $V : P \to B(\mathcal{H})$. Suppose that $V$ is tight. Let $F \in \mathcal{F}$ be given. Note that $F$ covers $P$ if and only if $\{M(t) : t \in F\}$ covers the set of projections in $\mathcal{N}$. Now the tightness of $V$ implies that $\sup_{t \in F}V(t)V(t)^* = 1$. This is equivalent to saying that $\prod_{x \in F}(1 - V(x)V(x)^*) = 0$. Thus we have the implication '$\Rightarrow$'.

$$\text{Proof.}$$

Consider a Nica-covariant representation $V : P \to B(\mathcal{H})$. Suppose that $V$ is tight. Let $F \in \mathcal{F}$ be given. Note that $F$ covers $P$ if and only if $\{M(t) : t \in F\}$ covers the set of projections in $\mathcal{N}$. Now the tightness of $V$ implies that $\sup_{t \in F}V(t)V(t)^* = 1$. This is equivalent to saying that $\prod_{x \in F}(1 - V(x)V(x)^*) = 0$. Thus we have the implication '$\Rightarrow$'.
Let $V$ be a Nica-covariant representation for which $\prod_{t \in F}(1 - V(t)V(t)^*) = 0$ for every $F \in \mathcal{F}$. We denote the set of projections in $\mathcal{N}$ by $E$. Then $E := \{M(t) : t \in P\} \cup \{0\}$. Let $\{M(t_1), M(t_2), \cdots, M(t_n)\} \subset [0, M(t)]$ be a finite cover. Then $M(t_i) \leq M(t)$ for every $i$. But this is equivalent to the fact that $t \leq t_i$.

We claim that $\{t^{-1}t_i : i = 1, 2, \cdots, n\}$ covers $P$. Let $s \in P$ be given. Then $t \leq ts$ which implies $M(ts) \leq M(t)$. Thus there exists a $t_i$ such that $M(ts)M(t_i) \neq 0$. This implies that $ts$ and $t_i$ have a common upper bound in $P$. In other words, $(ts)^{-1}t_i = s^{-1}t^{-1}t_i \in PP^{-1}$. Thus $s$ and $t^{-1}t_i$ have a common upper bound in $P$. This proves the claim.

By assumption it follows that $\prod_{i=1}^n(1 - L(t^{-1}t_i)) = 0$ where $L(s) := V(s)V(s)^*$. Now multiplying this equality on the left by $V(t)$ and on the right by $V(t)^*$, we get

$$\prod_{i=1}^n(V(t)V(t)^* - V(t)V(t^{-1}t_i)V(t^{-1}t_i)^*V(t)^*) = 0$$

$$\prod_{i=1}^n(V(t)V(t)^* - V(t)V(t_i)^*) = 0$$

But this is equivalent to $\sup_{i} L(t_i) = L(t)$. This completes the proof. \hfill \Box

**Remark 7.3.** The relations $\prod_{x \in F}(1 - V(t)V(t)^*) = 0$ for $F \in \mathcal{F}$ are the boundary relations considered in [CL07].

Let $Q_\mathbb{N}$ be the $C^*$-subalgebra of $U[\mathbb{Z}]$ generated by $u$ and $\{s_m : m > 0\}$. In [Cun08], it was proved that $Q_\mathbb{N}$ is simple and purely infinite. Moreover in [Cun08], it was shown that $U[\mathbb{Z}]$ is isomorphic to a crossed product of $Q_\mathbb{N}$ with $\mathbb{Z}/2\mathbb{Z}$. Let

$$P_\mathbb{N} := \left\{ \begin{bmatrix} 1 & 0 \\ k & m \end{bmatrix} : k \in \mathbb{N} \text{ and } m \in \mathbb{N}^* \right\}$$

Note that $P_\mathbb{N}$ is a semigroup of $P_Q$.

**Remark 7.4.** In [LR10], it was proved that $(P_Q, P_\mathbb{N})$ is a quasi-lattice ordered group. Moreover it was shown in [LR10] that for the quasi-lattice ordered group $(P_Q, P_\mathbb{N})$ Nica-covariance together with boundary relations is equivalent to Cuntz-Li relations and the universal $C^*$-algebra made out of Nica-covariant representations satisfying the boundary relations is in fact $Q_\mathbb{N}$.

### 8. The Cuntz-Li algebra for a general integral domain

We end this article by giving a few remarks of how to adapt the analysis in Section 1 – 6 for a general integral domain $R$. Now let $R$ be an integral domain such that $R/mR$ is finite for every non-zero $m \in R$. We also assume that $R$ is countable and $R$ is not a field.
Definition 8.1 ([CL10]). Let $U[R]$ be the universal $C^*$-algebra generated by a set of unitaries \( \{u^n : n \in R \} \) and a set of isometries \( \{s_m : m \in R^\times \} \) satisfying the following relations.

\[
\begin{align*}
    s_m s_n &= s_{mn} \\
    u^n u^m &= u^{n+m} \\
    s_m u^n &= u^{mn} s_m \\
    \sum_{n \in R/mR} u^n e_m u^{-n} &= 1
\end{align*}
\]

where \( e_m \) denotes the final projection of \( s_m \).

Now the problem is the product \( u^e_m u^{-r} u^e_n u^{-s} \) may not be of the form \( u^k e_c u^{-k} \) for some \( k \) and \( c \). Nevertheless it will be in the linear span of \( \{u^k e_m u^{-k} : k \in R/(mn)\} \). Let \( P \) denote the set of projections in \( U[R] \) which is in the linear span of \( \{u^r e_m u^{-r} : r \in R/(m)\} \) for some \( m \).

Explicitly, a projection \( e \in U[R] \) is in \( P \) if and only if there exists an \( m \in R^\times \) and \( a_r \in \{0, 1\} \) such that \( e = \sum a_r u^r e_m u^{-r} \).

Now it is easy to show that \( P \) is a commutative semigroup of projections containing 0. Moreover \( P \) is invariant under conjugation by \( u^r \), \( s_m \) and \( s_m^* \). One can prove the following Proposition just as in the case when \( R = \mathbb{Z} \).

Proposition 8.2. Let \( T := \{s_m u^n e u^{n'} s_m^* : e \in P, m, m' \neq 0, n, n' \in R\} \). Then \( T \) is an inverse semigroup of partial isometries. Moreover the set of projections in \( T \) coincide with \( P \). Also the linear span of \( T \) is dense in \( U[R] \).

Let \( \hat{R} := \{(r_m) \in \prod R/(m) : r_{mk} = r_m \text{ in } R/(m)\} \) be the profinite completion of the ring \( R \). For \( r \in \hat{R} \), define

\[
A_r := \{f \in P : f \geq u^m e_m u^{-r_m} \text{ for some } m\}
\]

Then \( A_r \) is an ultrafilter for every \( r \in \hat{R} \) and the map \( r \to A_r \) is a topological isomorphism from \( \hat{R} \) to \( \hat{P}_{\text{tight}} \).

Let \( Q(R) \) be the field of fractions of \( R \). For \( m \neq 0 \), let \( R_m := \hat{R} \). For every \( \ell \neq 0 \), let \( \phi_{m\ell, m} : R_m \to R_{\ell m} \) be the map defined by multiplication by \( \ell \). Then \( \phi_{m\ell, m} \) is only an additive homomorphism and it does not preserve the multiplication. We let \( \mathcal{R} \) be the inductive limit of \( (R_m, \phi_{m\ell, m}) \). Then \( \mathcal{R} \) is an abelian group and \( \hat{R} \) is a subgroup of \( \mathcal{R} \) via the inclusion \( R_1 \subset \mathcal{R} \).

Note that \( \mathcal{R} \) is a locally compact Hausdorff space. Moreover the group

\[
P_{Q(R)} := \left\{ \begin{bmatrix} 1 & 0 \\ b & a \end{bmatrix} : a \in Q(R)^\times, b \in Q(R) \right\}
\]

acts on \( \mathcal{R} \) by affine transformations. The action is described explicitly by the following formula. For \( x \in \mathcal{R}_p \)

\[
\begin{bmatrix} 1 & 0 \\ n/m & m/m \end{bmatrix} x = mx + np \in \mathcal{R}_{m'p}
\]
One can check that the above formula defines an action of $P_{Q(R)}$ on $\mathcal{G}_{\text{tight}}$. Let $\mathcal{G}_{\text{tight}}$ be the tight groupoid associated to the inverse semigroup $T$ defined in Proposition 8.2. Then as in the case when $R = \mathbb{Z}$, we have the following theorem.

**Theorem 8.3.** Let $\phi : \mathcal{R} \times P_{Q(R)}|_{\hat{R}} \to \mathcal{G}_{\text{tight}}$ be the map defined by

$$\phi((r, \begin{bmatrix} 1 & 0 \\ k & n \\ m \\ n \end{bmatrix})) = [(r, s_m^* u^k s_n)]$$

Then $\phi$ is a topological groupoid isomorphism. Moreover the $C^*$-algebra $U[\mathcal{R}]$ is isomorphic to the full (and the reduced) $C^*$-algebra of the groupoid $\mathcal{R} \times P_{Q(R)}|_{\hat{R}}$.

We end this article by showing that $U[\mathcal{R}]$ is simple.

**Proposition 8.4 ([CL10]).** The $C^*$-algebra $U[\mathcal{R}]$ is simple.

**Proof.** Let us denote the groupoid $\mathcal{R} \times P_{Q(R)}|_{\hat{R}}$ by $\mathcal{G}$. As in Proposition 6.1, we need to show that $\mathcal{G}$ is minimal and topologically principal. The proof of the minimality of $\mathcal{G}$ is exactly similar to that in Proposition 6.1. We now show that $\mathcal{G}$ is topologically principal. For $g \in P_{Q(R)} \setminus \{1\}$, let us denote the set of fixed points of $g$ in $\hat{R}$ by $F_g$. It follows from Baire category theorem that $\mathcal{G}$ is topologically principal if and only if $F_g$ has empty interior for every $g \neq 1$.

Let $g = \begin{bmatrix} 1 & 0 \\ k & n \\ m & n \end{bmatrix}$ be a non-identity element in $P_{Q(R)}$. Suppose that $F_g$ contains a non-empty open set say $U$. Now note that $R$ is dense in $\hat{R}$. Thus $U \cap R$ is non-empty. Moreover $U \cap R$ is infinite. Let $r_1, r_2$ be two distinct points of $R$ in $U$. Since $r_1, r_2 \in F_g$, it follows that $mr_1 - k = nr_1$ and $mr_2 - k = nr_2$. Thus we have $(m - n)r_1 = k = (m - n)r_2$. This forces $m = n$ and $k = 0$. This is a contradiction to the fact that $g \neq 1$. Thus for every $g \neq 1$, $F_g$ has empty interior which in turn implies that $\mathcal{G}$ is topologically principal. This completes the proof.

**Remark 8.5.** In [KLQ10], Cuntz-Li type relations arising out of a semidirect product $N \rtimes H$ where $N$ is a normal subgroup and $H$ is an abelian group satisfying certain hypothesis were considered. It was shown in [KLQ10] that the universal $C^*$-algebra generated by the Cuntz-Li type relations is isomorphic to a corner of a crossed product algebra. It is possible to apply inverse semigroups and tight representations to reconstruct this result. The details will be spelt out elsewhere.

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