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September 12, 2011
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Bounds for reliability of IFRA coherent systems using signatures

Jayant V. Deshpande
Isha Dewan

Indian Statistical Institute, Delhi Centre
7, SJSS Marg, New Delhi–110 016, India

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Jayant V. Deshpande [1] Isha Dewan [2]
[1]Indian Institute of Technology, Bombay
[2] Indian Statistical Institute, New Delhi

January 9, 2012

Abstract

The reliability of any coherent system is the probability that it completes a mission of time t without failure. To calculate it one needs to know the structure function of the system and the distribution of the component life times which we assume to be independent and identically distributed continuous positive valued random variables. In this paper we bring together (i) the single crossing property of IFRA distributions , (ii) the concept of a signature and (iii) the fact that the life distribution of a coherent system of IFRA components is itself IFRA (the celebrated IFRA closure theorem) to obtain useful functional bounds for the reliability of coherent systems of IFRA components. These bounds are in terms of the quantile of a specified order or the mean of the common component life distribution and the signature of the coherent system. This obviates the need of the knowledge of the entire survival function and the knowledge of the coherent system beyond its signature. We tabulate these reliability bounds for several 3, 4 and 5 component systems and show that they are very close to the true reliability in certain instances.

Keywords : *Coherent system, IFRA, signature, reliability bounds.*

AMS Classification 2010: 62 N05, 62 P30

1 Introduction

The exact reliability of coherent systems is notoriously difficult to calculate. It requires the distribution of component lifetimes as well as the exact structure function of the system. Therefore there have been efforts made to obtain useful bounds for it. Barlow and Proschan (1975 Ch 4) have given such bounds. Bodin (1970), Beichelt and Spross (1989) have also found useful bounds based on cut and path sets and module-decompositions of the coherent system. In Chaudhuri, Deshpande and Dharmadhikari (1991) bounds for coherent systems composed of independent IFRA components were derived. These bounds required the knowledge of the structure function of the system. From (2011) has found bounds for the reliability of sum of n New Better Than Used (NBUE) components. For other papers on bounds of reliability function see Koutras, Papastavridis and Patakos (1996) and Hsieh (2003) .

Recently Samaniego (1985), Kochar, Mukerjee and Samaniego (1999), Samaniego (2007) have introduced and utilized the concept of 'signature' of a coherent system to represent its reliability in the following manner. Let $h(x_1, x_2, \ldots, x_n)$ be the structure function of the system, that is to say,

$$\begin{aligned} h(x_1, x_2, \ldots, x_n) &= 1, \text{ if the system is working} \\ & \quad\ 0, \text{ if the system has failed,} \end{aligned} \tag{1}$$

where

$$\begin{aligned} x_i &= 1, \text{ if the ith component is working} \\ & \quad\ 0, \text{ if it has failed.} \end{aligned}$$

The reliability of any system for a mission of duration t is defined as the probability that the system does not fail upto time t. If the random variable T denotes the life time of the system, then the reliability at time t is $R(t) = P(T > t)$. It is well known that

$$R(t) = P(T > t) = h(\bar{F}_1(t), \bar{F}_2(t), \ldots, \bar{F}_n(t)), \tag{2}$$

where h is the structure function defined in (1) and $\bar{F}_i(t)$ is the continuous survival function of the ith component with random lifetime $X_i,\ i = 1, 2, \ldots, n$. If these random variables are independent and identically distributed with common survival function $\bar{F}(t)$, then the reliability of the system is

$$R(t) = P(T > t) = h(\bar{F}(t), \bar{F}(t), \ldots, \bar{F}(t)). \tag{3}$$

Samaniego and his associates in the above references have pointed out that any system fails only at the epoch of a component failure. Therefore, the lifetime of the system T is one of the component lifetimes. If we denote by $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ the ordered component life times, and

$$P[T = X_{(i)}] = s_i, \;\; i = 1, 2, \ldots, n,$$

then $\underline{s} = (s_1, s_2, \ldots, s_n)$ is a probability distribution, that is, $s_i \geq 0$ and $\sum_{i=1}^{n} s_i = 1$. The vector $\underline{s}$ is defined as the signature of the coherent system. It is seen that

$$R(t) = P(T > t) = \sum_{i=1}^{n} s_i \bar{F}_{(i)}(t), \tag{4}$$

where

$$\bar{F}_{(i)}(t) = P(X_{(i)} > t) = \sum_{j=0}^{i-1} \binom{n}{j} (1 - \bar{F}(t))^j (\bar{F}(t))^{n-j} \tag{5}$$

is the survival function of the ith order statistics of the random sample $X_1, X_2, \ldots, X_n$, the lifetimes of the n components which have i.i.d. life times. The survival function represented in (5) is the reliability function of the $(n-i+1)$-out-of-n system.

Navarro and Rychlik (2007) have derived bounds for the reliability function and expected lifetimes of coherent systems composed of exchangeable components in terms of the common margianl component life distribution and the signature. They (2010) derived bounds for the expected lifetimes of coherent or mixed systems with independent components having unknown distributions based on the expected lifetimes of the components and signature.

Now suppose that $F_i, \; i = 1, 2, \ldots, n$ or the common c.d.f F belong to the Increasing Failure Rate Average (IFRA) class of distributions. This class is of special significance in reliability theory as it is the smallest class of probability distributions which contains the exponential distribution and is closed under operations of (i) formation of coherent systems and (ii) taking limits of sequences of distributions. The Increasing Failure Rate (IFR) class is a subclass of the IFRA class which in turn is contained in New Better than Used (NBU) class of distributions. Thus all IFRA distributions exhibit the aspect of positive ageing (deteriorating effect of age) on the lifetimes. The distributions belonging to this class are appropriate models for a large number of components and systems occuring in electronics, engineering and other phenomena, which are subject to wear and tear with time.

It is a well known fact that any IFRA distribution function crosses every exponential distribution function at most once from below, if it does, and exactly once if the IFRA distribution and the exponential distribution have a common quantile ζ_p of any order $p, (0 < p < 1)$. Using this fact and the expressions (2) and (3), the paper by Chaudhuri, Deshpande and Dharmadhikari (1991) , referred to as CDD (1991) in the sequel, have provided a lower bound for $R(t)$ over $0 < t \leq t_0$ and an upper bound for $t_0 \leq t < \infty$. The calculation of these bounds require the knowledge of the structure function h and ζ_p, the quantile of a specific order p of the common component life time distribution. One need not know the entire distribution F, except for the fact that it belongs to the IFRA class. In CDD (1991) these bounds have been tabulated for the 5 component bridge sructure.

In the work presented in this paper we note that the expressions (4) and (5) provide a representation of the reliability function based on the common component life time survival function and the signature. The order statistic distribution $\bar{F}_{(i)}(t)$ is itself the reliability function of the $(n-i+1)$-out-of n system which is a coherent system. If F (or $\bar{F}$) belong to the IFRA class then so do the $\bar{F}_{(i)},\ i = 1, 2, \ldots, n$.

In Section 2 we derive the bounds for the reliabilty function $R(t)$ which are based on the signature vector $\underline{s}$ and the quantile ζ_p of a specified order p of the IFRA distribution F or the common mean μ of F. In section 3, we illustrate the bounds by calculating the bounds for (i) some of the 5 coherent structures of 3 components, (ii) for several coherent structures of 4 components and (iii) for the 5 component bridge structure. We show that the bounds are reasonably close to the exact reliability in these examples. In section 4 we provide some general remarks on the usefulness of these bounds in practical situations.

2 The Bounds for Reliability Function

Consider the representation (4) of the system reliability. Let us assume that $\bar{F}$ (or F) belongs to the IFRA class. Then, because of IFRA closure theorem $\bar{F}_{(i)}$, which is the survival function of the $(n-i+1)$-out-of-n system, being coherent systems, also belong to the IFRA class, and so does the survival function or the reliability function $R(t) = P(T > t)$ of the original coherent system. We have mentioned above the single crossing property of an IFRA distribution F. The failure rate average (FRA) function of a distribution function is

$$\frac{1}{t}\int_0^t r_F(u)du,$$

where $r_F(u)$ is the failure rate corresponding to F. Since $r_F(u) = \frac{d}{du}(-log\bar{F}(u))$, the FRA function is equal to $-\frac{1}{t}log\bar{F}(t)$ and it is increasing in t for an IFRA distribution . Let ζ_p be the quantile of order p of the distribution F. Then $F(\zeta_p) = p$ and $\bar{F}(\zeta_p) = 1 - p$. The increasing nature of FRA function implies that

$$\begin{aligned} -\frac{1}{t}log\bar{F}(t) &\leq -\frac{1}{\zeta_p}log(1-p) \text{ for } t \leq \zeta_p, \\ \text{and } -\frac{1}{t}log\bar{F}(t) &\geq -\frac{1}{\zeta_p}log(1-p) \text{ for } t \geq \zeta_p. \end{aligned} \tag{6}$$

The above inequalities may be rewritten as

$$\begin{aligned} \bar{F}(t) &\geq exp\{t(\frac{log(1-p)}{\zeta_p})\} \text{ for } t \leq \zeta_p, \\ \text{and } \bar{F}(t) &\leq exp\{t(\frac{log(1-p)}{\zeta_p})\} \text{ for } t \geq \zeta_p. \end{aligned} \tag{7}$$

It should be noted that the R.H.S. of the above inequalities is the survival function of the exponential distribution with mean $-\frac{\zeta_p}{log(1-p)}$, with the same quantile ζ_p of order p as the IFRA distribution F appearing on the LHS. This is the 'single crossing property' of an IFRA distribution which states that every IFRA survival function crosses every exponential distribution at most once; and if the crossing takes place it is from above. Of course the point where the crossing takes place is the quantile of the same order of both the IFRA and the exponential distribution. The subsequent result follows from this basic result, originally given in Barlow and Proschan (1975, Ch 4).

The structure function h as given in (3) above is an increasing function of its arguments. Hence combining it with (7) one may write

$$\begin{aligned} R(t) &\geq h(e^{\frac{tlog(1-p)}{\zeta_p}}, e^{\frac{tlog(1-p)}{\zeta_p}}, \ldots, e^{\frac{tlog(1-p)}{\zeta_p}}), \text{ for } t \leq \zeta_p, \\ \text{and } R(t) &\leq h(e^{\frac{tlog(1-p)}{\zeta_p}}, e^{\frac{tlog(1-p)}{\zeta_p}}, \ldots, e^{\frac{tlog(1-p)}{\zeta_p}}), \text{ for } t \geq \zeta_p. \end{aligned} \tag{8}$$

And specialising it to the reliability function of a $(n - i + 1)$-out-of-n system given by (3), we have the following inequality

$$\bar{F}_{(i)}(t) \geq \sum_{j=0}^{i-1} \binom{n}{j} (1 - e^{t\frac{log(1-p)}{\zeta_p}})^j (e^{t\frac{log(1-p)}{\zeta_p}})^{n-j}, \quad t \leq \xi_p,$$

$$\text{and } \bar{F}_{(i)}(t) \quad \leq \quad \sum_{j=0}^{i-1} \binom{n}{j} (1 - e^{t\frac{log(1-p)}{\zeta_p}})^j (e^{t\frac{log(1-p)}{\zeta_p}})^{n-j}, \quad t \geq \xi_p. \tag{9}$$

Effectively we are bounding the survival function or reliability of the $(n-i+1)$-out-of-n system of i.i.d. IFRA components by the corresponding system of independent exponential components which have the quantile ζ_p of order p. These inequalities put together in the mixture (4) and a change in the order of summation lead to the following theorem.

Theorem 2.1: Let $(s_1, s_2, \ldots, s_n)$ be the signature of a coherent system composed of n components with independent and identically distributed lifetimes with common survival function $\bar{F}$ having the quantile ζ_p of order p. Then, the reliability of the system is bounded as below:

$$\begin{aligned} R(t) \quad &\geq \quad \sum_{i=1}^{n} s_i \sum_{j=0}^{i-1} \binom{n}{j} (1 - e^{t\frac{log(1-p)}{\xi_p}})^j (e^{t\frac{log(1-p)}{\xi_p}})^{n-j}, \quad t \leq \xi_p, \\ &\leq \quad \sum_{i=1}^{n} s_i \sum_{j=0}^{i-1} \binom{n}{j} (1 - e^{t\frac{log(1-p)}{\xi_p}})^j (e^{t\frac{log(1-p)}{\xi_p}})^{n-j}, \quad t \geq \xi_p. \end{aligned} \tag{10}$$

These bounds are based on (i) the signature of the coherent system and (ii) the quantile of order p of the common component survival function. The bound obtained in CDD (1991) were based on the common quantile ζ_p and the structure function h of the coherent system. It is felt that the bounds being presented here will have wider applicability as the requirement of a known structure function is being weakened to that of a known signature. In any case the calculations will be easier, based on (10) rather than on the original structure function which can be very complicated.

These are one sided functional bounds, lower on the range $(0, \zeta_p]$ and upper on the range (ζ_p, ∞). One can choose ζ_p and hence the corresponding p to vary the ranges to suit the user. For example, choosing p small, thus $(1-p)$ large, will give us a quantile ζ_p which has a small magnitude leading to a lower bound over the small range $(0, \zeta_p]$. But this lower bound is for the high reliability region of the lifetime of the system. In situations where we do not know the structure function beyond the signature and only know the quantile ζ_p of order p of the common IFRA survival function of the component lifetimes, we have an effective lower bound to the exact reliability in this region. That these bounds are sharp is obvious, since these lower as well as upper bounds together is the exact reliability function of the coherent system under consideration, when it is composed of the specified exponentially distributed components that also belong to the IFRA class.

Further, if we know the mean μ of the common component lifetimes and the value of the common survival function at the mean, that is, the value of $\bar{F}(\mu)$, then we can take a slightly different path after the definition of IFRA distribution. Since $\bar{F}$ is the survival function of an IFRA distribution if $-\frac{1}{t}log\bar{F}(t)$ is an increasing function, it implies that

$$\begin{aligned} -\frac{1}{t}log\bar{F}(t) &\leq -\frac{1}{\mu}log\bar{F}(\mu), \text{ for } t \leq \mu, \\ -\frac{1}{t}log\bar{F}(t) &\geq -\frac{1}{\mu}log\bar{F}(\mu), \text{ for } t \geq \mu. \end{aligned} \tag{11}$$

These inequalities can be rewritten as

$$\begin{aligned} \bar{F}(t) &\geq exp\{t\tfrac{log\bar{F}(\mu)}{\mu}\}, \text{ for } t \leq \mu, \\ \bar{F}(t) &\leq exp\{t\tfrac{log\bar{F}(\mu)}{\mu}\}, \text{ for } t \geq \mu. \end{aligned} \tag{12}$$

The R.H.S. in the above inequalities is the survival function of the exponential distribution with mean $-\frac{\mu}{log\bar{F}(\mu)}$. $\bar{F}(t)$ and this exponential distribution cross at μ (a common quantile whose order will depend upon F). Then arguing as before we can bound the reliability function of $(n-i+1)$- out-of-n system as below:

$$\begin{aligned} \bar{F}_{(i)}(t) &\geq \sum_{j=0}^{i-1}\binom{n}{j}(1-e^{t\frac{log\bar{F}(\mu)}{\mu}})^j(e^{t\frac{log\bar{F}(\mu)}{\mu}})^{n-j}, \ t \leq \mu, \\ \bar{F}_{(i)}(t) &\leq \sum_{j=0}^{i-1}\binom{n}{j}(1-e^{t\frac{log\bar{F}(\mu)}{\mu}})^j(e^{t\frac{log\bar{F}(\mu)}{\mu}})^{n-j}, \ t \geq \mu. \end{aligned} \tag{13}$$

These inequalities then lead to the following theorem:

Theorem 2.2: Let $(s_1, s_2, \ldots, s_n)$ be the signature of a coherent system composed of n components with independent and identically distributed lifetimes with common survival function $\bar{F}$ having mean μ. Then the reliability of the system is bounded as below:

$$R(t) \geq \sum_{i=1}^{n} s_i \sum_{j=0}^{i-1}\binom{n}{j}(1-e^{t\frac{log\bar{F}(\mu)}{\mu}})^j(e^{t\frac{log\bar{F}(\mu)}{\mu}})^{n-j}, \ t \leq \mu,$$

$$R(t) \quad \leq \quad \sum_{i=1}^{n} s_i \sum_{j=0}^{i-1} \binom{n}{j} (1 - e^{t\frac{log\bar{F}(\mu)}{\mu}})^j (e^{t\frac{log\bar{F}(\mu)}{\mu}})^{n-j}, \; t \geq \mu. \tag{14}$$

In fact one may replace μ by any $a(\in (0,\infty))$ and the inequalities (11) still hold . Therefore, one can modify Theorem 2.2 to obtain bounds in terms of the reliability function of the coherent system composed of i.i.d. component lifetimes with exponential distribution with mean $-\frac{a}{log\bar{F}(a)}$. Thus, to actually use these bounds one needs to know (i) the signature of the system and (ii) the value of the common component life distribution at a. The theorem provides a lower bound upto a and an upper bound beyond a.

Remark 2.1: Note that the bound in (14) can also be expressed as

$$\begin{aligned} & \sum_{i=1}^{n} s_i \sum_{j=0}^{i-1} \binom{n}{j} (1 - e^{-t\frac{log\bar{F}(\mu)}{\mu}})^j (e^{-t\frac{log\bar{F}(\mu)}{\mu}})^{n-j} \\ = & \sum_{i=1}^{n-1} \binom{n}{i} (1 - e^{-t\frac{log\bar{F}(\mu)}{\mu}})^i (e^{-t\frac{log\bar{F}(\mu)}{\mu}})^{n-i} \sum_{j=i}^{n} s_j. \end{aligned}$$

Thus, given two signature vectors $\underline{s}$ and $\underline{s^*}$, if $\sum_{j=i}^{n} s_j \leq \sum_{j=i}^{n} s_j^*$, then we will have

$$\begin{aligned} & \sum_{i=1}^{n-1} \binom{n}{i} (1 - e^{-t\frac{log\bar{F}(\mu)}{\mu}})^i (e^{-t\frac{log\bar{F}(\mu)}{\mu}})^{n-i} \sum_{j=i}^{n} s_j \\ \leq & \sum_{i=1}^{n-1} \binom{n}{i} (1 - e^{-t\frac{log\bar{F}(\mu)}{\mu}})^i (e^{-t\frac{log\bar{F}(\mu)}{\mu}})^{n-i} \sum_{j=i}^{n} s_j^*. \end{aligned} \tag{15}$$

That is, majorisation of the signature vector leads to ordering of the bounds for the reliability function.

Remark 2.2: The basic single crossing propert between the IFRA distributions and the exponential distribution can be exploited for bounding the reliability of coherent systems composed of independent but not identically distributed components as well. The representation

$$R(t) = \sum_{i=1}^{n} s_i \bar{F}_{(i)}(t)$$

holds in this case also. However, there is no closed form expression for $\bar{F}_{(i)}(t)$, the relaibility of $(n-i+1)$-out of-n system. It becomes a sum of products of individual (non identical) component distribution functions and survival functions. Suppose we can specify numbers $a_1, a_2, \ldots, a_n$ such that the component lifetime's survival functions $\bar{G}_i(t)$ have known values $\bar{G}_i(a_i)$ at these points. Then, following arguments similar to those before, one can specify lower bound for $0 < t \le min(a_1, a_2, \ldots, a_n)$ and upper bounds for $max(a_1, a_2, \ldots, a_n) \le t < \infty$ in terms of the exponential distributions with mean $-\frac{a_i}{log\bar{G}_i(a_i)}$. By choosing $a_1 = a_2 = \ldots, a_n = a$, one can extend the range of the two inequalities to $(-\infty, a]$ and $[a, \infty)$ and not leaving any gap.

3 APPLICATIONS AND NUMERICAL RESULTS

There is a lot of interest in verifying whether certain coherent systems are highly reliable or not. In the literature 'highly' reliable systems are described to be those whose reliability exceeds a large fractional number, e.g., $0.9, 0.95, 0.99$, etc. See for example, Mease and Nair (2006) for concern in this issue. Any system which undergoes wear and tear is said to exhibit positive ageing. Since the reliability function of the system is the survival function of its lifetime, it is necessarily a monotone decreasing function. The systems which further exhibit the phenomenon of positive ageing would have the survival function belonging to one of the positive ageing classes such as IFR, IFRA, etc. Since IFRA class has the closure property under formation of coherent systems, it is a natural class to contain system lifetime distributions. Therefore, it is useful to find bounds for systems composed of IFRA components.

First we consider the bridge structure studied by CDD (1991). Each component has life distribution $G(\mu, 1)$ with survival function given by

$$\bar{F}_G(t) = \int_t^\infty \frac{x^{\mu-1}e^{-x}}{\Gamma(\mu)}dx,\ \mu > 1. \tag{16}$$

The reliability function of the bridge structure with i.i.d. components is given by

$$\bar{F}_{BS}(t) = (\bar{F}_G(t))^2[2 + 2\bar{F}_G(t) - 5(\bar{F}_G(t))^2 + 2(\bar{F}_G(t))^3].$$

Then, from Theorem 2.2 it follows that the bounds to system reliability of the bridge structure are given by

$$\begin{aligned}
\bar{F}_{BS}(t) &\ge \sum_{i=1}^{n} s_i \sum_{j=0}^{i-1} \binom{n}{j} (1 - e^{t\frac{log\bar{F}_G(\mu)}{\mu}})^j (e^{t\frac{log\bar{F}_G(\mu)}{\mu}})^{n-j},\ t \le \mu, \\
\bar{F}_{BS}(t) &\le \sum_{i=1}^{n} s_i \sum_{j=0}^{i-1} \binom{n}{j} (1 - e^{t\frac{log\bar{F}_G(\mu)}{\mu}})^j (e^{t\frac{log\bar{F}_G(\mu)}{\mu}})^{n-j},\ t \ge \mu,
\end{aligned} \tag{17}$$

where $\bar{F}_G(\mu)$ is given in (16) and the signature of the bridge structure is $(0, \frac{1}{5}, \frac{3}{5}, \frac{1}{5}, 0)$.

Table I gives the exact reliability and the reliability bounds for the bridge structure when the components have i.i.d. $G(2,1)$ distribution. The second and the fifth columns give the exact reliability of the bridge structure and the third and the sixth columns give the bounds based on signatures. Note that upto $t = 2$ we have the lower bounds to the reliability function and beyond $t = 2$ we have the upper bound to the relaibility function. It is interesting to note that for this example the CDD bounds and the bounds based on the signature coincide. CDD (1991) observed that their bounds are significantly better than Barlow and Proschan (1975) bounds. Hence the same is true for the proposed bounds based on signature of the system.

Table 1 : Reliabilty/ Reliability Bounds for Bridge system Based on Mean

time	exact rel.	Sign.bounds	time	exact rel.	sign. bounds
.1	.9994	.9841	2.1	.3095	.3223
.2	.9993	.9841	2.2	.2727	.2967
.3	.9972	.9652	2.3	.2393	.2729
.4	.9920	.9400	2.4	.2092	.2507
.5	.9825	.9097	2.5	.1823	.2301
.6	.9677	.8753	2.6	.1583	.2110
.7	.9466	.8378	2.7	.1371	.1934
.8	.9190	.7981	2.8	.1185	.1771
.9	.8850	.7570	2.9	.1021	.1622
1.	.8452	.7152	3.0	.0878	.1484
1.1	.8003	.6734	3.1	.0754	.1357
1.2	.7515	.6320	3.2	.0646	.1240
1.3	.7000	.5914	3.3	.0553	.1133
1.4	.6469	.5520	3.4	.0472	.1035
1.5	.5933	.5140	3.5	.0403	.0945
1.6	.5404	.4776	3.6	.0343	.0863
1.7	.4890	.4429	3.7	.0292	.0787
1.8	.4397	.4100	3.8	.0249	.0719
1.9	.39321	.3789	3.9	.0211	.0656
2.0	.3497	.3497	4.0	.0179	.0598

In Table 2 we report the bounds to the reliability of the bridge structure of i.i.d $G(2,1)$ random variables. But unilke the bounds reported in Table 1, these bounds are based on quantiles of the distribution. We consider $p = .1, .2, .3, .4, .5$ and the corresponding quantiles from $G(2,1)$ distributions. In each case the time (T), the exact reliability (Ex) and the bound based on signatures (Bd) has been reported, We have chosen 5 values of T less than the relevant quantile and 5 values more than the same quantile.

Table 2 : Reliabilty/ Reliability Bounds for Bridge system Based on Quantiles

$p = .1, \zeta_p = .5318$			$p = .2, \zeta_p = .8243$			$p = .3, \zeta_p = 1.0973$			$p = .4, \zeta_p = 1.3764$			$p = .5, \zeta_p = 1.6783$		
T	Bd	Ex	T	Bd	Ex	T	Bd	Ex	T	Bd	Ex	T	Bd	Ex
.1	.9992	.9999	.1	.9985	.9999	.2	.9916	.9993	.5	.9367	.9825	1.2	.6735	.7515
.2	.9968	.9993	.2	.9941	.9993	.4	.9676	.9920	.7	.8838	.9466	1.3	.6356	.7000
.3	.9930	.9971	.4	.9773	.9920	.6	.9306	.9677	.9	.8217	.8850	1.4	.5983	.6469
.4	.9876	.9920	.6	.9506	.9677	.8	.8836	.9190	1.1	.7546	.8003	1.5	.5619	.5933
.5	.9809	.9825	.8	.9160	.9190	1.0	.82961	.8452	1.3	.6857	.7000	1.6	.5267	.5404
.6	.9728	.9677	1.0	.8751	.8452	1.1	.8008	.8003	1.4	.6514	.6469	1.7	.4927	.4890
.7	.9634	.9466	1.2	.8298	.7515	1.3	.7414	.7000	1.6	.5844	.5404	1.8	.4602	.4397
.8	.9529	.9190	1.4	.7815	.6469	1.5	.6810	.5937	1.8	.5206	.4397	1.9	.4291	.3932
.9	.9412	.8850	1.6	.7316	.5404	1.7	.6213	.4890	2.0	.4611	.3497	2.0	.3996	.3497
1.0	.9286	.8452	1.8	.6813	.4397	1.9	.5636	.3932	2.1	.4330	.3095	2.1	.3716	.3095

These numerical contributions are more vividly presented in the several graphs that follow in Figures 1.1 - 1.5, . . . , 5.1 - 5.5. The exact reliability is given by the blue curve and the bounds by the red curve. Note that series(3) denotes a series system with 3 components. The other expressions are interpreted in a similar manner.

In the accompanying tables and graphs we see that for some 3, 4 and 5 component systems the bounds for the entire class give useful values. As an example we have calculated the exact relaibility of the corresponding systems with a specific IFRA distribution (viz, Gamma(2,1) distribution). Several points may be noted:

(i) The bounds (both the lower bound part and the upper bound part) are close to the exact reliability.

(ii) We have illustrated the fact that by choosing various p, $0 < p < 1$, (order of quantile) one can get very accurate bounds for values of the exact relaibility near ζ_p, the point of intersection of the exact relaibility and the bound.

(iii) By choosing values of p as small as $p = .1$, we have been able to obtain conservative (lower) bounds for the reliability which are extremely close to it in the region $0 < t < \zeta_p$.

(iv) It seems to us that for all practical purposes one can use the bound values over the region as the actual values of the relaibility, without any information about the component life times beyond the fact that they belong to the IFRA class and have a known quantile ζ_p of order p.

4 Conclusions

Here we have proposed bounds for the reliability of coherent systems consisting of independent IFRA components. In CDD (1991) the bounds for the reliability of a coherent system were obtained by applying the single crossing property of an IFRA distribution to the survival function of the entire system. However, if the signature of the system is available then one may apply this property to the survival function of the $(n-i+1)$-out-of-n systems to obtain bounds for these survival functions. Then these bounds could be combined in an over all bound for the entire system with the given signature via (4). Tables of signatures of all the coherent systems with $n = 3, 4$ components are given by Samaniego (2007). Navarro and Rubio (2009) have discussed an algorithm for finding the signature of $n \geq 5$ components.

The usefulness of the above tables in designing highly reliable systems is explained here by example of the bridge structure. It is composed of 5 components, say, with unknown i.i.d. IFRA distributions having a quantile of order 0.3 equal to 1.0973. Actually, these numbers correspond to the Gamma (2,1) distribution. Now the third part in this tableau gives the values of the lower bound for the reliability of any bridge structure composed of any i.i.d. IFRA distributed components. The value of the reliability of any of these structures cannot fall below the values given in the (Bd) column of column of this part upto $T = 1.0$ and cannot exceed the values in this column at $T \geq 1.1$. The values in the ninth column are the exact reliability values if the bridge structure happens to be composed of i.i.d Gamma (2,1) components. Thus while designing a high reliability structure, one gets an idea of the reliability of the components that one must use in terms of $(p,\ \zeta_p)$, even if the distributions are not entirely known provided they can be assumed to belong to the IFRA class.

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Figure 1.1 (Bridge system $p = .1$)

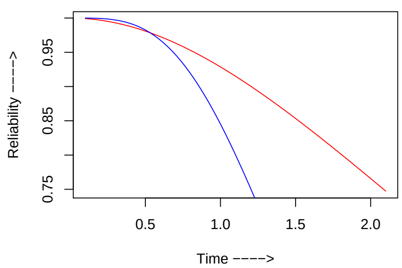


Figure 1.2 (Bridge system $p = .2$)

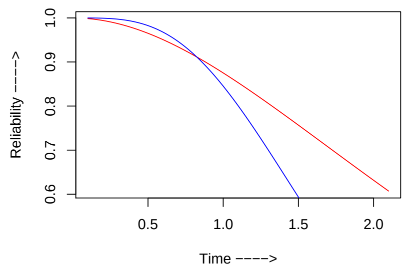


Figure 1.3 (Bridge system $p = .3$)

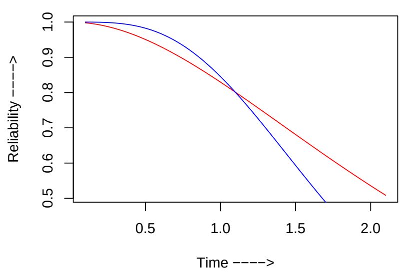


Figure 1.4 (Bridge system $p = .3$)

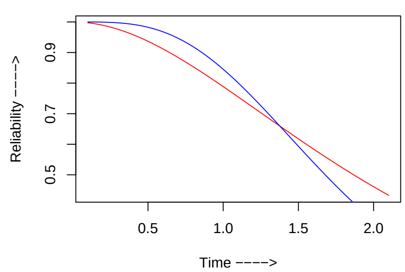


Figure 1.5 (Bridge system $p = .5$)

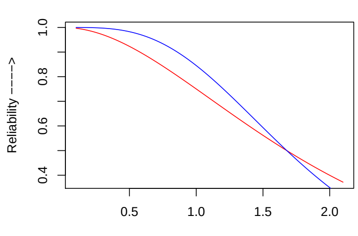
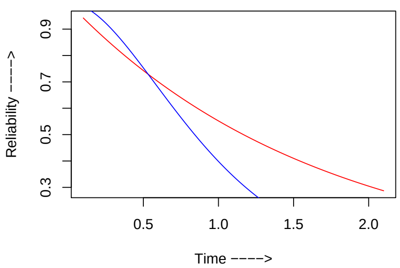


Figure 2.1 (Series system (3) $p = .1$)



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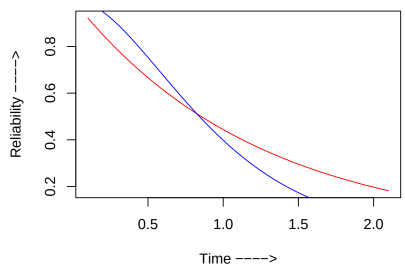


Figure 2.3 (Series system (3) $p = .3$)

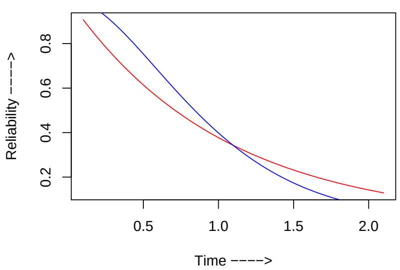
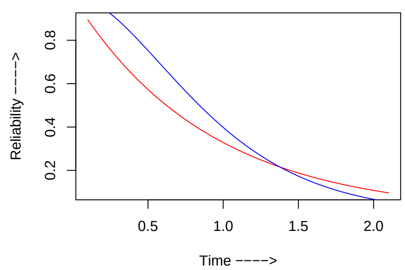


Figure 2.4 (Series system (3) $p = .4$)



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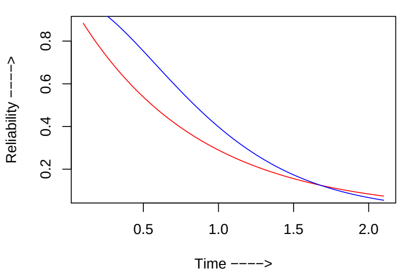
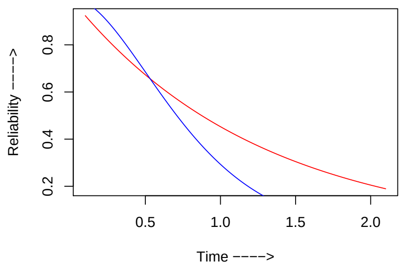


Figure 3.1 (Series system (4) $p = .1$)



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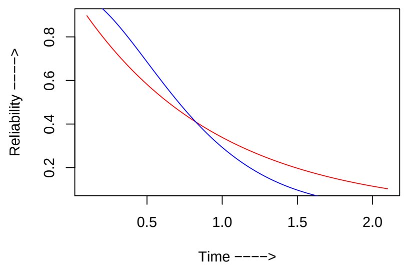

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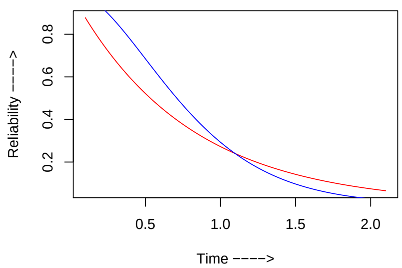

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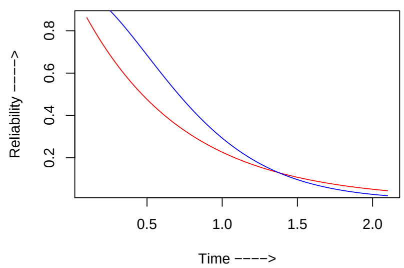

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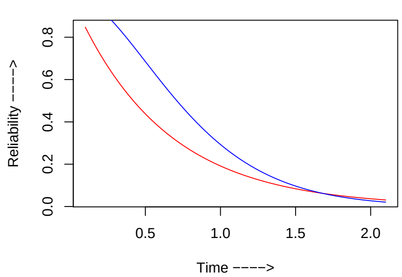
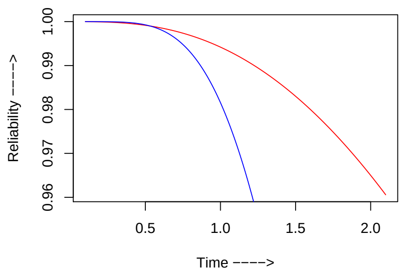


Figure 4.1 (Parallel system (3) $p = .1$)



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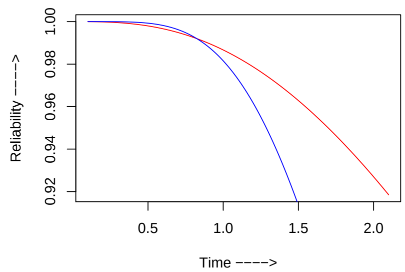

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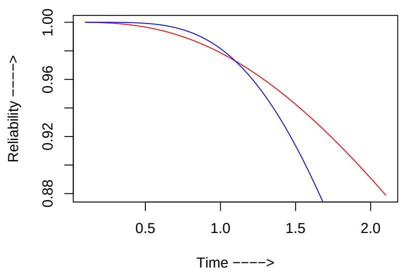

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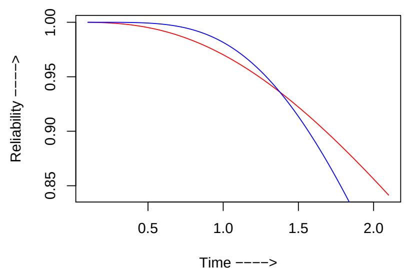
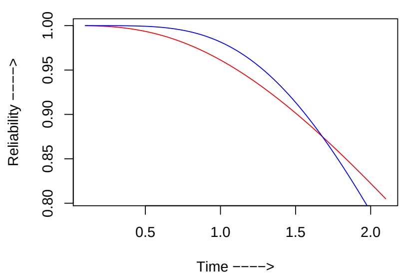


Figure 4.5 (Parallel system (3) $p = .51$)



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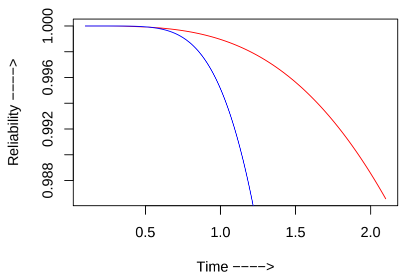

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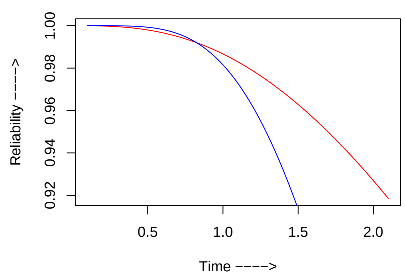

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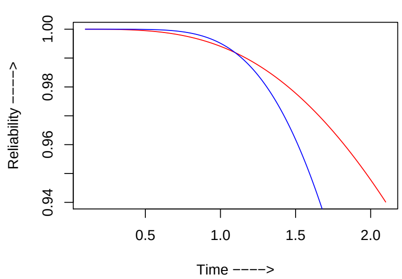

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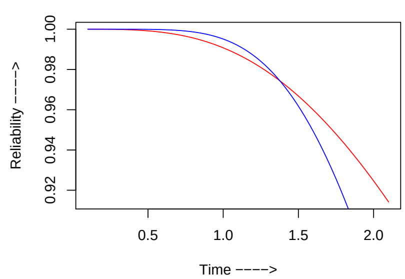

Figure 5.5 (Parallel system (4) $p = .5$)