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Estimation of parameters of two-dimensional sinusoidal signal in heavy-tailed errors

Swagata Nandi

Indian Statistical Institute, Delhi Centre
7, SJSS Marg, New Delhi–110 016, India

ESTIMATION OF PARAMETERS OF TWO-DIMENSIONAL SINUSOIDAL SIGNAL IN HEAVY-TAILED ERRORS

SWAGATA NANDI

Abstract. In this paper, we consider a two-dimensional sinusoidal model observed in a additive random field. The proposed model has wide applications in statistical signal processing. The additive noise has mean zero but the variance may not be finite. We propose the least squares estimators to estimate the unknown parameters. It is observed that the least squares estimators are strongly consistent. We obtain the asymptotic distribution of the least squares estimators under the assumption that the additive errors are from a symmetric stable distribution. Some numerical experiments are performed to see how the results work for finite samples.

1. Introduction

Estimation of the unknown parameters of a parametric model is a central problem. In this paper, we address the problem of estimation of parameters in the following two-dimensional (2-D) model;

$$y(m,n) = \sum_{k=1}^{p} \Big[A_k \cos(m\lambda_k + n\mu_k) + B_k \sin(m\lambda_k + n\mu_k) \Big] + \epsilon(m,n). \tag{1}$$

Here $y(m,n); m = 1,\ldots,M; n = 1,\ldots N$ are observed values at equidistant points on the (m,n) plane; A_k and B_k, $k = 1,\ldots,p$ are unknown real numbers called amplitudes; λ_k and μ_k are unknown frequencies such that $\lambda_k, \mu_k \in (0,\pi)$, $k = 1,\ldots,p$; $\{\epsilon(m,n)\}$ is a 2-D sequence of independent and identically distributed (i.i.d.) random variables with mean zero, but they may not have finite variance; p is the number of frequency pair present in the signal $y(m,n)$ and we assume that p is known in advance. The problem is to extract the unknown parameters, given a sample of size $M \times N$, having some desirable

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properties. In the present set-up, we allow the case when the second moment of $\epsilon(m,n)$ is not finite.

Zhang and Mandrekar (2001), Kundu and Gupta (1998), considered model (1) without the sine term, that is, $B_k = 0$, $k = 1, \ldots, p$. It is observed in Zhang and Mandrekar (2001) that 2-D sinusoidal model can be used quite effectively to model textures. In Model (1), $\epsilon(m,n)$ is a random field and first term in $y(m,n)$ corresponds to the regular deterministic textures, known as the signal component. The estimation and detection of the signal component in presence of additive noise is an important problem is statistical signal processing.

Zhang and Mandrekar (2001) discussed the consistent estimation of λ_k's and μ_k's, but their amplitude estimators are not consistent. Kundu and Gupta (1998) considered model (1) with $B_k = 0$ under the assumption that $\epsilon(m,n)$'s are i.i.d. random variables with mean zero and finite variance. Kundu and Nandi (2003) discussed the strong consistency and asymptotic normality of the least squares estimators (LSEs) of the unknown parameters when $\epsilon(m,n)$'s are from a stationary random field.

The problem is of interest in spectrography and is studied by Malliavan (1994a, 1994b) using group-theoretic methods. This is a basic model in many fields, such as antenna array processing, geophysical perception, biomedical spectral analysis, etc. See for example the work of Barbieri and Barone (1992), Cabrera and Bose (1993), Chun and Bose (1995), Hua(1992), Kundu and Gupta (1998) and Lang and McClellan (1982) for the different estimation procedures and for their properties. In an recent paper, Nandi, Prasad and Kundu (2010) propose an efficient algorithm which produces estimators of the unknown parameters of model (1) with the same rate of convergence as the LSEs.

The main aim of this paper is to consider the case when the error random variables have heavier tails. A heavy tailed distribution is one whose extreme probabilities approach zero relatively slowly. An important criterion of heavy tail distribution is the non-existence of second moment, pointed out by Mandelbrot (1963). We are using the same definition of Mandelbrot (1963), that is, a distribution is heavy tailed, if and only if the variance is infinite. It has been shown that under the assumption $E|\epsilon(m,n)|^{1+\delta} < \infty$, for some $\delta > 0$, the LSEs of the unknown parameters are strongly consistent. Additionally, if we assume that $\epsilon(m,n)$'s are from a symmetric α stable distribution, the asymptotic distribution of the LSEs is multivariate symmetric stable.

The rest of the paper is organized as follows. In section 2, we introduce the LSEs and state the assumptions required for model (1) when $p = 1$. The strong consistency of the LSEs is discussed in section 3 and asymptotic distribution is provided in section 4. The case of general p is discussed in section 5. Some numerical experiment results are discussed in section 6. Finally we conclude the paper in section 7. The proof of the consistency is given in Appendix.

2. Estimating the unknown parameters

In this section, we study the properties of the LSEs of the unknown parameters and the one obtained by maximizing the 2-D periodogram function. The second one is termed as the approximate LSE (ALSE) in the context of one-dimension (1-D) model and in some works of 2-D frequency model (see Walker (1969); Hannan (1971); Kundu and Nandi (2003) etc.). For simplicity of notation, we first assume that $p = 1$ in this section and next section. The model is

$$y(m,n) = A\cos(m\lambda + n\mu) + B\sin(m\lambda + n\mu) + \epsilon(m,n). \tag{2}$$

For model (2), the LSE of $\boldsymbol{\theta} = (A, B, \lambda, \mu)$, say $\widehat{\boldsymbol{\theta}} = (\widehat{A}, \widehat{B}, \widehat{\lambda}, \widehat{\mu})$ minimizes

$$Q(\boldsymbol{\theta}) = \sum_{m=1}^{M}\sum_{n=1}^{N}\Big[y(m,n) - A\cos(m\lambda + n\mu) - B\sin(m\lambda + n\mu)\Big]^2 \tag{3}$$

with respect to A, B, λ, μ. We write $\boldsymbol{\theta}^0 = (A^0, B^0, \lambda^0, \mu^0)$ as the true value of $\boldsymbol{\theta}$.

The ALSE of λ and μ can be obtained by maximizing the 2-D periodogram function

$$I(\lambda,\mu) = \Big|\sum_{m=1}^{M}\sum_{n=1}^{N} y(m,n)e^{-i(m\lambda+n\mu)}\Big|^2, \quad i = \sqrt{-1} \tag{4}$$

with respect to λ and μ. Let $\widetilde{\lambda}$ and $\widetilde{\mu}$ denote the ALSE of λ and μ, then the ALSEs of A and B, say $\widetilde{A}$ and $\widetilde{B}$ are obtained as

$$\widetilde{A} = \frac{2}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N} y(m,n)\cos(m\widetilde{\lambda} + n\widetilde{\mu}), \quad \widetilde{B} = \frac{2}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N} y(m,n)\sin(m\widetilde{\lambda} + n\widetilde{\mu}). \tag{5}$$

Alternatively, once the non-linear parameters are estimated, the linear parameters can be estimated by using the simple linear regression technique. The 2-D periodogram function defined in (4) is a simple extension of the periodogram function for 1-D data to 2-D data. For motivation of using ALSE, see Walker (1969) or Hannan (1971), where the authors used ALSE in a similar 1-D model as an equivalent estimator of LSE. Basically in ALSE, A and B are profiled out.

In the following, we state the definition of the symmetric α-stable ($S\alpha S$) distribution and state some assumption which are required in subsequent sections.

Definition 1. *(***Samorodnitsky and Taqqu (1994)***) A symmetric random variable X, symmetric around 0, is said to have the symmetric α stable ($S\alpha S$) distribution with scale parameter σ and stability index α, if the characteristic function of X is*

$$Ee^{itX} = e^{-\sigma^{\alpha}|t|^{\alpha}}.$$

The $S\alpha S$ distribution is a special case of general Stable distribution with non-zero shift and skewness parameters. For different properties of Stable and $S\alpha S$ distributions, see Samorodnitsky and Taqqu (1994).

Assumption 1. *The 2-D noise $\epsilon(m,n)$ are i.i.d. random variables with mean zero and $E|\epsilon(m,n)|^{1+\delta} < \infty$ for some $0 < \delta < 1$.*

Assumption 2. *The 2-D noise $\epsilon(m,n)$ are independent with mean zero and identically distributed as $S\alpha S$, defined above.*

Assumption 3. *A^0 and B^0 are not identically equal to zero.*

Under Assumption 1, the second moment does not exist, whereas the mean does. Assumption 3 ensures the presence of the frequency pair (λ, μ) in the data so that $y(m,n)$ are not pure noise. In the next section , we prove the consistency of the LSEs of the unknown parameters of model (2) under Assumptions 1 and 3 and in section 4, we develop the asymptotic distribution under assumptions 2 and 4, stated in section 5.

3. Strong Consistency of LSEs

In this section, we discuss the consistency properties of the LSEs. The following two lemmas are required to prove the results. Lemma 1 gives a sufficient condition for the strong consistency of the LSEs and Lemma 2 will be used to verify the condition in Lemma 1 under the moment condition given in Assumption 1.

Lemma 1. *If* $\{\epsilon(m,n), m,n \in Z\}$, *$Z$ the set of positive integers, are i.i.d. random variables with mean zero and* $E|\epsilon(m,n)|^{1+\delta} < \infty$, $0 < \delta < 1$, *then*

$$\sup_{\alpha,\beta} \left| \frac{1}{MN} \sum_{m=1}^{M} \sum_{n=1}^{N} \epsilon(m,n) \cos(m\alpha) \cos(n\beta) \right| \rightarrow 0 \ \ a.s.$$

as $\min\{M,N\} \rightarrow \infty$.

Corollary of Lemma 1:

$$\sup_{\alpha,\beta} \left| \frac{1}{M^{k+1} N^{l+1}} \sum_{m=1}^{M} \sum_{n=1}^{N} m^k n^l \epsilon(m,n) \cos(m\alpha) \cos(n\beta) \right| \rightarrow 0 \text{ a.s.}, \quad \text{for } \ k,l = 0,1,2\ldots$$

The result is true for all combination of cosine and sine functions.

Proof of Lemma 1: See the Appendix.

Lemma 2. *Write*

$$S_{c,K} \ \ = \ \ \{\boldsymbol{\theta} : \ \boldsymbol{\theta} = (A,B,\lambda,\mu), |\boldsymbol{\theta} - \boldsymbol{\theta}^0| \geq 4c, \ |A| \leq K, \ |B| \leq K\}.$$

If for any $c > 0$ *and for some* $K < \infty$,

$$\underline{\lim} \inf_{\boldsymbol{\theta} \in S_{c,K}} \frac{1}{MN} \left[Q(\boldsymbol{\theta}) - Q(\boldsymbol{\theta}^0) \right] > 0 \qquad a.s., \tag{6}$$

then $\widehat{\boldsymbol{\theta}}$, *the LSE of* $\boldsymbol{\theta}^0$, *is a strongly consistent estimator of* $\boldsymbol{\theta}^0$.

Proof of Lemma 2: It follows from Lemma 1 of Wu (1981).

Lemma 1 is a strong result. It generalizes some of the 1-D results, see Hannan(1971), Kundu (1993), Kundu and Mitra (1996), Nandi, Iyer and Kundu (2002) and the 2-D results given in Kundu and Gupta (1998). The following theorem states the consistency result.

Theorem 3.1. Under Assumptions 1 and 3, the LSEs of the parameters of model (2) are strongly consistent.

Proof of Theorem 3.1: See the Appendix.

4. Asymptotic Distribution of LSEs

In this section, we obtain the asymptotic distributions of the LSEs of the unknown parameters of model (2) under Assumptions 2 and 3. Now, the error random variables are from a $S\alpha S$ distribution. Then Assumption 1 is also satisfied with $1+\delta<\alpha<2$. Hence, from now on, we take $1+\delta<\alpha<2$.

Write $Q'(\boldsymbol{\theta})$ and $Q''(\boldsymbol{\theta})$ as the vector of first derivatives and the matrix of second derivatives of orders 1×4 and 4×4, respectively. Suppose $\mathbf{D_1}$ and $\mathbf{D_2}$ are two diagonal matrices of order 4×4 defined as follows:

$$\begin{aligned}
\mathbf{D_1} &= \operatorname{diag}\left\{\frac{1}{M^{\frac{1}{\alpha}}N^{\frac{1}{\alpha}}}, \frac{1}{M^{\frac{1}{\alpha}}N^{\frac{1}{\alpha}}}, \frac{1}{M^{\frac{1+\alpha}{\alpha}}N^{\frac{1}{\alpha}}}, \frac{1}{M^{\frac{1}{\alpha}}N^{\frac{1+\alpha}{\alpha}}}, \right\},\\
\mathbf{D_2} &= \operatorname{diag}\left\{\frac{1}{M^{\frac{\alpha-1}{\alpha}}N^{\frac{\alpha-1}{\alpha}}}, \frac{1}{M^{\frac{\alpha-1}{\alpha}}N^{\frac{\alpha-1}{\alpha}}}, \frac{1}{M^{\frac{2\alpha-1}{\alpha}}N^{\frac{\alpha-1}{\alpha}}}, \frac{1}{M^{\frac{\alpha-1}{\alpha}}N^{\frac{2\alpha-1}{\alpha}}}\right\}.
\end{aligned}$$

Expanding $Q'(\boldsymbol{\theta})$ at $\widehat{\boldsymbol{\theta}}$, the LSE of $\boldsymbol{\theta}$, around the true value $\boldsymbol{\theta}^0$ using multivariate Taylor series, we have

$$Q'(\widehat{\boldsymbol{\theta}}) - Q'(\boldsymbol{\theta}^0) = (\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}^0)Q''(\boldsymbol{\theta}^*), \tag{7}$$

where $\boldsymbol{\theta}^*$ is a point on the line joining $\widehat{\boldsymbol{\theta}}$ and $\boldsymbol{\theta}^0$.

It follows that $\widehat{\boldsymbol{\theta}}$ converges a.s. to $\boldsymbol{\theta}^0$ from Theorem 3.1 and $Q''(\boldsymbol{\theta})$ is a continuous function of $\boldsymbol{\theta}$. Therefore, it can be seen that

$$\lim_{M,N\to\infty} \mathbf{D_2}Q''(\boldsymbol{\theta}^*)\mathbf{D_1} = \lim_{M,N\to\infty} \mathbf{D_2}Q''(\boldsymbol{\theta}^0)\mathbf{D_1} = \begin{pmatrix} 1 & 0 & \frac{1}{2}B^0 & \frac{1}{2}B^0 \\ 0 & 1 & -\frac{1}{2}A^0 & -\frac{1}{2}A^0 \\ \frac{1}{2}B^0 & -\frac{1}{2}A^0 & \frac{1}{3}({A^0}^2+{B^0}^2) & \frac{1}{4}({A^0}^2+{B^0}^2) \\ \frac{1}{2}B^0 & -\frac{1}{2}A^0 & \frac{1}{4}({A^0}^2+{B^0}^2) & \frac{1}{3}({A^0}^2+{B^0}^2) \end{pmatrix},$$

and

$$\lim_{M,N\to\infty} \left[\mathbf{D_2}Q''(\boldsymbol{\theta}^*)\mathbf{D_1}\right]^{-1} = \begin{bmatrix} {A^0}^2+7{B^0}^2 & -6A^0B^0 & -6B^0 & -6B^0 \\ -6A^0B^0 & 7{A^0}^2+{B^0}^2 & 6A^0 & 6A^0 \\ -6B^0 & 6A^0 & 12 & 0 \\ -6B^0 & 6A^0 & 0 & 12 \end{bmatrix} = \boldsymbol{\Sigma}^{-1} \quad (say).$$

Since $Q'(\widehat{\boldsymbol{\theta}}) = \mathbf{0}$, and $\mathbf{D_2}Q''(\boldsymbol{\theta}^*)\mathbf{D_1}$ is an invertible matrix a.e. for large M and N, (7) can be written as

$$(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}^0){\mathbf{D_2}}^{-1} = -\left[Q'(\boldsymbol{\theta}^0)\mathbf{D_1}\right]\left[\mathbf{D_2}Q''(\boldsymbol{\theta}^*)\mathbf{D_1}\right]^{-1}.$$

In order to show that $Q'(\boldsymbol{\theta}^0)\mathbf{D_1}$ converges to a multivariate stable distribution, we write $Q'(\boldsymbol{\theta}^0) = (X^1_{MN}, X^2_{MN}, X^3_{MN}, X^4_{MN})$, where

$$X^1_{MN} = -\frac{2}{M^{\frac{1}{\alpha}}N^{\frac{1}{\alpha}}}\sum_{m=1}^{M}\sum_{n=1}^{N}\epsilon(m,n)\cos(m\lambda^0 + n\mu^0),$$
$$X^2_{MN} = -\frac{2}{M^{\frac{1}{\alpha}}N^{\frac{1}{\alpha}}}\sum_{m=1}^{M}\sum_{n=1}^{N}\epsilon(m,n)\sin(m\lambda^0 + n\mu^0),$$
$$X^3_{MN} = \frac{2}{M^{\frac{1+\alpha}{\alpha}}N^{\frac{1}{\alpha}}}\sum_{m=1}^{M}\sum_{n=1}^{N}m\epsilon(m,n)g(\boldsymbol{\theta}^0), \quad X^4_{MN} = \frac{2}{M^{\frac{1}{\alpha}}N^{\frac{1+\alpha}{\alpha}}}\sum_{m=1}^{M}\sum_{n=1}^{N}n\epsilon(m,n)g(\boldsymbol{\theta}^0),$$

where $g(\boldsymbol{\theta}^0) = A^0\sin(m\lambda^0 + n\mu^0) - B^0\cos(m\lambda^0 + n\mu^0)$. The trigonometric function $g(\boldsymbol{\theta})$ depends on m and n also, but we do not make it explicit. Then the joint characteristic function of $(X^1_{MN}, X^2_{MN}, X^3_{MN}, X^4_{MN})$ is

$$\begin{aligned}\phi_{MN}(\mathbf{t}) &= E\exp\{i(t_1X^1_{MN} + t_2X^2_{MN} + t_3X^3_{MN} + t_4X^4_{MN})\}, \quad \mathbf{t} = (t_1,t_2,t_3,t_4),\\ &= E\exp\Big\{i\frac{2}{M^{\frac{1}{\alpha}}N^{\frac{1}{\alpha}}}\sum_{m=1}^{M}\sum_{n=1}^{N}\epsilon(m,n)K_{\mathbf{t}}(m,n)\Big\},\end{aligned}$$

where

$$K_{\mathbf{t}}(m,n) = -t_1\cos(m\lambda^0 + n\mu^0) - t_2\sin(m\lambda^0 + n\mu^0) + \frac{mt_3}{M}g(\boldsymbol{\theta}^0) + \frac{nt_4}{N}g(\boldsymbol{\theta}^0). \qquad (8)$$

Because $\epsilon(m,n)$ are independent, we have

$$\begin{aligned}\phi_{MN}(\mathbf{t}) &= \prod_{m=1}^{M}\prod_{n=1}^{N}E\exp\Big\{i\frac{2}{M^{1/\alpha}N^{1/\alpha}}\epsilon(m,n)K_{\mathbf{t}}(m,n)\Big\}\\ &= \exp\Big\{-2^\alpha\sigma^\alpha\frac{1}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N}|K_{\mathbf{t}}(m,n)|^\alpha\Big\}.\end{aligned}$$

We could not prove theoretically that the sequence $\frac{1}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N}|K_{\mathbf{t}}(m,n)|^\alpha$ converges but extensive simulation suggests that it converges as M, N increase. If we assume that as $M,N\to\infty$, $\frac{1}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N}|K_{\mathbf{t}}(m,n)|^\alpha$ converges, then it can be shown that it converges to a non-zero limit for $\mathbf{t}\neq\mathbf{0}$. Note that for $\mathbf{t}\neq\mathbf{0}$

$$|K_{\mathbf{t}}(m,n)| \le |t_1| + |t_2| + (|t_3| + |t_4|)(A^0 + B^0) = S, \ \ (say),$$

for all m, M and n, N; $1 \leq m \leq M$, $1 \leq n \leq N$; $M, N = 1, 2, \ldots$. Therefore, $|K_{\mathbf{t}}(m,n)/S| \leq 1$ and

$$|K_{\mathbf{t}}(m,n)|^{\alpha} \geq \frac{S^{\alpha}}{S^2}|K_{\mathbf{t}}(m,n)|^2, \ \ 0 < \alpha \leq 2 \ \text{ and for all } \ M, N = 1, 2, \ldots.$$

Hence,

$$\lim_{M,N\to\infty} \frac{1}{MN} \sum_{m=1}^{M}\sum_{n=1}^{N} |K_{\mathbf{t}}(m,n)|^{\alpha} \geq \lim_{M,N\to\infty} \frac{S^{\alpha-2}}{MN} \sum_{m=1}^{M}\sum_{n=1}^{N} |K_{\mathbf{t}}(m,n)|^{2}.$$

Using

$$\lim_{M,N\to\infty} \frac{1}{MN} \sum_{m=1}^{M}\sum_{n=1}^{N} \cos^2(m\lambda + n\mu) = \frac{1}{2}, \quad \lim_{M,N\to\infty} \frac{1}{MN} \sum_{m=1}^{M}\sum_{n=1}^{N} \cos(m\lambda + n\mu) = 0,$$

and similar results involving sine functions, it follows that

$$\lim_{M,N\to\infty} \frac{1}{MN} \sum_{m=1}^{M}\sum_{n=1}^{N} |K_{\mathbf{t}}(m,n)|^{\alpha} > 0.$$

From now on we assume that $\frac{1}{MN} \sum_{m=1}^{M}\sum_{n=1}^{N} |K_{\mathbf{t}}(m,n)|^{\alpha}$ converges to a non-zero limit as $M, N \to \infty$ and let

$$\lim_{M,N\to\infty} \frac{1}{MN} \sum_{m=1}^{M}\sum_{n=1}^{N} |K_{\mathbf{t}}(m,n)|^{\alpha} = \tau_{\mathbf{t}}(A^0, B^0, \lambda^0, \mu^0, \alpha). \tag{9}$$

This implies

$$\lim_{M,N\to\infty} \phi_{MN}(\mathbf{t}) = e^{-2^{\alpha}\sigma^{\alpha}\tau_{\mathbf{t}}(A^0,B^0,\lambda^0,\mu^0,\alpha)}. \tag{10}$$

This limiting characteristic function (10), indicates that even if $M, N \to \infty$, any linear combination of X^1_{MN}, X^2_{MN}, X^3_{MN} and X^4_{MN}, follows a SαS distribution.

Now consider

$$\left[Q'(\boldsymbol{\theta}^0)\mathbf{D_1}\right]\left[\mathbf{D_2}Q''(\boldsymbol{\theta})\mathbf{D_1}\right]^{-1} = -\frac{2}{M^{\frac{1}{\alpha}}N^{\frac{1}{\alpha}}} \begin{bmatrix} \sum_{m=1}^{M}\sum_{n=1}^{N} \epsilon(m,n)U_A(m,n) \\ \sum_{m=1}^{M}\sum_{n=1}^{N} \epsilon((m,n)U_B(m,n) \\ \sum_{m=1}^{M}\sum_{n=1}^{N} \epsilon(m,n)U_{\lambda}(m,n) \\ \sum_{m=1}^{M}\sum_{n=1}^{N} \epsilon(m,n)U_{\mu}(m,n) \end{bmatrix}^T, \tag{11}$$

where

$$\begin{aligned}
U_A(m,n) &= \frac{1}{A^{0^2}+B^{0^2}}\Big[(A^{0^2}+7B^{0^2})\cos(m\lambda^0+n\mu^0) - 6A^0B^0\sin(m\lambda^0+n\mu^0) \\
&\quad - \frac{6mB^0}{M}g(\boldsymbol{\theta}^0) - \frac{6nB^0}{N}g(\boldsymbol{\theta}^0)\Big], \\
U_B(m,n) &= \frac{1}{A^{0^2}+B^{0^2}}\Big[(-6A^0B^0\cos(m\lambda^0+n\mu^0) + (7A^{0^2}+B^{0^2})\sin(m\lambda^0+n\mu^0) \\
&\quad + \frac{6mA^0}{M}g(\boldsymbol{\theta}^0) + \frac{6nA^0}{N}g(\boldsymbol{\theta}^0)\Big], \\
U_\lambda(m,n) &= \frac{1}{A^{0^2}+B^{0^2}}\Big[(-6B^0\cos(m\lambda^0+n\mu^0) + 6A^0\sin(m\lambda^0+n\mu^0) + \frac{12m}{M}g(\boldsymbol{\theta}^0)\Big], \\
U_\mu(m,n) &= \frac{1}{A^{0^2}+B^{0^2}}\Big[(-6B^0\cos(m\lambda^0+n\mu^0) + 6A^0\sin(m\lambda^0+n\mu^0) + \frac{12n}{N}g(\boldsymbol{\theta}^0)\Big].
\end{aligned}$$

Note that each element of $[Q'(\boldsymbol{\theta}^0)\mathbf{D_1}]\,[\mathbf{D_2}Q''(\boldsymbol{\theta})\mathbf{D_1}]^{-1}$ is a linear combination of X^1_{MN}, X^2_{MN}, X^3_{MN} and X^4_{MN} and hence distributed as symmetric α-stable distribution. Therefore, using Theorem 2.1.5 of Samorodnitsky and Taqqu (1994) that *a random vector is symmetric in $\mathbb{R}^d$, if and only if any linear combination is symmetric stable in $\mathbb{R}^1$, where d is the order of the vector*, it follows that

$$\lim_{M,N\to\infty}\left[Q'(\boldsymbol{\theta}^0)\mathbf{D_1}\right]\left[\mathbf{D_2}Q''(\boldsymbol{\theta}^*)\mathbf{D_1}\right]^{-1},$$

converges to a symmetric stable random vector in $\mathbb{R}^4$ which has the characteristic function

$$\Phi(\mathbf{t}) = e^{-2^\alpha\sigma^\alpha\tau_{\mathbf{v}}(A^0,B^0,\lambda^0,\mu^0,\alpha)}. \tag{12}$$

Here $\boldsymbol{\tau}_{\mathbf{v}}$ is defined as in (9), $\mathbf{t}$ replaced by $\mathbf{v}=(v_1,v_2,v_3,v_4)$, with

$$\begin{aligned}
v_1(t_1,t_2,t_3,t_4,A^0,B^0) &= \frac{1}{A^{0^2}+B^{0^2}}\Big[(A^{0^2}+7B^{0^2})t_1 - 6A^0B^0t_2 - 6B^0t_3 - 6B^0t_4\Big], \\
v_2(t_1,t_2,t_3,t_4,A^0,B^0) &= \frac{1}{A^{0^2}+B^{0^2}}\Big[-6A^0B^0t_1 + (A^{0^2}+7B^{0^2})t_2 + 6A^0t_3 - 6A^0t_4\Big], \\
v_3(t_1,t_2,t_3,t_4,A^0,B^0) &= \frac{1}{A^{0^2}+B^{0^2}}\left[-6B^0t_1 + 6A^0t_2 + 12t_3\right] = v_4(t_1,t_2,t_3,t_4,A^0,B^0).
\end{aligned}$$

Therefore, we have the following theorem;

Theorem 4.1. In model (2), if $\epsilon(m,n)$ satisfy Assumptions 2 and 3, then $(\widehat{\boldsymbol{\theta}}-\boldsymbol{\theta}^0)\mathbf{D_2}^{-1}$
$= \Big(M^{\frac{\alpha-1}{\alpha}}N^{\frac{\alpha-1}{\alpha}}(\widehat{A}-A^0), M^{\frac{\alpha-1}{\alpha}}N^{\frac{\alpha-1}{\alpha}}(\widehat{B}-B^0), M^{\frac{2\alpha-1}{\alpha}}N^{\frac{\alpha-1}{\alpha}}(\widehat{\lambda}-\lambda^0), M^{\frac{\alpha-1}{\alpha}}N^{\frac{2\alpha-1}{\alpha}}(\widehat{\mu}-\mu^0)\Big)$

converges to a multivariate symmetric stable distribution in $\mathbb{R}^4$ having characteristic function as defined in (12).

5. Consistency and Asymptotic Distributions for General Model

In this section, we provide the asymptotic results of the LSEs of the unknown parameters for model (1). Write $\boldsymbol{\xi}_k = (A_k, B_k, \lambda_k, \mu_k), k = 1, \ldots, p$ and $\boldsymbol{\xi} = (\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_p)$ as the parameter vector. Let $\boldsymbol{\xi}^0$ and $\widehat{\boldsymbol{\xi}}$ denote the true value and the LSE of $\boldsymbol{\xi}$, respectively. The LSE of $\boldsymbol{\xi}$ is obtained by minimizing the residual sum of squares for model (1), say $R(\boldsymbol{\xi})$, defined similarly as $Q(\boldsymbol{\theta})$ in (3). The following assumption is required instead of Assumption 3 in this section.

Assumption 4. *$A_1^0, \ldots, A_p^0$ and $B_1^0, \ldots, B_p^0$ are arbitrary real numbers such that both A_j^0 and B_j^0 are not simultaneously equal to zero for all j.*

The consistency of $\widehat{\boldsymbol{\xi}}$ follows similarly as the consistency of $\widehat{\boldsymbol{\theta}}$. We discuss the asymptotic distribution of $\widehat{\boldsymbol{\xi}}$ here.

Let $R'(\boldsymbol{\xi})$ and $R''(\boldsymbol{\xi})$ be the vector and the matrix of the first and the second derivatives of orders $(1 \times 4p)$ and $(4p \times 4p)$ respectively, as was defined in section 4. Define two diagonal matrices of order $4p \times 4p$ using $\mathbf{D_1}$ and $\mathbf{D_2}$

$$\boldsymbol{\Gamma}_1 = \begin{pmatrix} \mathbf{D_1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{D_1} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{D_1} \end{pmatrix}, \qquad \boldsymbol{\Gamma}_2 = \begin{pmatrix} \mathbf{D_2} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{D_2} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{D_2} \end{pmatrix},$$

where $\mathbf{D_1}$ and $\mathbf{D_2}$ are defined in previous section. Along the same lines as in section 4, using multivariate Taylor series expansion and $R'(\boldsymbol{\xi}^0) = \mathbf{0}$, we have

$$(\widehat{\boldsymbol{\xi}} - \boldsymbol{\xi}^0)\boldsymbol{\Gamma}_2^{-1} = -\left[R'(\boldsymbol{\xi}^0)\boldsymbol{\Gamma}_1\right]\left[\boldsymbol{\Gamma}_2 R''(\boldsymbol{\xi}^*)\boldsymbol{\Gamma}_1\right]^{-1},$$

because $[\boldsymbol{\Gamma}_2 R''(\boldsymbol{\xi}^*)\boldsymbol{\Gamma}_1]$ is an invertible matrix for large M and N. Similarly as in case of model (2), it can be shown that

$$\lim_{M,N\to\infty}[\boldsymbol{\Gamma}_2 R''(\boldsymbol{\xi}^*)\boldsymbol{\Gamma}_1] = \lim_{M,N\to\infty}[\boldsymbol{\Gamma}_2 R''(\boldsymbol{\xi}^0)\boldsymbol{\Gamma}_1] = \begin{pmatrix} \boldsymbol{\Sigma}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Sigma}_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \boldsymbol{\Sigma}_p \end{pmatrix} = \boldsymbol{\Delta}, \ (say)$$

where $\mathbf{\Sigma}_k$ is a 4×4 matrix obtained from $\mathbf{\Sigma}$ by replacing A^0 and B^0 by A_k^0 and B_k^0, respectively. Consider $\mathbf{t} = (\mathbf{t_1}, \ldots, \mathbf{t_p})$, $\mathbf{t_j} = (t_{1j}, t_{2j}, t_{3j}, t_{4j})$ and write $R'(\boldsymbol{\xi})\mathbf{\Gamma_2} = (\mathbf{X_1}, \ldots, \mathbf{X_p})$, $\mathbf{X_j} = (X_{MN}^{1j}, X_{MN}^{2j}, X_{MN}^{3j}, X_{MN}^{4j})$. Here $X_{MN}^{kj}, k = 1, \ldots, 4$ are defined similarly as $X_{MN}^k, k = 1, \ldots, 4$; A^0, B^0, λ^0 and μ^0 are replaced by A_j^0, B_j^0, λ_j^0 and μ_j^0. Then the joint characteristic function of the elements of $R'(\boldsymbol{\xi})\mathbf{\Gamma}_2$ is

$$\phi_{MN}^p(\mathbf{t}) = E \exp\Big\{ i \frac{2}{M^{1/\alpha} N^{1/\alpha}} \sum_{m=1}^{M} \sum_{n=1}^{N} \epsilon(m,n) K_{\mathbf{t}}^p(m,n) \Big\}, \quad K_{\mathbf{t}}^p(m,n) = \sum_{j=1}^{p} K_{\mathbf{t_j}}(m,n).$$

For $j = 1, , \ldots, p$, $K_{\mathbf{t_j}}(m,n)$ is $K_{\mathbf{t}}(m,n)$ with A^0, B^0, λ^0 and μ^0, replaced by A_j^0, B_j^0, λ_j^0 and μ_j^0, respectively. This form enables us to write

$$\begin{aligned}
\phi_{MN}^p(\mathbf{t}) &= \prod_{j=1}^{p} \prod_{m=1}^{M} \prod_{n=1}^{N} E \exp\Big\{ i \frac{2}{M^{\frac{1}{\alpha}} N^{\frac{1}{\alpha}}} \epsilon(m,n) K_{\mathbf{t_j}}(m,n) \Big\} \\
&= \prod_{j=1}^{p} \prod_{m=1}^{M} \prod_{n=1}^{N} \exp\Big\{ -\frac{2^\alpha \sigma^\alpha}{MN} |K_{\mathbf{t_j}}(m,n)|^\alpha \Big\} \\
&= \prod_{j=1}^{p} \exp\Big\{ -\frac{2^\alpha \sigma^\alpha}{MN} \sum_{m=1}^{M} \sum_{n=1}^{N} |K_{\mathbf{t_j}}(m,n)|^\alpha \Big\}.
\end{aligned}$$

Taking limit as $M, N \to \infty$, we obtain

$$\begin{aligned}
\lim_{M,N\to\infty} \phi_{MN}^p(\mathbf{t}) &= \prod_{j=1}^{p} \exp\Big\{ -2^\alpha \sigma^\alpha \tau_{\mathbf{t_j}}(A_j^0, B_j^0, \lambda_j^0, \mu_j^0, \alpha) \Big\} \\
&= \prod_{j=1}^{p} \Big\{ \text{joint characteristic function of } X_{MN}^{1j}, X_{MN}^{2j}, X_{MN}^{3j}, X_{MN}^{4j} \Big\}.
\end{aligned}$$

This implies that $X_{MN}^{1j}, X_{MN}^{2j}, X_{MN}^{3j}, X_{MN}^{4j}$ and $X_{MN}^{1k}, X_{MN}^{2k}, X_{MN}^{3k}, X_{MN}^{4k}$, $j \neq k$ are asymptotically independently distributed.

Now considering linear combinations similarly as in section 4, we find that as $M, N \to \infty$, $(\widehat{\boldsymbol{\xi}} - \boldsymbol{\xi}^0)\mathbf{\Gamma}_2^{-1} = \Big((\widehat{\boldsymbol{\xi}}_1 - \boldsymbol{\xi}_1^0)\mathbf{D_2}^{-1}, \ldots, \widehat{\boldsymbol{\xi}}_p - \boldsymbol{\xi}_p^0)\mathbf{D_2}^{-1} \Big)$ converges to a symmetric stable random vector in $\mathbb{R}^{4p}$ having the following characteristic function

$$\mathbf{\Phi}_{\mathbf{t}}^p = \exp\Big\{ -2^\alpha \sigma^\alpha \sum_{j=1}^{p} \tau_{\mathbf{w_j}}(A_j^0, B_j^0, \lambda_j^0, \mu_j^0, \alpha) \Big\}, \tag{13}$$

$$\begin{aligned}
&\mathbf{w_j} = (w_{1j}(t_{1j}, t_{2j}, t_{3j}, t_{4j}), w_{2j}(t_{1j}, t_{2j}, t_{3j}, t_{4j}), w_{3j}(t_{1j}, t_{2j}, t_{3j}, t_{4j}), w_{4j}(t_{1j}, t_{2j}, t_{3j}, t_{4j})), \\
&w_{kj}(t_{1j}, t_{2j}, t_{3j}, t_{4j}) = v_k(t_{1j}, t_{2j}, t_{3j}, t_{4j}, A_j^0, B_j^0), \quad k = 1, \ldots, 4; \quad j = 1, \ldots, p.
\end{aligned} \tag{14}$$

Therefore, we have the following theorem regarding the asymptotic distribution of the LSEs of the unknown parameters, present in model (1).

Theorem 5.1. In model (1), if $\epsilon(m,n)$'s satisfy Assumption 2 and amplitudes A_j^0's and B_j^0's satisfy Assumption 4, then for $j = 1, \ldots, p$, $(\widehat{\boldsymbol{\xi}}_j - \boldsymbol{\xi}_j^0)\mathbf{D_2}^{-1}$ converges to a multivariate stable distribution in $\mathbb{R}^4$ whose characteristic function is given by $\exp\left\{-2^\alpha \sigma^\alpha \tau_{\mathbf{w_j}}(A_j^0, B_j^0, \lambda_j^0, \mu_j^0, \alpha)\right\}$; w_{kj}'s are defined in (14); $(\widehat{\boldsymbol{\xi}}_j - \boldsymbol{\xi}_j^0)\mathbf{D_2}^{-1}$ and $(\widehat{\boldsymbol{\xi}}_k - \boldsymbol{\xi}_k^0)\mathbf{D_2}^{-1}$, $j \neq k$ are asymptotically independently distributed.

According to Theorems 4.1 and 5.1, the LSEs of the frequencies λ_j and μ_j are of order $O_p(M^{-\frac{2\alpha-1}{\alpha}} N^{-\frac{\alpha-1}{\alpha}})$ and $O_p(M^{-\frac{\alpha-1}{\alpha}} N^{-\frac{2\alpha-1}{\alpha}})$, respectively, whereas the linear parameters A_i and B_i are of order $O_p(M^{-\frac{\alpha-1}{\alpha}} N^{-\frac{\alpha-1}{\alpha}})$. These orders depends on the unknown stability index α of the stable error process, which needs to be estimated.

Ideally Theorems 4.1 and 5.1 can be used for interval estimation by inverting the asymptotic joint characteristic function but is not very easy in practice. The distribution depends on limiting quantities like $\tau_{\mathbf{t}}(A^0, B^0, \lambda^0, \mu^0, \alpha)$ which are functions of the true parameters and the unknown stability index. Also for a moderate sample size, the approximation of $\tau_{\mathbf{t}}(A^0, B^0, \lambda^0, \mu^0, \alpha)$ will not be a good one. One can aim to obtain the marginal distributions of the estimators by inverting the corresponding characteristic functions. Otherwise some established bootstrap method, say percentile bootstrap or bootstrap-t can be used.

6. Numerical Experiment

In this section, we provide results of some numerical experiments based on simulation to see how the proposed estimator works for finite samples. We consider the following model;

$$y(m,n) = A\cos(m\lambda + n\mu) + B\sin(m\lambda + n\mu) + \epsilon(m,n), \tag{15}$$

with $A = B = 1.0$, $\lambda = .5$ and $\mu = .25$. $\epsilon(m,n)$'s are i.i.d. random variables distributed as symmetric $S\alpha S$ with mean zero, scale parameter $\sigma = 1.0$ and stability index $1 < \alpha < 2$. The error random variables $\epsilon(m,n)$'s are generated using the stable random number generator of Samorodnitsky and Taqqu (1994). The subroutines of Press et al. (1993) are used for optimization. Different values of the stability index α are used. We have written $\alpha = 1 + \delta$ and for simulation $\delta = .2(.9)(.1)$ are considered. Therefore, $\alpha = 1.2(1.9)(.1)$ which is according to the assumption $1 < \alpha < 2$. The sample size is fixed at $(M, N) = (20, 20)$ and $(30, 30)$. For each choice of (α, M, N), we generate a sample of size $M \times N$ and compute the LSEs of the unknown parameters, namely, A, B, λ and μ by minimizing the residual sum of squares defined in (3). We note that using linear separation technique, minimization takes place in two dimension. We replicate the procedure 1000 time and obtain the average estimates (AVEST) and the mean absolute deviations (MAD) of the LSEs of the unknown parameters over these replications. The results are reported in Tables 1 and 2 for different stability index α and sample sizes.

In section 4, we have obtained the asymptotic distribution of the LSEs as multivariate symmetric stable and ideally that can be used in interval estimation of the unknown parameters. But due to the complexity involved in the distribution, these are hard to implement in practice. The asymptotic distribution involves $\tau_{\mathbf{t}}(A^0, B^0, \lambda^0, \mu^0, \alpha)$ which is defined as $M, N \to \infty$ and for a finite sample size, say (20×20) or (30×30), the estimate $\hat{\tau}_{\mathbf{t}}(\hat{A}, \hat{B}, \hat{\lambda}, \hat{\mu}, \alpha)$ is quite unstable and in many cases very large values of M and N are required for convergence of $\dfrac{1}{MN}\displaystyle\sum_{m=1}^{M}\sum_{n=1}^{N}|K_{\mathbf{t}}(m,n)|^{\alpha}$. For this reason, we have used percentile bootstrap method for interval estimation of the different parameters as suggested by Nandi and Kundu (2010). In each replication of our experiment, we generate 1000 bootstrap resamples using the estimated parameters and then the bootstrap confidence intervals using the bootstrap quantiles at the 95% nominal level are obtained. As a result, we have 1000 intervals for each parameter from the replicated experiment. We estimate the 95% bootstrap coverage probability by calculating the proportion covering the true parameter value. We report them as COVP in Tables 1 and 2. We also report the average length of the bootstrap confidence interval as B-AVEL. So, in each table, we report the average estimate, the mean absolute deviation, and the 95% bootstrap coverage probability and the average length of the intervals.

We observe that the average estimates are quite good as they are quite close to the true parameter values. The mean absolute deviations are reasonably small and as the sample size increases, the mean absolute deviations of all the parameter estimator decrease. This has been observed in case of all parameter estimators. For a fixed sample size, as α increases, the bias and MAD decrease in general. Typically, the same trend has been observed in case of average length of bootstrap confidence intervals. When $(M, N) = (20, 20)$, in most of the cases, the bootstrap coverage probabilities do no attain the nominal level 95%. Whereas when the sample size increases to $(30, 30)$, the coverage probabilities are close to the nominal value except μ with α close to one. The asymptotic distribution suggests that rates of convergence of $\widehat{A}$, $\widehat{B}$, $\widehat{\lambda}$ and $\widehat{\mu}$ are of orders $M^{-\frac{\alpha-1}{\alpha}}N^{-\frac{\alpha-1}{\alpha}}$, $M^{-\frac{\alpha-1}{\alpha}}N^{-\frac{\alpha-1}{\alpha}}$, $M^{-\frac{2\alpha-1}{\alpha}}N^{-\frac{\alpha-1}{\alpha}}$, and $M^{-\frac{\alpha-1}{\alpha}}N^{-\frac{2\alpha-1}{\alpha}}$ respectively. These are reflected in the bootstrap intervals to some extent. Moreover, the order of the MADs approximately match the order given in the asymptotic distribution of the LSEs as expected for finite samples of moderate size.

Table 1. LSE, corresponding MAD, average length and coverage probability of the bootstrap percentile method for Model 1 when $\sigma = 1.0$ and $\alpha = 1.2,\ 1.3,\ 1.4,$ and 1.5.

		$\alpha = 1.2$		$\alpha = 1.3$		$\alpha = 1.4$		$\alpha = 1.5$	
	(M, N)	$(20, 20)$	$(30, 30)$	$(20, 20)$	$(30, 30)$	$(20, 20)$	$(30, 30)$	$(20, 20)$	$(30, 30)$
A	AVEST	.9701	1.0111	.9476	.9832	1.0124	.9879	1.0328	.9975
	MAD	.2592	.1616	.2690	.1579	.2273	.1403	.2222	.1323
	AVLEN	1.1723	.8006	1.1930	.8172	1.0674	.7828	1.0028	.7156
	COVP	.881	.936	.901	.958	.898	.967	.885	.956
B	AVEST	.9407	.9688	.9789	.9867	.9335	.9964	.9170	.9858
	MAD	.2805	.1484	.2631	.1458	.2411	.1233	.2514	.1098
	AVLEN	1.1885	.8149	1.1803	.8026	1.1574	.7736	1.1265	.7224
	COVP	.901	.970	.921	.967	.929	.989	.917	.987
λ	AVEST	.4957	.4988	.4954	.4996	.4945	.5001	.4940	.4998
	MAD	1.8692e-2	7.2754e-3	2.1294e-2	7.0024e-3	1.9069e-2	6.8101e-3	1.9192e-2	6.6807e-3
	AVLEN	7.1492e-2	3.2557e-2	8.0903e-2	3.2721e-2	6.6625e-2	3.1440e-2	6.3071e-2	2.9172e-2
	COVP	.887	.940	.872	.938	.864	.943	.847	.930
μ	AVEST	.2519	.2496	.2558	.2502	.2509	.2503	.2492	.2501
	MAD	1.8420e-2	5.6380e-3	1.7533e-2	6.6717e-3	1.4476e-2	5.6091e-3	1.4016e-2	4.9893e-3
	AVLEN	7.2552e-2	3.2483e-2	8.3907e-2	3.2661e-2	6.8435e-2	3.1465e-2	6.6091e-2	2.9137e-2
	COVP	.817	.962	.840	.939	.854	.978	.859	.973

We have analysed a single data set. The data was generated using the model given in (15) with the same values of the parameters used in simulation study. We consider the case when $\alpha = 1.9$, $\sigma = 1.0$. The 2-D image plot of the generated data without the additive error is given in Figure 1 as the first plot. The middle one in Figure 1 is the image plot of the contaminated data observed in presence of additive error. The last one in Figure 1 is the image plot of the fitted values. We observe that first plot and the last plot matches quite well. Therefore, we can infer that the proposed method extract the signal satisfactorily from the contaminated one.

Table 2. LSE, corresponding MAD, average length and coverage probability of the bootstrap percentile method for Model 1 when $\sigma = 1.0$ and $\alpha = 1.6,\ 1.7,\ 1.8,$ and 1.9.

		$\alpha = 1.6$		$\alpha = 1.7$		$\alpha = 1.8$		$\alpha = 1.9$	
	(M, N)	$(20, 20)$	$(30, 30)$	$(20, 20)$	$(30, 30)$	$(20, 20)$	$(30, 30)$	$(20, 20)$	$(30, 30)$
A	AVEST	1.0448	1.0108	1.0458	1.0064	1.0357	1.0004	1.0391	1.0035
	MAD	0.2024	.1291	.1875	.1136	.1749	.1133	.1678	.1079
	AVLEN	.9103	.6527	.8406	.6106	.8119	.5752	.7696	.5575
	COVP	.873	.948	.879	.949	.887	.941	.881	.946
B	AVEST	.9188	.9730	.9256	.9823	.9403	.9846	.9333	.9875
	MAD	.2305	.1131	.2087	.1013	.1876	9.9479e-2	.1870	8.8188e-2
	AVLEN	1.0387	.6764	.9620	.6231	.9037	.5817	.8614	.5639
	COVP	.913	.984	.916	.990	.922	.979	.909	.982
λ	AVEST	.4946	.4996	.4952	.4999	.4964	.5000	.4955	.4999
	MAD	1.7363e-2	6.4321e-3	1.5443e-2	5.6697e-3	1.4177e-2	5.6021e-3	1.3700e-2	5.7263e-3
	AVLEN	5.7447e-2	2.6975e-2	5.3149e-2	2.4945e-2	5.0745e-2	2.3350e-2	4.8467e-2	2.2639e-2
	COVP	.841	.921	.844	.918	.847	.904	.848	.890
μ	AVEST	.2485	.2496	.2482	.2496	.2483	.2498	.2485	.2498
	MAD	1.2266e-2	4.7976e-3	1.1010e-2	4.2126e-3	1.0295e-2	3.9837e-3	1.0352e-2	3.5485e-3
	AVLEN	5.9131e-2	2.6979e-2	5.4703e-2	2.4912e-	5.2096e-2	2.3368e-2	4.9569e-2	2.2594e-2
	COVP	.866	.972	.871	.982	.860	.984	.859	.991

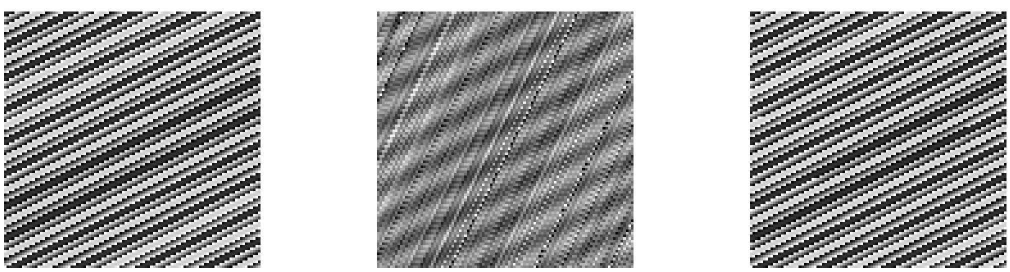

Figure 1. Image plots of the data without error (left), data with error (middle) and fitted values (right).

7. Concluding Remarks

In this paper, we consider the 2-D frequency model under the assumption of additive i.i.d. errors which are heavy tailed. We propose the LSEs and prove the strong consistency. Any distributional assumption is not required to prove the consistency. We obtain the asymptotic distribution as multivariate symmetric stable when the errors are from a symmetric stable distribution. Due to the involvement of complicated limiting quantities in the asymptotic distribution, we have propose the percentile bootstrap method for interval estimation. Although we address the problem when errors are i.i.d., the results can be extended when the errors are 2-D moving average type. Another important point is that we have not considered the problem of estimation of p. We may need to use some cross validation technique or information theoretic criterion. Further work is required in this direction.

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Appendix

Proof of Lemma 1: Define for $1 < \frac{p}{q} < 1+\delta$,

$$Z(m,n) = \begin{cases} \epsilon(m,n), & \text{if} \quad |\epsilon(m,n)| < (mn)^{\frac{p}{q(1+\delta)}}, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned}
\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} P\left[Z(m,n) \neq \epsilon(m,n)\right] &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} P\left[|\epsilon(m,n)| > (mn)^{\frac{p}{q(1+\delta)}}\right] \\
&\leq \sum_{t=1}^{\infty} \sum_{2^{t-1}\leq r<2^t} P\left[|\epsilon(m,n)| \geq r^{\frac{p}{q(1+\delta)}}\right] \\
&\leq \sum_{t=1}^{\infty} t2^t \frac{E|\epsilon(1,1)|^{1+\delta}}{2^{\frac{p}{q}(t-1)}} \leq C\sum_{t=1}^{\infty} \frac{t}{2^{(\frac{p}{q}-1)t}} < \infty.
\end{aligned}$$

Here C is a constant. Hence, $\epsilon(m,n)$ and $Z(m,n)$ are equivalent sequences.

Let $U(m,n) = Z(m,n) - E(Z(m,n))$, then for large M and N,

$$\sup_{\alpha,\beta}\left|\frac{1}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N} E(Z(m,n))\cos(m\alpha)\cos(n\beta)\right| \leq \frac{1}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N} |E(Z(m,n)| \to 0.$$

Thus, we only need to show that $\sup_{\alpha,\beta}\left|\frac{1}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N}U(m,n)\cos(m\alpha)\cos(n\beta)\right| \to 0$ a.s.
Now for fixed $\epsilon > 0$, $-\pi < \alpha, \beta < \pi$ and $0 < h \le \frac{1}{2(MN)^{\frac{p}{q(1+\delta)}}}$,

$$P\left[\left|\frac{1}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N}U(m,n)\cos(m\alpha)\cos(n\beta)\right| \ge \epsilon\right] \le 2e^{-hMN\epsilon}\prod_{m=1}^{M}\prod_{n=1}^{N}(1+2Ch^{1+\delta}),$$

since $|hU(m,n)\cos(m\alpha)\cos(n\beta)| \le \frac{1}{2}$, $e^x \le 1+x+2|x|^{1+\delta}$ for $|x| \le \frac{1}{2}$ and $E|U(m,n)|^{1+\delta} < C$, for some $C > 0$. Hence

$$2e^{-hMN\epsilon}\prod_{m=1}^{M}\prod_{n=1}^{N}(1+2Ch^{1+\delta}) = 2e^{-hMN\epsilon}(1+2Ch^{1+\delta})^{MN} \le 2e^{-hMN\epsilon+2CMNh^{1+\delta}}.$$

Now choose $h = (2(MN)^{\frac{p}{q(1+\delta)}})^{-1}$ and write $\frac{p}{q} = 1+k < 1+\delta$, then for large M and N,

$$\begin{aligned} & P\left[\left|\frac{1}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N}U(m,n)\cos(m\alpha)\cos(n\beta)\right| \ge \epsilon\right] \\ \le\ & 2e^{\left(-\frac{\epsilon}{2}(MN)^{1-\frac{p}{q(1+\delta)}}+\frac{2C}{(MN)^{\frac{p}{q}-1}}\right)} \le 2e^{\left(-\frac{\epsilon}{2}(MN)^{\frac{\delta-k}{1-\delta}}\right)}, \end{aligned}$$

Let $r = M^2N^2$, choose r points $\boldsymbol{\theta}_1 = (\alpha_1, \beta_1)$, $\boldsymbol{\theta}_2 = (\alpha_2, \beta_2)$, ..., $\boldsymbol{\theta}_r = (\alpha_r, \beta_r)$ such that for each point $\boldsymbol{\theta} = (\alpha, \beta) \in (0, \pi) \times (0, \pi)$, we have a point $\boldsymbol{\theta}_j = (\alpha_j, \beta_j)$, satisfying

$$|\alpha_j - \alpha| + |\beta_j - \beta| \le \frac{2\pi}{M^2N^2}.$$

Note that
$$\begin{aligned} & \left|\frac{1}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N}U(m,n)\left\{\cos(m\alpha)\cos(n\beta) - \cos(m\alpha_j)\cos(n\beta_j)\right\}\right| \\ \le\ & C\frac{1}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N}(MN)^{\frac{p}{q(1+\delta)}}\frac{1}{M^2N^2}[m+n] \to 0 \text{ as } M, N \to \infty. \end{aligned}$$

Therefore, for large M and N, we have

$$\begin{aligned} & P\left\{\sup_{\alpha,\beta}\left|\frac{1}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N}U(m,n)\cos(m\alpha)\cos(n\beta)\right| \ge 2\epsilon\right\} \\ \le\ & P\left\{\max_{j\le M^2N^2}\left|\frac{1}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N}U(m,n)\cos(m\alpha_j)\cos(n\beta_j)\right| \ge \epsilon\right\} \le CM^2N^2e^{-\frac{\epsilon}{2}(MN)^{\frac{\delta-k}{1-\delta}}}. \end{aligned}$$

Since $\sum_{t=1}^{\infty} t^2 e^{-\frac{\epsilon}{2}t^{\frac{\delta-k}{1-\delta}}} < \infty$, using Borel Cantelli Lemma, the result follows ∎

Proof of Theorem 3.1: In this proof, we denote $\widehat{\boldsymbol{\theta}}$ by $\widehat{\boldsymbol{\theta}}_{MN}$ to make it clear that $\widehat{\boldsymbol{\theta}}$ depends on M and N. If $\widehat{\boldsymbol{\theta}}_{MN}$ is not consistent for $\boldsymbol{\theta}^0$, then either

CASE I: for all sub-sequences $\{M_k, N_k\}$ of $\{M, N\}$, $|\widehat{A}_{MN}| + |\widehat{B}_{MN}| \to \infty$. which implies $\frac{1}{M_k N_k}\Big[Q(\widehat{\boldsymbol{\theta}}_{M_k N_k}) - Q(\boldsymbol{\theta}^0)\Big] \to \infty$. At the same time, $\widehat{\boldsymbol{\theta}}_{M_k N_k}$ is the LSE of $\boldsymbol{\theta}^0$ at $(M, N) = (M_k, N_k)$, hence $Q(\widehat{\boldsymbol{\theta}}_{M_k N_k}) - Q(\boldsymbol{\theta}^0) < 0$. This indicates to a contradiction.
CASE II:for at least one sub-sequence $\{M_k, N_k\}$ of $\{M, N\}$, $\widehat{\boldsymbol{\theta}}_{M_k N_k} \in S_{c,K}$ for some $c > 0$ and a $0 < K < \infty$. Write $\frac{1}{MN}\Big[Q(\boldsymbol{\theta}) - Q(\boldsymbol{\theta}^0)\Big] = f_1(\boldsymbol{\theta}) + f_2(\boldsymbol{\theta})$, where

$$
\begin{aligned}
f_1(\boldsymbol{\theta}) &= \frac{1}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N}\Big[A^0\cos(m\lambda^0 + n\mu^0) - A\cos(m\lambda_k + n\mu_k) \\
&\quad + B^0\sin(m\lambda^0 + n\mu^0) - B\sin(m\lambda_k + n\mu_k)\Big]^2, \\
f_2(\boldsymbol{\theta}) &= \frac{1}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N}\epsilon(m,n)\Big[A^0\cos(m\lambda^0 + n\mu^0) - A\cos(m\lambda_k + n\mu_k) \\
&\quad + B^0\sin(m\lambda^0 + n\mu^0) - B\sin(m\lambda_k + n\mu_k)\Big].
\end{aligned}
$$

Define sets $S^j_{c,K}$, $j = 1, \ldots, 4$ as follows: $S^j_{c,K} = \{\boldsymbol{\theta} : |\theta_j - \theta_j^0| > c, |A| \le K, |B| \le K\}$, where θ_j, $j = 1, \ldots, 4$ are elements of $\boldsymbol{\theta}$, that is, A, B, λ and μ and θ_j^0 is the corresponding true value. Then $S_{c,K} \subset \cup_{j=1}^4 S^j_{c,K} =$ S, say and

$$
\liminf_{M,N\to\infty}\inf_{S_{c,K}}\frac{1}{MN}\Big[Q(\boldsymbol{\theta}) - Q(\boldsymbol{\theta}^0)\Big] \ge \liminf_{M,N\to\infty}\inf_{S}\frac{1}{MN}\Big[Q(\boldsymbol{\theta}) - Q(\boldsymbol{\theta}^0)\Big].
$$

Using Lemma 1, we have $\lim_{M,N\to\infty}\sup_{\boldsymbol{\theta}\in S_{c,K}} f_2(\boldsymbol{\theta}) = 0$, a.s. Now we show in the following that

$$
\liminf_{M,N\to\infty}\inf_{S^j_{c,K}}\frac{1}{MN}\Big[Q(\boldsymbol{\theta}) - Q(\boldsymbol{\theta}^0)\Big] = \liminf_{M,N\to\infty}\inf_{S^j_{c,K}} f_1(\boldsymbol{\theta}) > 0 \ \text{ a.s. for } \ j = 1, \ldots, 4,
$$

and this would imply that $\liminf_{M,N\to\infty} \inf_{S_{c,K}} \frac{1}{MN}\Big[Q(\boldsymbol{\theta}) - Q(\boldsymbol{\theta}^0)\Big] > 0$ a.s. For $j = 1$,

$$
\begin{aligned}
& \liminf_{M,N\to\infty} \inf_{S^j_{c,K}} f_1(\boldsymbol{\theta}) \\
= \;& \liminf_{M,N\to\infty} \inf_{|A-A^0|>c} \frac{1}{MN} \sum_{m=1}^{M}\sum_{n=1}^{N} \Big[\Big\{A^0\cos(m\lambda^0 + n\mu^0) - A\cos(m\lambda_k + n\mu_k)\Big\}^2 \\
& + \Big\{B^0\sin(m\lambda^0 + n\mu^0) - B\sin(m\lambda_k + n\mu_k)\Big\}^2 + 2\Big\{A^0\cos(m\lambda^0 + n\mu^0) - \\
& A\cos(m\lambda_k + n\mu_k)\Big\}\Big\{B^0\sin(m\lambda^0 + n\mu^0) - B\sin(m\lambda_k + n\mu_k)\Big\}\Big] \\
= \;& \liminf_{M,N\to\infty} \inf_{|A-A^0|>c} \frac{1}{MN} \sum_{m=1}^{M}\sum_{n=1}^{N} \Big[\Big\{A^0\cos(m\lambda^0 + n\mu^0) - A\cos(m\lambda_k + n\mu_k)\Big\}^2 \; > \frac{1}{2}c^2 \; > 0 \text{ a.s.}
\end{aligned}
$$

Similarly, the inequality holds for other i and hence, the theorem is proved.

[1]Theoretical Statistics and Mathematics Unit, Indian Statistical Institute, 7, S.J.S. Sansanwal Marg, New Delhi - 110016, India, *E-mail*: nandi@isid.ac.in