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Quantum dynamical semigroups involving separable and entangled states

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QUANTUM DYNAMICAL SEMIGROUPS INVOLVING SEPARABLE AND ENTANGLED STATES

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Abstract. We study the evolvement or supression of separability or entanglement in its various levels in quantum dynamical semigroups. We give examples depending mainly on generalized Choi maps and unitary equivalence of a matrix to its transpose.

1. introduction

Quantum inseparability or entanglement plays a significant role in quantum communication. The concept goes back to A. Einstein, E. Schrödinger and their contemporaries way back in the 1930s. Important practical applications have been envisaged in recent years by computer scientists, mathematicians and physicists. Various necessary and sufficient conditions were given by M. Horodecki, P. Horodecki and R. Horodecki [25], A. Peres [44], R. Simon [49], for instance. B. M. Terhal and P. Horodecki [54], came up with different levels in terms of Schmidt numbers. E. Størmer [53], has strengthened and formulated the theory in the context of operator algebras. The dynamics of entanglement in continuous variable open systems has been a recent phenomenon. Particular emphasis is on Gaussian states, to mention a few, T. Yu and J. H. Eberly [58], M. N. Wolf and J. I Cirac [56], A. Isar ([28], [29], [30], [31]), G. Adesso and Animesh Datta [1].

The next section is devoted to the basics of separable and entangled states and maps as well as of quantum dynamical semigroups. It also includes a few simple new results. In the third section we study the separability and entanglement of the so-called Choi maps ([11],[12]) and their generalisations introduced and studied mainly by Seung-Hyeok Kye, Sung Je Cho and Sa Ge Lee in ([37], [10]). The have corresponding Choi matrices can have all possible ranks. We may mention that recently entangled states of low rank have been studied in detail by M. B. Ruskai and E. M. Werner [RW], L. Chen and D. M. Dokovic ([8], [9]) and others. We utilise the recent work of S. R. Garcia and J. E. Tener [19] on unitary equivalence of a matrix to its transpose in Section 4. Finally we come to our objective in the last section.

2. Basics of separable and entangled states and maps and quantum dynamical semigroups

2.1. **Notation.** Let $\mathbb{N}$ be the set of natural numbers $1, 2, \ldots$; $\mathbb{Z}$ that of integers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let $\mathbb{R}$ be the field of real numbers and $\mathbb{C}$ that of complex numbers. For $n \in \mathbb{N}$, let M_n denote the C^*-algebra of $n \times n$ complex matrices. For $1 \leq j,\ k \leq n$, let E_{jk} be the elementary $n \times n$ matrix with 1 at the (j, k)-th place and zero elsewhere. Let I_n denote the identity matrix present in M_n. For $A \in M_n$, A^t and A^*(or $A^\dagger$) denote the transpose of A and the adjoint of A respectively. Let τ denote the transpose map on M_n to itself taking A to A^t. Let M_n^+ denote the positive cone in M_n, viz., the set of positive semi-definite matrices A. A density matrix is an A in M_n^+ with $\operatorname{tr} A = 1$, where tr denotes the trace. A density ρ gives rise to a positive functional ω_ρ on M_n with $\omega_\rho(I_n) = 1$, also called a state, given by $\omega_\rho(X) = \operatorname{tr}(\rho X)$ for X in M_n. In fact, the correspondence $\rho \to \omega_\rho$ is bijective and we often use the name state for ρ as well.

2.2. **Separable and entangled states.** For $n, m \in \mathbb{N}$, we consider the tensor product $H = \mathbb{C}^m \otimes \mathbb{C}^n$. A state on H can be viewed as an $m \times n$ state and can be represented as $\rho = [A_{jk}]_{1 \leq j,k \leq m}$ with $n \times n$ matrices A_{jk} acting on $\mathbb{C}^n$. The state ρ may be called a mixed state.
(i) The density ρ is said to be *separable* if it is in $M_m^+ \otimes M_n^+$. Choi [11] gave examples of non-separable states and also necessary conditions for ρ to be separable; for example, its partial transpose $\rho^t = [A_{jk}^t]$ is a state. In the literature the condition is known as the *Peres test* because of significant work by A. Peres [44] or *Positive Partial transpose* (PPT) and, thus, states that satisfy it can be called Peres states or *PPTS.*
(ii) Non-separable states are called *entangled states.* The *PPTES* is also used for an entangled PPT state in the literature with interesting applications to Quantum Information theory appearing mainly in Physics Journals [15], [22], [25], [26] [27], [35], [46], [50]. For an expository account of this and further developments one may consult ([24], [32], [39], [40],[43]).

2.3. **Separability à la Horodecki et al.** M. Horodecki, P. Horodecki and R. Horodecki [25] provided necessary and sufficient conditions for the separability of mixed states and gave examples to illustrate them.
(i) *Theorem* ([25], Theorem 2) : Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces of finite dimension and ρ a state acting on $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ i.e., ρ is a linear operator acting on $\mathcal{H}$ with $\operatorname{tr} \rho = 1$ and $\operatorname{tr} \rho P \geq 0$ for any projection P. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ denote the set of linear operators acting on $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. Then ρ is *separable* (i.e., can be written or approximated in

the trace norm by the states of the form $\rho = \sum_{i=1}^{k} p_i \rho_i \otimes \widetilde{\rho}_i$ where ρ_i and $\widetilde{\rho}_i$ are states on $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively) if and only if for any positive (linear) map $\Lambda : \mathcal{A}_2 \to \mathcal{A}_1$ the operator $(I \otimes \Lambda)\rho$ is positive.
(ii)([25], Remark on p.5) says that one can put $\widetilde{\Lambda} \otimes \Lambda$ or $\widetilde{\Lambda} \otimes I$ instead of $I \otimes \Lambda$ (involving any positive $\widetilde{\Lambda} : \mathcal{A}_1 \to \mathcal{A}_2$, $\Lambda : \mathcal{A}_2 \to \mathcal{A}_1$). The same applies to the PPT condition.
(iii) Next, ([25], Theorem 3) can be reworded as: a state acting on $\mathbb{C}^2 \otimes \mathbb{C}^2$ or $\mathbb{C}^2 \otimes \mathbb{C}^3$ is separable if and only if it is PPT. The proof uses results of E. Størmer [51] and S. Woronowicz [57].

2.4. **Entanglement breaking channels, Separable and entangled maps.** We refer mainly to M. Horodecki, P. W. Shor and Ruskai[26], Størmer [53] for this subsection.
(i) Horodecki, Shor and Ruskai [26] study entanglement breaking channels. A *quantum channel* is a stochastic map, i.e., a map on M_n to itself which is both completely positive and trace preserving.
(a) A. S. Holevo [24] introduced channels of the form $\varphi(\rho) = \sum_k R_k \operatorname{Tr}(F_k \rho)$, where each R_k is a density matrix and $\{F_k\}$ form a positive operator valued measure POVM. The expression for φ is called the *Holevo form* in [26].
(b) ([24], [26], Definition 1) A stochastic map φ is called *entanglement breaking* if $(Id \otimes \varphi)$ A is separable for any density matrix A, i.e., any entangled density matrix A is mapped to a separable one.
(c) For $m, n \in \mathbb{N}$ and a linear map φ on M_n to M_m, the *Choi matrix* C_φ for φ is $C_\varphi = \sum_{j,k} E_{j,k} \otimes \varphi(E_{j,k}) = (Id \otimes \varphi)\rho \in M_n \otimes M_n$, where $\frac{1}{n}\rho$ is the 1-dimensional projection $\frac{1}{n} \sum_{j,k} E_{jk} \otimes E_{jk}$, the so-called maximally entangled state. M. D. Choi ([11], [12], [13]) proved that φ is completely positive if and only if C_φ is positive. Physicists usually use $\frac{1}{n} C_\varphi$ for a trace-preserving completely positive map φ and call it a Jamiolkowski state, (See for instance, [48], [56]) following A. Jamiolkowski [33].

(d) A part of ([26], Theorem 4) says that the following are equivalent for a channel φ.
(α) φ is entanglement breaking.
(β) φ has the Holevo form with F_k positive definite.
(γ) $\frac{1}{n} C_\varphi$ is separable.
(δ) $\psi \circ \varphi$ is completely positive for all positivity preserving maps ψ.
($\in$) $\varphi \circ \Lambda$ is completely positive for all positivity preserving maps Λ.

(ii) ([53],§1) Let $\mathcal{A}$ be an operator system, i.e., a norm-closed self-adjoint linear space $\mathcal{A}$ of bounded operators on a Hilbert space $\mathcal{K}$ containing the identity. Let $\mathcal{H}$ be a Hilbert space and $\mathcal{B}(\mathcal{H})$, its operator algebra and $\mathcal{T}(\mathcal{H})$, the space of the trace class operators on $\mathcal{H}$. Let τ be the transpose map on $\mathcal{B}(\mathcal{H})$ (respectively $\mathcal{B}(\mathcal{K})$) with respect to some orthonormal basis for $\mathcal{H}$ (respectively $\mathcal{K}$). At times for $a \in \mathcal{B}(\mathcal{H})$ or $\mathcal{B}(\mathcal{K})$, $\tau(a)$ will be denoted by a^t. The BW-topology on the space of bounded linear maps on $\mathcal{A}$ to $\mathcal{B}(\mathcal{H})$ is the topology of bounded pointwise weak convergence, i.e., a net (φ_ν) converges to φ if it is uniformly bounded, and $\varphi_\nu(a) \longrightarrow \varphi(a)$ weakly for all $a \in \mathcal{A}$. Let $S(\mathcal{H})$ be the BW-closed cone generated by maps of the form

$$x \to \sum_{j=1}^{n} \omega_j(x) a_j$$

where ω_j is a normal state on $\mathcal{B}(\mathcal{H})$ and $a_j \in$ the positive cone $\mathcal{B}(\mathcal{H})^+$. Here, $\mathcal{A} = \mathcal{B}(\mathcal{H})$.

The theory of completely positive maps is now a folklore. One may consult any standard book containing the topic; we mention some sources referred to here ([2], [4], [7], [11], [12], [13], [18], [41], [51], [52], [57]). M. B. Ruskai, S. Szarek and E. Werner [47] give an interesting analysis in the simplest set-up of 2×2 matrices with applications to Quantum Information theory using Pauli matrices.

For the sake of convenience we recall some conditions for a linear map φ on $\mathcal{A}$ to $\mathcal{B} = \mathcal{B}(\mathcal{H})$. Here $r \in \mathbb{N}$, and φ is a $*$-*map* in the sense that $\varphi(x^*) = \varphi(x)^*$ for $x \in \mathcal{A}$.

(a) The map φ is said to be r-*positive* if the map $\varphi_r = \varphi \otimes I : \mathcal{A} \otimes M_r \to \mathcal{B} \otimes M_r$ is positive.
(b) The map φ is said to be *completely positive* if φ is r-positive for all $r \in \mathbb{N}$.
(c) The map φ is said to be r-*copositive (respectively, completely co-positive)*, if $\tau \circ \varphi$ is r-positive (respectively, completely positive).
(d) The map φ is said to be a *Schwarz* map if

$$\varphi(x^*x) \geq \varphi(x^*)\varphi(x) \text{ for } x \in \mathcal{A} \text{ with } x^*x \in \mathcal{A}$$

(e) The map φ is said to be r-*Schwarz* if φ_r is a Schwarz map i.e.,

$$\begin{aligned} \varphi_r\left([x_{jk}]^*[x_{jk}]\right) \quad \geq \quad & \varphi_r\left([x_{jk}]^*\right)\varphi_r\left([x_{jk}]\right) \\ & \text{for } x_{jk} \in \mathcal{A}, 1 \leq j,k \leq r \text{ with } x_{jk}^* x_{pq} \in \mathcal{A} \\ & \text{ for } 1 \leq j,k,p,q \leq r. \end{aligned}$$

(f) For a C^*-algebra $\mathcal{A}$, and φ unital in the sense that φ take the identity of φ to that of $\mathcal{B}(\mathcal{H})$, φ is 2-positive implies that φ is a Schwarz map. As a consequence, a unital φ

is completely positive if and only if φ is r-Schwarz for each r.
(g) For a unital φ, the inequality in (d) above is satisfied for normal elements x when φ is positive.
(h) Many more assorted inequalities hold for positive maps (see [12], for instance).
(i) For φ satisfying any condition as in (a) to (e), ψ completely positive on $\mathcal{B}(\mathcal{H})$ to itself, and Λ completely positive on $\mathcal{A}$ to itself, the maps $\varphi \circ \Lambda$ and $\psi \circ \varphi$ satisfy the corresponding condition.
(j) Similar terminology as in (c) above applies to other conditions like (d) and (e).

(iii) ([53], Lemma 1) sets up an isometric isomorphism $\varphi \to \widetilde{\varphi}$ between the set $\mathcal{B}(\mathcal{A}, \mathcal{B}(\mathcal{H}))$ of bounded linear maps of $\mathcal{A}$ into $\mathcal{B}(\mathcal{H})$ and the dual $(\mathcal{A}\widehat{\otimes}\mathcal{T}(\mathcal{H}))^*$ of the projective tensor product of $\mathcal{A}$ and $\mathcal{T}(\mathcal{H})$ given by

$$\widetilde{\varphi}(a \otimes b) = \mathrm{Tr}\,(\varphi(a) b^t)$$

where Tr denotes the usual trace on $\mathcal{B}(\mathcal{H})$ taking the value 1 on minimal projections. Furthermore, φ is a positive linear operator if and only if $\widetilde{\varphi}$ is positive on the cone $\mathcal{A}^+\widehat{\otimes}\mathcal{T}(H)^+$ generated by operators of the form $a \otimes b$ with a and b positive.

(iv) As noted in [53], p. 2305, it follows from ([52], Theorem 3.2) that φ is completely positive if and only if $\widetilde{\varphi}$ is positive on the cone $(\mathcal{A}\widehat{\otimes}\mathcal{T}(\mathcal{H}))^+$, the closure of the positive operators in the algebraic tensor product $\mathcal{A} \odot \mathcal{T}(\mathcal{H})$.

(v) A positive linear functional ρ on $\mathcal{A}\widehat{\otimes}\mathcal{T}(\mathcal{H})$ is said to be *separable* if it belongs to the norm closure of positive sums of states of the form $\sigma \otimes \omega$ where σ is a state of $\mathcal{A}$ and ω a normal state of $\mathcal{B}(\mathcal{H})$. Otherwise ρ is called *entangled*.

(vi) A part of ([53], Theorem 2) says that the following are equivalent for a $\varphi \in \mathcal{B}(\mathcal{A}, \mathcal{B}(\mathcal{H}))$.

(a) $\widetilde{\varphi}$ is a separable positive linear functional.
(b) φ is a BW-limit of maps of the form $x \to \sum_{j=1}^{n} \omega_j(x) b_j$ with ω_j a state of $\mathcal{A}$ and $b_j \in \mathcal{B}(\mathcal{H})^+$.

(vii) **Definition** A completely positive map φ in $\mathcal{B}(\mathcal{A}, \mathcal{B}(\mathcal{H}))$ will be called *separable* (respectively, *entangled)* if $\widetilde{\varphi}$ is so. A separable map may also be called entanglement breaking, if we like.

(viii) ([53], Corollary 3) can now be reworded as : Let $\mathcal{H}$ be separable and $\varphi \in \mathcal{B}(\mathcal{A}, \mathcal{B}(\mathcal{H}))$ be positive. If $\varphi(\mathcal{A})$ is contained in an abelian C^*-algebra then φ is separable.

(ix) We shall say a positive linear functional ρ on $\mathcal{A}\widehat{\otimes}\mathcal{T}(\mathcal{H})$ is *PPT* (i.e. satisfies the Peres condition) if $\rho \circ (Id \otimes \tau))$ is positive. In line with (ii) above $\varphi \in \mathcal{B}(\mathcal{A}, \mathcal{B}(\mathcal{H}))$ will be said to be *PPT* if $\widetilde{\varphi}$ is so. ([53], Proposition 4) can now be interpreted as: φ is PPT if and only if φ is both completely positive and completely co-positive.

We are now ready to prove a simple result in line with item (i)(d) above.

Theorem 2.5 Let $\varphi \in \mathcal{B}(\mathcal{A}, \mathcal{B}(\mathcal{H}))$ be a separable map. Then for any positive bounded map ψ on $\mathcal{B}(\mathcal{H})$ to itself and any positive unital map Λ on $\mathcal{A}$ to itself, the maps $\psi \circ \varphi$ and $\varphi \circ \Lambda$ are both separable.

Proof. We note that for a state ω of $\mathcal{A}$ and $b \in \mathcal{B}(\mathcal{H})^+$, $\omega \circ \Lambda$ is a state of $\mathcal{A}$ and $\psi(b) \in \mathcal{B}(\mathcal{H})^+$. We can now apply the item 2.4 (vi) above for φ and conclude that maps $\varphi \circ \Lambda$ and $\psi \circ \varphi$ satisfy the condition (b) in 2.4 (vi). By 2.4 (vi) we obtain that $\varphi \circ \Lambda$ and $\psi \circ \varphi$ are separable.

Corollary 2.6 Let φ be a completely positive map on $\mathcal{A}$ to $\mathcal{B}(\mathcal{H})$. If φ is separable then for each positive bounded map ψ on $\mathcal{B}(\mathcal{H})$ to itself and each positive unital map Λ on $\mathcal{A}$ to itself, $\psi \circ \varphi$ and $\varphi \circ \Lambda$ are completely positive.

2.7 Horodecki's-Størmer Theorem. Let $m, n \in \mathbb{N}$, $\mathcal{A} = M_n$, $\mathcal{H} = \mathcal{C}^m$, $\varphi \in \mathcal{B}(A, \mathcal{B}(\mathcal{H}))$, and C_φ the Choi matrix for φ. The map $\varphi^t = \tau \circ \varphi \circ \tau$ (where τ is the transpose map in either M_n or M_m) is completely positive if and only if φ is so. ([53], Lemma 5) says that C_{φ^t} is the density matrix for $\widetilde{\varphi}$.

Størmer continues with his study and gives, amongst other things, his infinite-dimensional extension of Horodecki's Theorem, which we may call Horodecki's-Størmer Theorem, and methods to construct PPTES.

2.8 Pure Product states and Schmidt number (i) P. Horodecki [27] proved that a separable state on the Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ (with $\mathcal{H}_1$ and $\mathcal{H}_2$ both finite-dimensional) can be written as a convex combination of N pure product states with $N \leq (\dim \mathcal{H})^2$ and gave a new separability criterion in terms of the range of the density matrix. This was carried further in different ways by several mathematicians (cf. [15], [22], [34], [35], [42], [54], [55]).

(ii) K. R. Parthasarathy [42] called a subspace of $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_r$ containing no non-zero product vector of the form $u_1 \otimes u_2 \otimes \cdots \otimes u_r$ to be *completely entangled* and determined the maximal dimension of such a space. He also introduced a more delicate notion of perfectly entangled subspace for a multipartial quantum system. The notion of completely entangled subspaces is related to notions of unextendible product bases and uncompletable product bases, which are well studied by D. P. Di Vincenzo et al [16] and further studied by N. Alon and L. Lovász [3], A. O. Pittenger [45], B. V. R. Bhat [6], L. Skowronek [50] and others. They have been used to construct entangled PPT densities.

(iii) B. M. Terhal and P. Horodecki [54] extended the notion of the Schmidt rank of a pure state to the domain of bipartite density matrices. To motivate our next notions, we quote their characterization viz. Theorem 1 [54]:
Let ρ be a density matrix on $\mathcal{H}_n \otimes \mathcal{H}_n$, i.e., $\mathbb{C}^n \otimes \mathbb{C}^n$. The density matrix has the Schmidt number at least $r+1$ if and only if there exists an r-positive linear map $\Lambda : M_n \to M_n$ such that $(I \otimes \Lambda_r)(\rho) \not\geq 0$.

Definition 2.9 Let $\mathcal{A}$, $\mathcal{B}(\mathcal{H})$ etc. be as in subsection 2.4 and $\varphi \in \mathcal{B}(\mathcal{A}, \mathcal{B}(\mathcal{H}))$ be a completely positive map. We say it has *Schmidt number at least* $r+1$ if either there exists an r-positive linear map $\psi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ such that $\psi \circ \varphi$ is not completely positive or there exists an r-positive linear map $\Lambda : \mathcal{A} \to \mathcal{A}$ such that $\varphi \circ \Lambda$ is not completely positive.

2.10 Quantum dynamical semigroups We may refer to any standard source for this folklore material, particularly ([14], [17], [18] [21], [38], [41]) mentioned in the list of references, if we like, rather than original sources.

(i) Algebraically speaking, a dynamical system is a family $(T(t))_{t\geq 0}$ or, for short $(T_t)_{t\geq 0}$, of mappings on a set $\mathcal{X}$ satisfying

$$T(t+s) = T(t)T(s) \text{ for all } t, s \geq 0$$

$$T(0) = Id.$$

In fact even if we just confine our attention to the first condition, $T_0^2 = T_0$, the range $\mathcal{R}_0$ of T_0 contains the range $\mathcal{R}_t$ of T_t for all t, and T_0 restricted to $\mathcal{R}_0$ is the identity. So we may replace $\mathcal{X}$ by $\mathcal{R}_0$, and then the second condition holds for $(S_t)_{t\geq 0}$, where $S_t = T_t|\mathcal{R}_0$. Then $T_t = S_t T_0 = T_t T_0 = T_0 T_t$ and $S_t S_s = S_s S_t = S_{t+s}$. Thus, $(S_t)_{t\geq 0}$ is a dynamical system and we call $(T_t)_{t\geq 0}$ a T_0-*constricted dynamical system.*

(ii) Usually $\mathcal{X}$ is taken to be a Banach space, T_t a bounded linear operator on $\mathcal{X}$ for each t and the system to be strongly continuous on $\mathcal{X}$. Then $(T_t)_{t\geq 0}$ is called a *strongly continuous* (*one parameter*) *semigroup* or a C_0-*semigroup*. Again, if we relax the condition $T_0 = Id$, we call $(T_t)_{t\geq 0}$ a T_0-*constricted* C_0-*semigroup*. We note that the continuity of T_0 and the fact that $T_0 x = x$ for x in $\mathcal{R}_0$ forces $T_0 x = x$ for x in the closure $\bar{\mathcal{R}}_0$ of $\mathcal{R}_0$. This, in turn, gives that $\bar{\mathcal{R}}_0 \subset \mathcal{R}_0$. Hence $\mathcal{R}_0$ is closed and, therefore, a Banach space.

(iii) When $\mathcal{X}$ is an operator system $\mathcal{A}$, and maps T_t satisfy conditions like those in the item 2.4 (ii) (a) to (e) above, we term a C_0-semigroup $(T_t)_{t\geq 0}$ as a *quantum dynamical semigroup or system.* In practice, T_t's are all taken to be completely positive and $\mathcal{A}$ to be M_n or $\mathcal{B}(\mathcal{H})$. Once again, the term T_0-constricted quantum dynamical semigroup will be used when we relax the condition $T_0 = Id$.

(iv) For a C_0-semigroup $(T_t)_{t\geq 0}$, the infinitesimal generator L is the operator which has the domain

$$D(L) = \left\{ x \in \mathcal{X} : \lim_{t\to 0+} \frac{1}{t}(T_t x - x) \text{ exists} \right\}$$

and is given by $Lx = \lim\limits_{t\to 0+} \frac{1}{t}(T_t x - x)$ for x in $D(L)$. Then L is a closed and densely-defined linear operator that determines the semigroup uniquely.

(v) For an $A \in \mathcal{B}(\mathcal{X})$, $T_t = \exp(tA)$, L coincides with A. For this reason T_t as in (iv) above is written as $\exp(tL) = e^{tL}$ as well.

(vi) It follows from the proof of and the Proposition itself on p.73 [17] that if there exists some $t_0 > 0$ such that $T(t_0)$ is invertible, then

(a) for $0 \leq t < t_0$, $T(t_0) = T(t_0 - t)T(t) = T(t)T(t_0 - t)$ and for $t = nt_0 + s$ for $n \in \mathbb{N}$, $s \in [0, t_0)$, $T(t) = T(t_0)^n T(s)$, and therefore, $T(t)$ is invertible for all $t \geq 0$; $T(t)^{-1} = T(t_0 - t)T(t_0)^{-1}$ for $0 \leq t < t_0$ and $T(t)^{-1} = T(s)^{-1}T(t_0)^{-n}$ for $t = nt_0 + s$ for $n \in \mathbb{N}$, $s \in [0, t_0)$;

(b) $(T(t))_{t\geq 0}$ can be embedded in a group $(T(t))_{t\in\mathbb{R}}$ on $\mathcal{X}$.

Theorem 2.11 Let $(T_t)_{t\geq 0}$ be a quantum dynamical system of completely positive maps. If there exists some $t_0 > 0$ such that $T(t_0)$ is invertible and $T(t_0)^{-1}$ satisfies any of the conditions 2.4 (iv) (a) to (c) then each $T(t)^{-1}$ satisfies the corresponding condition.

Proof. This is obvious from 2.10 (vi)(a) above.

2.12 Generator of a quantum dynamical semigroup When T_t's satisfy any of the conditions in 2.4 (ii) (a) to (e) or corresponding "co" parts as indicated in 2.4(ii) (j) above, L satisfies a corresponding variant of the condition. Fundamental theoretical work in this direction is by V. Gorini, A. Kossakowski and E. C. G. Sudarashan [21] and G. Lindblad [38] though history can be traced back to specific irreversible processes or quantum stochastic processes of open systems by many like R. V. Kadison or E. B. Davies. For further basic developments one can see [14], [18] and [41].

3. Separability and entanglement of Generalized Choi maps

3.1. **A foliation.** Let D_n be the linear span of $\{E_{jj} : 1 \leq j \leq n\}$ and F_n the linear span of $\{E_{jk} : 1 \leq j \neq k \leq n\}$. We note that as a linear space $M_n = D_n \oplus F_n$. Also any linear map Λ on M_n to itself can be expressed in the form $\begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{bmatrix}$ where $\Lambda_{11} : D_n \to D_n$, $\Lambda_{12} : F_n \to D_n$, $\Lambda_{21} : D_n \to F_n$ and $\Lambda_{22} : F_n \to F_n$ are linear maps.

(i) Let C_Λ be the so-called Choi matrix of map Λ. It is given by the block matrix $[\Lambda(E_{jk})]$ written as an $n^2 \times n^2$ matrix with entries in $\mathbb{C}$, in fact. The diagonal of C_Λ is same as the diagonal of the block matrix with $\Lambda_{11}(E_{jj})$ at the jj^{th} block. As a consequence, $\operatorname{tr}(C_\Lambda) = \sum_{j=1}^{n} \operatorname{tr}\Lambda_{11}(E_{jj}) = \operatorname{tr}\Lambda_{11}(I_n) = \operatorname{tr}\Lambda(I_n)$. So C_Λ is a density matrix if and only if Λ is completely positive with $\operatorname{tr}\Lambda_{11}(I_n) = 1$. See ([11], [12], [13]) for more details.

(ii) We consider the class $\mathcal{L}$ of maps Λ with $\Lambda_{12} = 0$ and $\Lambda_{21} = 0$ and write $\Lambda = \Lambda_1 \oplus \Lambda_2$ with $\Lambda_1 = \Lambda_{11}$ and $\Lambda_2 = \Lambda_{22}$. Addition and product of maps in $\mathcal{L}$ is component-wise and as a consequence, for $\Lambda \in \mathcal{L}$, $e^\Lambda = e^{\Lambda_1} \oplus e^{\Lambda_2}$. A large number of examples in the study of positive, k-positive and completely positive maps, of dynamical semigroups and of separability, entanglement and Schmidt number for density matrices are of this form. One may observe this tendency in [10], [11], [12], [13], [15], [25], [27], [36], [37], [46], [47], [54], for instance. Quite often Λ_2 is just in the one-dimensional linear space spanned by $\mathcal{I}_{F_n}$, the identity operator on F_n, or else in the two dimensional algebra generated by $\mathcal{I}_{F_n}$ and the restriction τ_{F_n} of the transpose map τ on M_n. Also Λ_1's are usually taken to be upper (or lower) triangular matrices (cf.[36]) or matrices with rows being just permutations of each other ([10], [12]). We test such maps for more properties like separability, entanglement and Schmidt numbers. We begin with a few easy consequences for the sake of motivation.

3.2. **Generalized Choi maps.**

(i) For $a, b, c \in \mathbb{C}$, let

$$D(a,b,c) = \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix}.$$

(a) The set $\mathcal{D} = \{D(a,b,c) : a,b,c \in \mathbb{C}\}$ is a commutative semigroup with identity $D(1,0,0) = I_3$ simply because $D(a',b',c')\, D(a,b,c) = D(a'a+b'c+c'b, c'c+a'b+b'a, b'b+c'a+a'c)$. Further $\mathcal{D} \cap GL(3,\mathbb{C})$ is a subgroup of the general linear group $GL(3,\mathbb{C})$. To see this it is enough to note that

$$\begin{aligned} & D(a,b,c)D(a^2-bc, c^2-ab, b^2-ac) \\ = \;\; & (a^3+b^3+c^3-3abc)I_3 = \det D(a,b,c)I_3. \end{aligned}$$

(b) For $a,b,c \in \mathbb{R}_+ = [0,\infty)$, the map $\Phi[a,b,c]$ defined on p.214 [10] is the same as $D(a-1,b,c) \oplus (-\mathcal{I}_{F_3})$. We prefer to consider for $a,b,c \in \mathbb{R}_+$, the variants

$$\rho[a,b,c] = \Phi[a+1,b,c] = D(a,b,c) \oplus (-\mathcal{I}_{F_3})$$

and generalizations,

$$\begin{aligned} \rho[a,b,c,d] &= D(a,b,c) \oplus d\mathcal{I}_{F_3}, \quad \text{and} \\ \tau[a,b,c,d] &= D(a,b,c) \oplus d\,\tau_{F_3} = \rho[a,b,c,d]\,\tau = \tau\rho[a,b,c,d]. \end{aligned}$$

(c) We note that $\rho[a,b,c,d]$ is unital if and only if $a+b+c=1$ if and only if $\tau[a,b,c,d]$ is unital. Also $\rho[a,b,c,d]$ is trace-preserving if and only if $a+b+c=1$ if and only if $\tau[a,b,c,d]$ is trace-preserving.

The map $\rho[a,b,c,d]$ is a $*$-map if and only if $a,b,c,d \in \mathbb{R}$ if and only if $\tau[a,b,c,d]$ is a $*$-map.

Finally, if $\rho[a,b,c,d]$ or $\tau[a,b,c,d]$ is a positive map then a,b,c are all non-negative simply because the image of E_{11} is $\begin{bmatrix} a & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & b \end{bmatrix}$.

(ii) We note that $\rho[1,0,\mu]$ with $\mu \geq 1$ is the same as the map Φ in ([12], Appendix B, Example) and it is positive but not decomposable. S.-H. Kye [37] defined and studied another generalisation of $\rho[1,0,\mu]$ which we present in a bit different notation in line with the above.

For non-negative real numbers a, c_1, c_2, c_3 let

$$T(a, c_1, c_2, c_3) = \begin{bmatrix} a & 0 & c_1 \\ c_2 & a & 0 \\ 0 & c_3 & a \end{bmatrix}$$

and $\theta[a, c_1, c_2, c_3] = T(a, c_1, c_2, c_3) \oplus (-\mathcal{I}_{F_3})$. Then $\theta[a, c_1, c_2, c_3]$ is the same as $\Theta(a + 1, c_1, c_2, c_3)$ of [37] and $\theta[1, \mu, \mu, \mu] = \rho[1, 0, \mu]$ and, thus, is the same as Φ of ([12], Appendix B, Example).

For the sake of convenience we recall the results in [10] and [37] in our notation in the following remarks.

Remark 3.3 Let $a, b, c \in \mathbb{R}_+$, $d \in \mathbb{R}$.

(i)By Theorem 2.1 [10], the map $\rho[a, b, c]$ is positive if and only if $a+b+c \geq 2$ together with $bc \geq (1-a)^2$ in case $0 \leq a \leq 1$.

(ii) (a) By [10], Lemma 3.1, the map $\rho[a, b, c, d]$ is completely positive if and only if $a \geq d$ and $a \geq -2d$.
(b) In particular, $\rho[a, b, c]$ is completely positive if and only if $a \geq 2$. This is Proposition 3.2 [10].
(iii) (a) By [10] Lemma 3.1, second part, $\tau[a, b, c, d]$ is completely positive if and only if $bc \geq d^2$. As a consequence, $\rho[a, b, c, d]$ is positive if $bc \geq d^2$.
(b) In particular, $\tau \circ \rho[a, b, c]$ is completely positive (i.e. $\rho[a, b, c]$ is completely copositive) if and only if $bc \geq 1$. This is a part of Proposition 3.3 of [10].
(iv) (a) Theorem 3.9 [10] gives that for $0 \leq a < 2$, $\rho[a, b, c]$ is decomposable if and only if $bc \geq (1 - \frac{a}{2})^2$.
(b) Let $a = 0$. Then (a) above combined with (ii) (b) and (iii)(b) gives that $\rho[0, b, c]$ is not completely positive and it is decomposable if and only if $bc \geq 1$ if and only if it is completely copositive.
(c) Let $0 < a < 2$ and $bc < 1$. Then (a) above combined with (ii)(b) and (iii)(b) gives that $\rho[a, b, c]$ is neither completely positive nor completely copositive, but is nevertheless decomposable if and only if $bc \geq (1 - \frac{a}{2})^2$.

Remark 3.4 (i) Theorem 2.1 of [37] gives that $\theta[a, c_1, c_2, c_3]$ is positive if and only if $a \geq 1$ and $c_1 c_2 c_3 \geq (2-a)^3$.

(ii) By Theorem 2.3 [37], $\theta[a, c_1, c_2, c_3]$ is completely positive if and only if it is 2-positive if and only if $a \geq 2$.

(iii) Let $1 \le a \le 2$ and $c_1c_2c_3 \ge (2-a)^3$. By ([37], Theorem 3.2) $\theta[a, c_1, c_2, c_3]$ is atomic in the sense that it cannot be expressed as the sum of a 2-positive and 2-copositive map.

3.5 Corollary 2.6 combined with the above remarks give us a multitude of completely positive maps and states that are PPT, non-PPT or PPTE. We illustrate it by recording a few special non-clumsy cases.

Theorem 3.6 Let $a, b, c, d \in \mathbb{R}$ with $a, b, c \ge 0$. Let $\varphi = \rho[a, b, c, d]$ and $\psi = \tau[a, b, c, d]$.

(i) If $d = 0$, then $\varphi = \psi$ is a separable map.

(ii) Let $d > 0$.

  (a) If $a \ge d$ but $bc < d^2$, then φ is a non-PPT completely positive map.

  (b) If $bc \ge d^2$ but $a < d$, then ψ is a non-PPT completely positive map.

  (c) If $a \ge d$, $bc \ge d^2$, then φ and ψ are PPT maps.

  (d) If $a \ge d$, $bc \ge d^2$ but $a + b < 2d$ or $a + c < 2d$, then φ and ψ are PPTE maps.

  In particular, it is so if $a = 1 = d$, $0 < b < 1$, $c = \frac{1}{b}$.

(iii) Let $d < 0$.

  (a) If $a \ge -2d = 2|d|$ but $bc < d^2$, then φ is a non-PPT completely positive map.

  (b) If $bc \ge d^2$ but $a < 2|d|$, then ψ is a non-PPT completely positive map.

  (c) If $a \ge 2|d|$, $bc \ge d^2$, then φ and ψ are PPT maps.

Proof Items (i), (ii) (a), (ii)(b), (ii)(c), (iii)(a), (iii)(b) follow immediately from Remark 3.3 above. For item (ii)(d) we consider the map $\xi = \rho[a', b', c'] \equiv \rho[a', b', c', -1]$ with a', b', c' to be suitably chosen yet to be specified. We have

$$\xi\varphi = \rho[a'a + b'c + c'b, c'c + a'b + b'a, b'b + c'a + a'c, -d].$$

By Remark 3.3 (ii)(a), $\xi\varphi$ is completely positive if and only if $a'a + b'c + c'b \ge 2d$. If $a + b < 2d$ then we take $a' = 1 = c'$ and $b' = 0$. Then $a' + b' + c' = 2$, $b'c' = 0 = (1 - a')^2$ and $a'a+b'c+c'b = a+b < 2d$. So by Remark 3.3 (i) and (ii)(a) respectively, ξ is positive and $\xi\varphi$ is not completely positive. Similarly, in case $a + c < 2d$, we take $a' = 1 = b'$ and $c' = 0$ and conclude that ξ is positive but $\xi\varphi$ is not completely positive. By Corollary 2.6, φ and ψ are not separable. So φ and ψ are PPTE maps.

Theorem 3.7 Let $a, b, c, d \in \mathbb{R}$ with $a, b, c \ge 0$ and $a + b + c = \frac{1}{3}$. Let $\varphi = \rho[a, b, c, d]$, $\psi = \tau[a, b, c, d]$. Let $A = C_\varphi$ and $B = C_\psi$ be the Choi matrices of φ and ψ respectively.

(i) If $d = 0$ then $A = B$ is a separable state.

(ii) Let $d > 0$.

(a) If $a \geq d$ but $bc < d^2$, then A is a non-PPT state.

(b) If $bc \geq d^2$ but $a < d$, then B is a non-PPT state.

(c) If $a \geq d$, $bc \geq d^2$ then A and B are PPT states.

(d) If $a \geq d$, $bc \geq d^2$ but $a + b < 2d$ or $a + c < 2d$ then A and B are PPTES.

In particular, this is true if for an arbitrary $0 < \beta < 1$, we take $\lambda = \frac{1}{3}\beta/(\beta^2 + \beta + 1)$, $a = \lambda = d$, $b = \lambda\beta$, $c = \frac{\lambda}{\beta}$.

(e) If $d < a < 2d$ and $2(b + c) < 2d - a$ then A has Schmidt number > 2. This is equivalent to requiring $\frac{2}{9} < d \leq \frac{1}{3}$ and then putting $b + c = \frac{1}{3} - a$.

(iii) Let $d < 0$.

(a) If $a \geq -2d = 2|d|$ but $bc < d^2$ then A is a non-PPT state.

(b) If $bc \geq d^2$ but $a < 2|d|$, then B is a non-PPT state.

(c) If $a \geq 2|d|$, $bc \geq d^2$, then A and B are PPT states.

Proof This follows immediately from Theorem 3.6 above.

3.8 The range and the rank of $C_{\rho[a,b,c,d]}$, $C_{\tau[a,b,c,d]}$ and $C_{\theta[a,c_1,c_2,c_3]}$

(i) We first note that

$$C_{\rho[a,b,c,d]} = \begin{bmatrix} a & 0 & 0 & 0 & d & 0 & 0 & 0 & d \\ 0 & c & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b & 0 & 0 & 0 & 0 & 0 \\ d & 0 & 0 & 0 & a & 0 & 0 & 0 & d \\ 0 & 0 & 0 & 0 & 0 & c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & b & 0 \\ d & 0 & 0 & 0 & d & 0 & 0 & 0 & a \end{bmatrix},$$

$$C_{\tau[a,b,c,d]} = \begin{bmatrix} a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & c & 0 & d & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 & 0 & d & 0 & 0 \\ 0 & d & 0 & b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c & 0 & d & 0 \\ 0 & 0 & d & 0 & 0 & 0 & c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & d & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a \end{bmatrix}.$$

(a) So the range of $C_{\rho[a,b,c,d]}$ is the linear span of

$$\begin{aligned} &ae_1 + de_5 + de_9, \quad && ce_2,\ be_3,\ be_4,\\ &de_1 + ae_5 + de_9, && ce_6,\ ce_7,\ be_8 \text{ and}\\ &de_1 + de_5 + ae_9. \end{aligned}$$

Here e_j, $1 \leq j \leq 9$ is the arrangement of product vectors $e^3_p \otimes e^3_q$ in the lexicographic order, $(e^3_p : p = 1, 2, 3)$ being the standard ordered basis of $\mathbb{C}^3$.

Now the matrix $\begin{bmatrix} a & d & d \\ d & a & d \\ d & d & a \end{bmatrix}$ has determinant $= (a-d)^2(a+2d)$.

Therefore $C_{\rho[a,b,c,d]}$ attains all ranks from 0 to 9 depending on the values of a, b, c, d in $\mathbb{C}$. For instance, for a, b, c all non-zero, the matrix $C_{\rho[a,b,c,d]}$ has rank 7 if $a = d$, rank 8 if $a = -2d$ and rank 9 if $a \neq d$ and $a \neq -2d$.

(b) Next, the range of $C_{\tau[a,b,c,d]}$ is the linear span of ae_1, $ce_2 + de_4$, $be_3 + de_7$, $de_2 + be_4$, ae_5, $ce_6 + de_8$, $de_3 + ce_7$, $de_6 + be_8$ and ae_9. The matrices $\begin{bmatrix} c & d \\ d & b \end{bmatrix}$ and $\begin{bmatrix} b & d \\ d & c \end{bmatrix}$ both have determinant $= bc - d^2$. So the matrix $C_{\tau[a,b,c,d]}$ has rank 9 if $a \neq 0 \neq bc - d^2$, rank 6 if $a \neq 0 = bc - d^2$ and $d \neq 0$, rank 6 if $a \neq 0 = bc - d^2 = d$ and $b^2 + c^2 \neq 0$, and rank 3 if $a \neq 0 = b = c = d$.

(ii) We note that

$$C_{\theta[a.c_1,c_2,c_3]} = \begin{bmatrix} a & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 \\ 0 & c_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & a & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & c_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & a \end{bmatrix}$$

So the range of $C_{\theta[a,c_1,c_2,c_3]}$ is the linear span of $ae_1 - e_5 - e_9$, c_2e_2, $-e_1 + ae_5 - e_9$, c_3e_6, c_1e_7 and $-e_1 - e_5 + ae_9$. The determinant of the matrix $\begin{bmatrix} a & -1 & -1 \\ -1 & a & -1 \\ -1 & -1 & a \end{bmatrix}$ is $(a+1)^2(a-2)$.

Therefore, $C_{\theta[a,c,_1,c_2,c_3]}$ attains all ranks from 1 to 6 depending on values of a, c_1, c_2, c_3.

For instance, for c_1, c_2, c_3 all non-zero, $C_{\theta[a,c_1,c_2,c_3]}$ has rank 6 if $-1 \neq a \neq 2$, and rank 5 if $a = 2$.

3.9 Schmidt number of related densities Let $a, b, c, c_1, c_2, c_3 \geq 0$ and $d \in \mathbb{R}$ with $d \neq 0$. Let $A = C_{\rho[a,b,c,d]}$, $B = C_{\tau[a,b,c,d]}$ and $K = C_{\theta[a,c_1,c_2,c_3]}$. We refer to their expanded form in 3.8.

(i) In view of Remark 3.3(ii)(a) or Theorem 3.7, A is a density matrix if and only if $a + b + c = \frac{1}{3}$ and $a \geq \max\{d, -2d\}$. We now consider only this case. Then $a > 0$, and, therefore, the Schmidt number of A is ≥ 1. Further, from the expanded form of A, it is clear that to determine the Schmidt number of A, it is enough to look at its only non-trivial sub-block $A_1 = \begin{bmatrix} a & d & d \\ d & a & d \\ d & d & a \end{bmatrix}$ acting on the span of e_1, e_5 and e_9. It has eigenvalues $a + 2d$, $a - d$, $a - d$. Since $d \neq 0$, we have $a + 2d \neq a - d$. Let $\xi = \frac{1}{\sqrt{3}}(e_1 + e_5 + e_9)$. Then $A_1 = (a + 2d)P_\xi + (a - d)P$, where P_ξ is the projection determined by ξ and P, the (orthogonal) projection on $\xi^\perp$. Then $\xi^\perp$ is the linear span of $e_1 - e_5$ and $e_5 - e_9$ and it contains no non-zero product vectors. So we have the following conclusions:

(a) If $a + 2d = 0$, then the Schmidt number of A is 2.

If $a + 2d \neq 0$, then the Schmidt number of A is 3.

(ii) By Remark 3.3(ii)(a) or Theorem 3.7, we have that B is a density matrix if and only if $a + b + c = \frac{1}{3}$ and $bc \geq d^2$. We now consider only this case. Then $b > 0$, $c > 0$. So the Schmidt number of B is ≥ 1. Further, it is clear from the expanded form of B that to determine the Schmidt number of B, it is enough to consider its non-trivial sub-blocks $B_1 = \begin{bmatrix} c & d \\ d & b \end{bmatrix}$, $B_2 = \begin{bmatrix} b & d \\ d & c \end{bmatrix}$, $B_3 = \begin{bmatrix} c & d \\ d & b \end{bmatrix}$, acting on linear spans L_1, L_2, L_3 respectively of pairs of product vectors (e_2, e_4), (e_3, e_7) and (e_6, e_8) respectively. Because $d \neq 0$, B_1 has distinct eigenvalues. Further, any corresponding eigenvector has the Schmidt rank 2. Hence B has the Schmidt number 2.

(iii) By Remark 3.4(ii), K is positive if and only if $a \geq 2$. We now consider only this case. Let $\lambda = \frac{1}{(3a+c_1+c_2+c_3)}$ and $H = \lambda K$. Then H is a density matrix and its Schmidt number is ≥ 1. As argued in (i) and (ii) above, it is enough to consider the non-trivial sub-block $\lambda \begin{bmatrix} a & -1 & -1 \\ -1 & a & -1 \\ -1 & -1 & a \end{bmatrix}$ of H acting on the linear span of e_1, e_5 and e_9. And as in (i) above we have the following results

(a) If $a = 2$, then the Schmidt number of H is 2.

(b) If $a > 2$, then the Schmidt number of H is 3.

4. Peres condition and unitary equivalence of a matrix to its transpose

We follow the notation and terminology of S. R. Garcia and J. E. Tener [19] who obtained a canonical decomposition for complex matrices T which are UET, i.e., unitarily equivalent to their transpose $T^t(UET)$.

Remark 4.1 We collect a few facts from [19] for ready reference.
(i) [19, §1]. In his problem book ([23], Pr. 159) Halmos asks whether every square matrix is UET and in his discussion gives the counterexample $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$, which is not UET. Every Toeplitz matrix is UET via the permutation matrix which reverses the order of the standard basic vectors.
(ii) [19, Theorem 1.1] A matrix T in M_n is UET if and only if it is unitarily equivalent to a direct sum of (some of the summands may be absent):
(a) irreducible complex symmetric matrices (CSMs),
(b) irreducible skew-Hamiltonian matrices (SHMs) (such matrices are necessarily 8×8 or larger, a SHM is a $2d \times 2d$ block matrix of the form $\begin{pmatrix} A & B \\ D & A^t \end{pmatrix}$ with $B^t = -B$ and $D^t = -D$),
(c) $2d \times 2d$ blocks of the form $\begin{pmatrix} A & 0 \\ 0 & A^t \end{pmatrix}$ where A is irreducible and neither unitarily equivalent to a complex symmetric matrix (UECSM) nor unitarily equivalent to a skew-Hamiltonian matrix (UESHM) (such matrices are necessarily 6×6 or larger).

Moreover, the unitary orbits of the three classes described above are pairwise disjoint.

(iii) ([19], Corollary 2.3). If T is UET and has order $n \times n$ with $n \leq 7$, then T is UECSM.
(iv) ([19], 8.3 and 8.4) S is UET if and only if S is unitarily equivalent to a matrix T that satisfies $TQ = QT^t$, where Q is a unitary matrix of the special form (some of the blocks may be absent and empty blocks are all zero):

$$Q = \left(\begin{array}{c|c|cc|c|cc} Q_+ & & & & & & \\ \hline & Q_- & & & & & \\ \hline & & 0 & \lambda_1 X_1^t & & & \\ & & X_1 & 0 & & & \\ \hline & & & & \ddots & & \\ \hline & & & & & 0 & \lambda_r X_r^t \\ & & & & & X_r & 0 \end{array}\right),$$

where (a) $Q_+ = Q_+^t$ is complex symmetric and unitary,
(b) $Q_- = -Q_-^t$ is skewsymmetric and unitary,
(c) $\lambda_i \neq \pm 1$ and X_i is unitary for $i = 1, 2, \ldots, r$.
(v) ([19],8.5) Given Q as in (iv) above, T is as in (iv) above if and only if

$$T = \left(\begin{array}{c|c|cc|c|cc} T_+ & & & & & & \\ & T_- & & & & & \\ \hline & & A_1 & 0 & & & \\ & & 0 & X_1 A_1^t X_1^* & & & \\ \hline & & & & \ddots & & \\ \hline & & & & & A_r & 0 \\ & & & & & 0 & X_r A_r^t X_r^* \end{array}\right),$$

where
(a) $T_+ = Q_+ T_+^t Q_+^*$ (such a T_+ is UECSM),
(b) $T_- = Q_- T_-^t Q_-^*$ (such a T_- is UESHM),
(c) $A_1, \ldots, A_r$ are arbitrary.

In fact this is the final step of the proof of (ii) in [19].

Definition 4.2 A tuple $(Y_1, \ldots, Y_s)$ of $n \times n$ matrices is said to be *collectively unitarily equivalent to the respective transposes (CUET)* if there is a unitary U with $Y_j = U Y_j^t U^*$ for $1 \leq j \leq s$.

Remarks 4.3 (i) W. B. Arverson [4, Lemma A.3.4], gives that $\left(\begin{pmatrix} 0 & \lambda & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & \mu \\ 0 & 1 & 0 \\ 0 & -\lambda & 0 \end{pmatrix}\right)$ is not CUET where λ is a non-real complex number and μ is a complex number with $|\mu| = (1 + |\lambda|^2)^{\frac{1}{2}}$.
(ii) Remark 4.1 tells us how to construct CUET tuples viz., choose a Q as in item 4.1 (iv) and then Y_j's as in item 4.1 (v) by varying T_+, T_-, A_k for $1 \leq k \leq r$.

Theorem 4.4 Let $[A_{jk}]$ be a positive block matrix such that $(A_{jk} : 1 \leq j, k \leq n)$ is CUET. Then $[A^t_{jk}]$ is positive.

Proof. There is a unitary matrix U such that $A_{jk} = UA^t_{jk}U^*$ for $1 \leq j, k \leq n$. Let $\widetilde{U}$ be the block matrix $[\delta_{jk}U]$, with $\delta_{jk} = 0$ for $j \neq k$ and 1 for $j = k$. Then $\widetilde{U}$ is unitary and $[A^t_{jk}] = \widetilde{U}^*[A_{jk}]\widetilde{U}$. So $[A^t_{jk}]$ is positive.

Construction 4.5 Remark 4.3 together with Theorem 4.4 tell us how to construct PPT matrices.

Step 1: Let $n \geq 2$ and put $m = \frac{n(n+1)}{2}$. Use Remark 4.3 (ii) to construct a CUET m-tuple $(Y_j : 1 \leq j \leq m)$ with $Y_j \geq 0$ for $1 \leq j \leq n$ (we may take all Y_j, $1 \leq j \leq n$ to be zero, for instance). For $1 \leq j \leq m$, we have $Y_j = QY^t_jQ^*$ and, therefore $Y^*_j = QY^{*t}_jQ^*$. We set $B_{jj} = Y_j$ for $1 \leq j \leq n$, arrange Y_j for $n+1 \leq j \leq \frac{n(n+1)}{2}$ as B_{pq}, $1 \leq p < q \leq n$ and take $B_{qp} = B^*_{pq}$ for $1 \leq p < q \leq n$. Thus, we obtain a block matrix $B = [B_{jk}]$ which is Hermitian and $\{B_{jk} : 1 \leq j, k \leq n\}$ is CUET.

Step 2: The set $\{a \in \mathbb{R} : B + a\ I_{n^2} \geq 0\}$ is an interval $[a_0, \infty)$ for some $a_0 \in \mathbb{R}$. We take any a in this interval and set $A = B + a\ I_{n^2}$ i.e., $A_{jk} = B_{jk}$ for $j \neq k$, whereas $A_{jj} = B_{jj} + aI_n$ for $1 \leq j, k \leq n$. Then $\{A_{jk} : 1 \leq j, k \leq n\}$ is CUET and $A \geq 0$. So we can apply Theorem 4.4 to conclude that A is a PPT matrix.

It is my pleasure to thank my students Priyanka Grover and Tanvi Jain. Priyanka came to discuss UET in some other context and Tanvi found a former version of [19] on the internet for that context.

5. Quantum dynamical semigroups involving separable and entangled states

Let $\mathcal{H}$ be a Hilbert space and τ the transpose map on $\mathcal{B}(\mathcal{H})$ with respect to some orthonormal basis for $\mathcal{H}$. Let $*$ or $\dagger$ be the adjoint map on $\mathcal{B}(\mathcal{H})$ that takes x to x^*. Let $\mathcal{X}$ be a linear subspace of $\mathcal{B}(\mathcal{H})$ which is closed under τ as well as $*$. We shall consider C_0-semigroups $(T_t)_{t\geq 0}$ as well as T_0-constricted C_0-semigroups $(T_t)_{t\geq 0}$ of operators on $\mathcal{X}$ to itself.

We begin with a few examples.

5.1. **Examples.**

(i) This is modelled on Størmer's Example 8.13 [51] and is in a foliated form with $\Lambda_{t1} = \begin{bmatrix} 1 & 0 \\ 1-e^{-t} & e^{-t} \end{bmatrix}$, $\Lambda_{t2} = e^{-\frac{t}{2}}\mathcal{I}_{F_2}$ and $\Lambda_t = \Lambda_{t1} \oplus \Lambda_{t2}$ in the notation of item 3.1. It is a non-PPT quantum dynamical semigroup.

(ii) If we are interested in separable maps we have to do away with the condition $T_0 = \mathcal{I}d$, which we now do.

This example is modelled on the example of the two spin $\frac{1}{2}$-states given by Horodecki et al [25]. It is in a foliated form with $\Lambda_1^{p,a,b} = \begin{bmatrix} pa^2 & (1-p)b^2 \\ (1-p)a^2 & pb^2 \end{bmatrix}$ and $\Lambda_2^{p,a,b} = \begin{bmatrix} pab & (1-p)ab \\ (1-p)ab & pab \end{bmatrix}$ with $0 \le p \le 1$, $a > 0$, $b > 0$ and $\Lambda^{p,a,b} = \Lambda_1^{p,a,b} \oplus \Lambda_2^{p,a,b}$.

Taking Pauli matrices $\sigma_0 = I_2$, $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ as a basis, the map $\Lambda^{p,a,b}$ has the simple form

$$\begin{pmatrix} \frac{a^2+b^2}{2} & 0 & 0 & \frac{a^2+b^2}{2} \\ 0 & ab & 0 & 0 \\ 0 & 0 & 2(\frac{1}{2}-p)ab & 0 \\ (p-\frac{1}{2})(a^2-b^2) & 0 & 0 & (p-\frac{1}{2})(a^2+b^2) \end{pmatrix}.$$

As noted by Horodecki et al, it is a separable map if and only if $p = \frac{1}{2}$ and, in that case, the matrix becomes $\begin{pmatrix} \frac{a^2+b^2}{2} & 0 & 0 & \frac{a^2-b^2}{2} \\ 0 & ab & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and thus the range is the commutative algebra spanned by σ_0 and σ_1. Taking $a^2 + b^2 = 1$, $a = \cos\theta$, $b = \sin\theta$, $0 < \theta \le \frac{\pi}{4}$, $u = \sin 2\theta$ we have the semigroup

$$T_t = \left(\frac{1}{2}\right)^t \begin{bmatrix} 1 & 0 & 0 & \sqrt{1-u^2} \\ 0 & u^t & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad t \ge 0.$$

We note that T_0 is the idempotent

$$\begin{bmatrix} 1 & 0 & 0 & \sqrt{1-u^2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We may consider the variant (with $a^2+b^2=2$) for $t \geq 0$,

$$S_t = \begin{bmatrix} 1 & 0 & 0 & \sqrt{1-u^2} \\ 0 & u^t & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

All these take I_2 to itself and also $S_0 = T_0$.

Theorem 5.2(Trichotomy) Let $(T_t)_{t\geq 0}$ be a T_0 constricted quantum dynamical semigroup. Then one and only one of the following holds.

(i) For each $t \geq 0$, T_t is separable.
(ii) There exists $t_0 > 0$ such that T_t is entangled for $t < t_0$ but T_t is separable for $t \geq t_0$.
(iii) For each $t \geq 0$, T_t is entangled.
Moreover, (i) holds if and only if T_0 is separable.

Proof The three conditions are mutually exclusive. If (i) and (iii) do not hold, then the set $S = \{t \geq 0 : T_t \text{ is separable}\} \neq \phi$. If $t \in S$ then for $s > t$, $T_s = T_{s-t}T_t$. By Theorem 2.5 T_s is separable. So S is an interval of the form (t_0, ∞) or $[t_0, \infty)$. By item 2.4 (vi) and the condition of strong continuity on $(T_t)_{t\geq 0,}$ T_{t_0} is separable. So $S = [t_0, \infty)$. Since (i) does not hold, we have $t_0 > 0$. Thus (ii) holds.

Now suppose T_0 is separable. Then the set $S = \{t \geq 0 : T_t \text{ is separable}\}$ contains 0. As seen above, if $S \neq \phi$, then $S = [t_0, \infty)$ for some $t_0 \in [0, \infty]$. So $S = [0, \infty)$ i.e. (i) holds.

Definition 5.3 The space $\mathcal{X}$ will be said to be *normal* if for $x \in \mathcal{X}$, $\#\{x, x^*, xx^*, x^*x\} \cap \mathcal{X} \leq 3$. In other words, each $x \in \mathcal{X}$ is either normal or else at most one of xx^* and x^*x is in $\mathcal{X}$.

Proposition 5.4 Let φ be an idempotent $*$-map on $\mathcal{X}$. If φ is co-Schwarz then the range of φ is normal.

Proof Let $\mathcal{Y} = \varphi(\mathcal{X})$. Since φ is a $*$-map for $x \in \mathcal{Y}$, x^* is in $\mathcal{Y}$. Since $\varphi^2 = \varphi$ we have $\varphi|\mathcal{Y} = \mathcal{I}d_{\mathcal{Y}}$. Let, if possible, there exist $y \in \mathcal{Y}$ with y^*y, $yy^* \in \mathcal{Y}$. Since φ is co-Schwarz, we have $\tau\varphi(y^*y) \geq \tau\varphi(y^*)\tau\varphi(y)$, i.e. $\tau(y^*y) \geq \tau(y^*)\tau(y)$. So $\tau(y)\tau(y^*) \geq \tau(y^*)\tau(y)$. We may interchange the role of y and y^* and get $\tau(y^*)\tau(y) \geq \tau(y)\tau(y^*)$. So $\tau(yy^*) = \tau(y^*y)$. Therefore $yy^* = y^*y$.

Remark 5.5 Let $(T_t)_{t\geq 0}$ be a T_0-constricted quantum dynamical semigroup.

(i) If the range $\mathcal{R}_0$ of T_0 is normal, then the range $\mathcal{R}_t$ of each T_t is normal simply because $\mathcal{R}_t \subset \mathcal{R}_0$ for $t > 0$.

(ii) One can have more Trichotomy results by replacing "separable" by
(a) PPT, or
(b) has Schmidt rank $\leq r$, or
(c) has normal range
and then "entangled" by the corresponding negations like non-PPT, has Schmidt rank $> r$ and has non-normal range.

(iii) In fact, the first condition in any such Trichotomy holds if and only if it holds for T_0. By 2.4(viii), it holds if $\mathcal{R}_0$ is contained in an abelian C^* algebra acting on a separable Hilbert space $\mathcal{H}$.

(iv) A non-commutative C^*-algebra is not normal. So for an interesting theory, we can give up the condition (i) of Trichotomy and instead take $T_0 = \mathcal{I}d$.

Theorem 5.6 Let $\mathcal{X}$ be a non-commutative C^*-algebra and $(T(t))_{t\geq 0}$ be a quantum dynamical semigroup of unital completely positive maps. If for some $t_0 > 0$, $T(t_0)^{-1}$ exists and is a Schwarz map, then for each $t > 0$, $T(t)$ is non-PPT.

Proof We refer to item 2.10 (vi)(a) as for the proof of Theorem 2.11. We use the fact that the product of two Schwarz maps is a Schwarz map. For $0 < t < t_0$, $T(t)^{-1} = T(t_0 - t)(T(t_0))^{-1}$, and therefore, $T(t)^{-1}$ is a Schwarz map. Also for $n \in \mathbb{N}$, $0 < s < t_0$, $t = nt_0 + s$, $T(t)^{-1} = T(s)^{-1}(T(t_0)^{-1})^n$, and therefore, $T(t)^{-1}$ is a Schwarz map. Let, if possible, for some $t > 0$, $T(t)$ be PPT. Then $\tau T(t)$ is completely positive. So $\tau = \tau T(t)(T(t))^{-1}$ is a Schwarz map, which is not so because $\mathcal{X}$ is non-commutative.

We now illustrate results in this section with examples of generalized Choi maps discussed in the third section above.

Example 5.7(i) This may be thought of as continuation of §3. We begin by recalling relevant details, which are well-known from the theory of circulant matrices (cf. [7], [5], [20]).

(ii) Let $\alpha \in \mathbb{R}$, $\beta \in \mathbb{C}$. Then

$$\begin{aligned} D(\alpha,\beta,\bar\beta) &= \alpha I_3 + \beta(E_{12}+E_{23}+E_{31}) + \bar\beta(E_{21}+E_{32}+E_{13}) \\ &= \alpha I_3 + \beta L + \bar\beta L^*, \text{ where} \\ L &= E_{12}+E_{23}+E_{31}. \end{aligned}$$

We note that $L^2 = L$, $LL^* = L^*L = I_3$. So L is a unitary matrix with eigenvalues $1,\omega,\omega^2$ and is expressible as $L = W\mathrm{Diag}(1,\omega,\omega^2)W^*$ with $W = \frac{1}{\sqrt3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}$.
Here $\omega = -\frac12 + \frac{\sqrt3}{2}i$, a cube root of unity.
So $D(\alpha,\beta,\bar\beta) = W\mathrm{Diag}(\alpha+\beta+\bar\beta, \alpha+\beta\omega+\bar\beta\omega^2, \alpha+\beta\omega^2+\bar\beta\omega)W^*$.

(iii) For $a,b,c \in \mathbb{C}$, $D(a,b,c)^* = D(\bar a,\bar c,\bar b)$ and thus, $D(a,b,c)$ is normal. Further $D_1 = \frac12(D+D^*) = D(\mathrm{Re}\,a,\beta,\bar\beta)$ with $\beta = \frac12(b+\bar c)$, and $D_2 = \frac12(D-D^*) = D(\mathrm{Im}\,a,\gamma,\bar\gamma)$ with $\gamma = \frac{1}{2i}(b-\bar c)$. So by (ii),

$$D(a,b,c) = D_1 + iD_2$$

$$\begin{aligned} &= W\mathrm{Diag}(a+(\beta+i\gamma)+(\bar\beta+i\bar\gamma), a+(\beta+i\gamma)\omega+(\bar\beta+i\bar\gamma)\omega^2, \\ &\quad a+(\beta+i\gamma)\omega^2+(\bar\beta+i\bar\gamma)\omega^2)W^* \\ &= W\mathrm{Diag}(a+b+c, a+b\omega+c\omega^2, a+b\omega^2+c\omega)W^*. \end{aligned}$$

(iv) For $n \in \mathbb{N}$, $a,b,c \in \mathbb{C}$
$(D(a,b,c))^n = W\,\mathrm{Diag}((a+b+c)^n, (a+bw+cw^2)^n, (a+bw^2+cw)^n))W^*$, and therefore, for $t \in \mathbb{C}$,
$e^{tD(a,b,c,)} = W\,\mathrm{Diag}(e^{t(a+b+c)}, e^{t(a+bw+cw)}, e^{t(a+bw^2+cw^2)})W^*$. We note that all these matrices are in $GL(3,\mathbb{C})$ and $(e^{tD(a+b+c)})^{-1} = e^{-tD(a+b+c)} = e^{tD(-a,-b,-c)}$ for $a,b,c,t \in \mathbb{C}$.

(v) Let $a,b,c,\,t \in \mathbb{C}$.
By (iii)
$e^{tD(a,b,c)} = D(\boldsymbol{a}(t),\boldsymbol{b}(t),\boldsymbol{c}(t))$ with

$$\begin{pmatrix} \boldsymbol{a}(t) \\ \boldsymbol{b}(t) \\ \boldsymbol{c}(t) \end{pmatrix} = \frac{1}{\sqrt3}W^* \begin{pmatrix} e^{t(a+b+c)} \\ e^{t(a+b\omega+c\omega^2)} \\ e^{t(a+b\omega^2+c\omega)} \end{pmatrix}.$$

Therefore,

$$\begin{aligned}
\boldsymbol{a}(t) &= \frac{1}{3}\left[e^{t(a+b+c)} + e^{t(a+b\omega+c\omega^2)} + e^{t(a+b\omega^2+c\omega)}\right] \\
\boldsymbol{b}(t) &= \frac{1}{3}\left[e^{t(a+b+c)} + \omega^2\, e^{t(a+b\omega+c\omega^2)} + \omega\ e^{t(a+b\omega^2+c\omega)}\right] \text{ and} \\
\boldsymbol{c}(t) &= \frac{1}{3}\left[e^{t(a+b+c)} + \omega\ e^{t(a+b\omega+c\omega^2)} + \omega^2\ e^{t(a+b\omega^2+c\omega)}\right]
\end{aligned}$$

We set $\boldsymbol{d}(t) = e^{td}$. We note that $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$ are all entire functions and $\boldsymbol{a}(0) = 1 = \boldsymbol{d}(0)$ whereas $\boldsymbol{b}(0) = 0 = \boldsymbol{c}(0)$. Further, for a, b, c, d, t all real, $\boldsymbol{a}(t)$, $\boldsymbol{b}(t)$, $\boldsymbol{c}(t)$, $\boldsymbol{d}(t)$ are all real.

(vi) Let $a, b, c, d \in \mathbb{C}$. For $t \in \mathbb{C}$,

$$\begin{aligned}
\boldsymbol{a}'(t) &= \frac{1}{3}[(a+b+c)e^{t(a+b+c)} + (a+b\omega+c\omega^2)e^{t(a+b\omega+c\omega^2)}. \\
&\quad +(a+b\omega^2+c\omega)e^{t(a+b\omega^2+c\omega)}] \\
\boldsymbol{b}'(t) &= \frac{1}{3}[(a+b+c)e^{t(a+b+c)} + \omega^2\,(a+b\omega+c\omega^2)e^{t(a+b\omega+c\omega^2)} \\
&\quad +\omega\,(a+b\omega^2+c\omega)e^{t(a+b\omega^2+c\omega)}] \\
\boldsymbol{c}'(t) &= \frac{1}{3}[(a+b+c)e^{t(a+b+c)} + \omega\,(a+b\omega+c\omega^2)e^{t(a+b\omega+c\omega^2)} \\
&\quad +\omega^2\,(a+b\omega^2+c\omega)e^{t(a+b\omega^2+c\omega)}] \text{ and} \\
\boldsymbol{d}'(t) &= d\boldsymbol{e}^{td}.
\end{aligned}$$

In particular, $\boldsymbol{a}'(0) = a$, $\boldsymbol{b}'(0) = b$, $\boldsymbol{c}'(0) = c$, and $\boldsymbol{d}'(0) = d$. As a consequence, if $\boldsymbol{a}(t_n)$ (respectively $\boldsymbol{b}(t_n)$, $\boldsymbol{c}(t_n)$) are all real for a real sequence (t_n) convergent to zero then a (respectively b, c) is real.

Thus in view of the last line of (v) we may say that $\boldsymbol{a}(t_n)$, $\boldsymbol{b}(t_n)$, $\boldsymbol{c}(t_n)$ are all real for a real sequence (t_n) convergent to zero if and only if a, b, c are all real if and only $\boldsymbol{a}(t)$, $\boldsymbol{b}(t)$, $\boldsymbol{c}(t)$ are all real for all real t. A similar statement holds for the function $\boldsymbol{d}$ as well.

(vii) Let $a, b, c, d \in \mathbb{C}$ and set $\rho = \rho[a, b, c, d]$. Then by (v) above, for $t \in \mathbb{C}$, $\boldsymbol{\rho}(t) = e^{t\rho}$ coincides with $\rho[\boldsymbol{a}(t), \boldsymbol{b}(t), \boldsymbol{c}(t), \boldsymbol{d}(t)]$. We first note that in view of (iv) above, each $\boldsymbol{\rho}(t)$ is a bijective map on M_3 to itself. Further, for $b = 0 = c$, $\boldsymbol{a}(t) = e^{at}$ and $\boldsymbol{b}(t) = 0 = \boldsymbol{c}(t)$, so that

$$\boldsymbol{\rho}(t) = e^{ta}\mathcal{I}d_{D_n} \oplus e^{td}\mathcal{I}d_{F_n} \text{ for } t \in \mathbb{C}.$$

(a) Item (i) and (vi) may be combined to give: $\boldsymbol{\rho}(t_n)$ are all $*$-maps for a real sequence (t_n) convergent to 0 if and only if a, b, c, d are all real if and only if $\boldsymbol{\rho}(t)$

are all $*$-maps for all real t.

From now onwards we consider only real a, b, c, d, t.

(b) By 3.2(i) (c) and (v) above, for any $t \neq 0$, $\boldsymbol{\rho}(t)$ is unital if and only if $a+b+c=0$ and in that case all $\boldsymbol{\rho}(t)$ are unital as t varies in $\mathbb{R}$. Similar statements hold with unital replaced by trace-preserving.

(viii) Let a, b, c, d be real. Set $u = \frac{1}{2}(b+c)$, $v = \frac{1}{2}(b-c)$. Then

$$\begin{aligned} a + b\omega + c\omega^2 &= a - u + i\sqrt{3}v, \\ a + b\omega^2 + c\omega &= = a - u - i\sqrt{3}v. \end{aligned}$$

Let $t \in \mathbb{R}$. Then

$$\begin{aligned} \boldsymbol{a}(t) &= \frac{1}{3}\left[e^{t(a+b+c)} + e^{t(a+b\omega+c\omega^2)} + e^{t(a+b\omega^2+c\omega)}\right] \\ &= \frac{1}{3}\left[e^{t(a+2u)} + e^{t(a-u+i\sqrt{3}v)} + e^{(a-u-i\sqrt{3}v)}\right] \\ &= \frac{1}{3}e^{t(a-u)}\left[e^{3tu} + 2\cos(\sqrt{3}vt)\right], \\ \boldsymbol{b}(t) &= \frac{1}{3}\left[e^{t(a+2u)} + e^{t(a-u+i\sqrt{3}v)-\frac{2}{3}\pi i} + e^{t(a-u-i\sqrt{3}v)+\frac{2}{3}\pi i}\right] \\ &= \frac{1}{3}e^{t(a-u)}\left[e^{3tu} + 2\cos(\sqrt{3}vt - \frac{2}{3}\pi)\right], \\ \boldsymbol{c}(t) &= \frac{1}{3}e^{t(a-u)}\left[e^{3tu} + 2\cos(\sqrt{3}vt + \frac{2}{3}\pi)\right], \\ \boldsymbol{d}(t) &= e^{td} > 0. \end{aligned}$$

We recall from (v) above that $\boldsymbol{a}(0) = 1$, $\boldsymbol{b}(0) = 0 = c(0)$, $\boldsymbol{d}(0) = 1$.

If ρ is a positive map then by 3.2(i)(c) $\boldsymbol{a}(t)$, $\boldsymbol{b}(t)$, $\boldsymbol{c}(t)$, are ≥ 0.

We begin by finding out when $\boldsymbol{a}(t)$, $\boldsymbol{b}(t)$, $\boldsymbol{c}(t)$, are ≥ 0 and then go on to find conditions under which $\boldsymbol{\rho}(t)$ is completely positive, PPT, separable etc.

(ix) We can argue as in (vi) above and have that if $\boldsymbol{b}(t_n)$ (respectively $\boldsymbol{c}(t_n)$) are all non-negative for a sequence (t_n) in $(0, \infty)$ convergent to zero, then b (respectively c) is ≥ 0. So from now onwards we take $b, c \geq 0$.

(x) Let $b = 0 = c$. Then $\boldsymbol{a}(t) = e^{ta}$ and $\boldsymbol{d}(t) = e^{td} > 0$ for all $t \in \mathbb{R}$ whereas $\boldsymbol{b}(t) = 0 = \boldsymbol{c}(t)$ for all $t \in \mathbb{R}$. By Remark 3.3 (iii) (a) no $\boldsymbol{\rho}(t)$ is completely copositive. By Remark 3.3 (ii)(a), $\boldsymbol{\rho}(t)$ is completely positive if and only if $e^{ta} \geq e^{td}$ if and only if $ta \geq td$.

(a) For $a = d, \{\boldsymbol{\rho}(t) : t \in \mathbb{R}\} \equiv \{e^{td} Id_{M_3} : t \in \mathbb{R}\}$ is a group of completely positive maps that are all non-PPT, which illustrates Theorem 5.6. For $a = d = 0$ it is the trivial group $\{Id_{M_3}\}$ for $t \in \mathbb{R}$.

(b) The family $\{\boldsymbol{\rho}(t) : t \geq 0\}$ is a quantum dynamical semigroup if and only if $a \geq d$ and all the maps are non-PPT and therefore, entangled. This illustrates the condition (iii) of the Trichotomy in Theorem 5.2 and Remark 5.5 (ii) (a).

(c) Let $a = 0 > d$. Then by (vii)(b) above each $\boldsymbol{\rho}(t)$ is unital and trace-preserving. It follows from (3.9)(i)(b) that the Choi matrix $\frac{1}{3}C_{\underline{\rho}(t)}$ is a density with Schmidt number 3.

(d) It follows from 3.8 (i)(a) that for $a = d$, the Choi matrix $C_{\boldsymbol{\rho}(t)}$ has rank 1 for all $t \in \mathbb{R}$ and, on the other hand, for $a > d$, $t > 0$, the Choi matrix $C_{\boldsymbol{\rho}(t)}$ has rank 3.

(xi) Let $(b, c) \neq (0, 0)$, b, $c \geq 0$. We refer to (viii) above.

Then $u > 0$, $u \geq |v|$. So for $t < 0$, $e^{3tu} < 1$. Also $2\cos(\sqrt{3}vt + \frac{2}{3}\pi)$ assumes value -1 for some $t < 0$ and thus $\boldsymbol{b}(t) < 0$. Similar conclusions hold for $\boldsymbol{c}(t)$. So we consider only the case $t \geq 0$. As already noted in (v) $\boldsymbol{a}(0) = 1$, $\boldsymbol{b}(0) = 0$, $\boldsymbol{c}(0) = 0$.

(a) In case $b = c$, i.e., $v = 0$, we immediately have for $t > 0$,

$$\begin{aligned}
\boldsymbol{a}(t) &= \frac{1}{3}e^{t(a-u)}\left[e^{3ut} + 2\right] > 0, \\
\boldsymbol{b}(t) &= \frac{1}{3}e^{t(a-u)}\left[e^{3ut} - 1\right] > 0, \\
\boldsymbol{c}(t) &= \frac{1}{3}e^{t(a-u)}\left[e^{3ut} - 1\right] = \boldsymbol{b}(t) > 0.
\end{aligned}$$

For the general case some computations are needed.

(b) Let $\alpha = 0, \frac{2\pi}{3}, \frac{-2\pi}{3}$. Set $f_\alpha(t) = e^{3ut} + 2\cos(\sqrt{3}vt + \alpha)$, $t \in \mathbb{R}$. Then f_α is infinitely differentiable and $f_\alpha(0) = 1 + 2\cos\alpha \geq 0$. Further, for $t \in \mathbb{R}$, $f'_\alpha(t) = 3ue^{3ut} - 2\sqrt{3}v\sin(\sqrt{3}vt + \alpha)$. Therefore, for $t \in \mathbb{R}$, $f''_\alpha(t) = (3u)^2e^{3ut} - 2(\sqrt{3}v)^2\cos(\sqrt{3}vt + \alpha) \geq 9u^2e^{3ut} - 6v^2 = 9u^2(e^{3ut} - 1) + (9u^2 - 6v^2)$. So for $t > 0$, $f''_\alpha(t) > 0$.

As a consequence f'_α is strictly increasing on $[0, \infty)$. Now $f'_\alpha(0) = 3u - 2\sqrt{3}v\sin\alpha$, which is $3u$, $3u - 3v$, $3u + 3v$ respectively for $\alpha = 0, \frac{2\pi}{3}, \frac{-2\pi}{3}$ respectively i.e. $3u$, $3b$, $3c$ respectively. But $3u$, $3b$, $3c$ are all ≥ 0. So $f'_\alpha(t) > 0$ for $t > 0$. Therefore, f_α is strictly increasing on $[0, \infty)$. Consequently $f_\alpha(t) > 0$ for $t > 0$ and hence $\boldsymbol{a}(t)$, $\boldsymbol{b}(t)$, $\boldsymbol{c}(t)$ are all > 0 for $t > 0$.

(c) Now $\boldsymbol{a}(t) \geq \boldsymbol{d}(t)$ if and only if $\frac{1}{3}e^{t(a-u)}\left[e^{3ut}+2\cos(\sqrt{3}vt)\right] \geq e^{td}$ if and only if $e^{-ut}\left[e^{3ut}+2\cos(\sqrt{3}vt)\right] \geq 3e^{t(d-a)}$. Set $w = a-d$. The condition $\boldsymbol{a}(t) \geq \boldsymbol{d}(t)$ is equivalent to

$$e^{-ut}\left[e^{3ut}+2\cos(v\sqrt{3}t)\right] \geq 3e^{-wt}.$$

Let $g(t) = e^{2ut} + 2e^{-ut}\cos(v\sqrt{3}t) - 3e^{-wt},\ t \in \mathbb{R}$. Then g is infinitely differentiable and $g(0) = 0$. Also for $t \in \mathbb{R}$,

$$\begin{aligned} g'(t) &= 2ue^{2ut} + 2e^{-ut}\left(-u\cos(v\sqrt{3}t) - v\sqrt{3}\sin(v\sqrt{3}t)\right) + 3we^{-wt} \\ &= 2u\left(e^{2ut} - e^{-ut}\cos(v\sqrt{3}t)\right) - 2v\sqrt{3}e^{-ut}\sin(v\sqrt{3}t) + 3we^{-wt} \\ &= 2u\left[(e^{2ut} - e^{-ut}) + 2e^{-ut}\sin^2(\frac{v\sqrt{3}}{2}t)\right] - 2v\sqrt{3}e^{-ut}\sin(v\sqrt{3}t) \\ &\quad +3we^{-wt}. \end{aligned}$$

In particular, $g'(0) = 3w$. So if $g(t_n) \geq 0$ for a sequence (t_n) in $(0,\infty)$ with t_n convergent to 0 then $g'(0) \geq 0$, i.e., $w \geq 0$. Now assume $w \geq 0$. Then for $t > 0$, using $|\sin t| \leq |t|$ for all t,

$$\begin{aligned} g'(t) &\geq 2u(e^{2ut} - e^{-ut}) - 2v\sqrt{3}e^{-ut}(v\sqrt{3}t) \\ &= 2e^{-ut}\left[u(e^{3ut}-1) - 3v^2t\right]. \end{aligned}$$

Let $h(t) = u(e^{3ut}-1) - 3v^2t$, $t \in \mathbb{R}$. Then h is infinitely differentiable and $h(0) = 0$. Also for $t > 0$,

$$\begin{aligned} h'(t) &= u.3ue^{3ut} - 3v^2 \\ &= 3u^2(e^{3ut}-1) + 3(u^2 - v^2) > 0 \end{aligned}$$

So $h(t) > 0$ for $t > 0$. As a consequence, $g'(t) > 0$ for $t > 0$. This gives $g(t) > 0$ for all $t > 0$. Thus $g(t_n) \geq 0$ for a sequence (t_n) in $(0,\infty)$ with t_n convergent to zero if and only if $w \geq 0$ if and only if $g(t) > 0$ for all $t > 0$. Hence $\boldsymbol{\rho}(t_n)$ are all completely positive maps for a sequence (t_n) in $(0,\infty)$ convergent to 0 if and only if $w \geq 0$ if and only if $\boldsymbol{\rho}(t)$ are all completely positive maps for all $t > 0$.

(d) Moreover, 3.8 then gives that for $a \geq d$, $t > 0$, the Choi matrix $C_{\boldsymbol{\rho}(t)}$ has rank 9. Item 3.9(i)(b) then gives that in case $a+b+c = 0$, the Schmidt number of the density $\frac{1}{3}C_{\underline{\rho}(t)}$ is 3 for all $t \geq 0$.

(e) Suppose $w \geq 0$; then $h(t) = \boldsymbol{b}(t)\boldsymbol{c}(t) - \boldsymbol{d}(t)^2$

$$\begin{aligned}
&= \frac{1}{9}e^{2t(a-u)}\left[(e^{3ut} - \cos(\sqrt{3}vt))^2 - 3\sin^2(\sqrt{3}vt)\right] - e^{2dt} \\
&= \frac{1}{9}e^{2t(a-u)}\left[e^{6ut} - 2e^{3ut}\cos(\sqrt{3}vt) + \cos^2(\sqrt{3}vt) - 3\sin^2(\sqrt{3}vt)\right] - e^{2dt} \\
&= \frac{1}{9}e^{2t(a-u)}\left[e^{6ut} - 2e^{3ut}\cos(\sqrt{3}vt) - 1 + 2\cos(2\sqrt{3}vt)\right] - e^{2dt} \\
&= \frac{1}{9}e^{2ta}\left[e^{4ut} - 2e^{ut}\cos(\sqrt{3}vt) - e^{-2ut} + 2e^{-2ut}\cos(2\sqrt{3}vt) - 9e^{-2wt}\right] \\
&= \frac{1}{9}e^{2ta}\, g(t), \text{where}
\end{aligned}$$

$g(t) = e^{4ut} - e^{-2ut} - 9e^{-2wt} - 2e^{ut}\cos(\sqrt{3}vt) + 2e^{-2ut}\cos(2\sqrt{3}vt), t \in \mathbb{R}$. We note that g is infinitely differentiable on $\mathbb{R}$ and $g(0) = -9$.

Since $u > 0$, $g(t) \to \infty$ as $t \to \infty$. So there is an $s_0 \in (0, \infty)$ satisfying $g(t) > 0$ for $t > s_0$. This, in turn, gives that $h(t) > 0$ for $t > s_0$. By Remark 3.3(iii) $\boldsymbol{\rho}(t)$ is PPT for $t > s_0$. As $T_0 = Id$ is not PPT, an application of the Trichotomy result as envisaged in Remark 5.5 (ii) (a) immediately gives that there exists a unique $t_0 \in (0, \infty)$ such that for $0 \leq t < t_0$, $\boldsymbol{\rho}(t)$ is not PPT but for $t \geq t_0$, $\boldsymbol{\rho}(t)$ is PPT. This, in view of Remark 3.3 (iii), entails that there exists a $t_0 \in (0, \infty)$ satisfying, $h(t) < 0$ for $0 \leq t < t_0$ and $h(t) \geq 0$ for $t \geq t_0$.

We now proceed to refine this observation.

(f) For $t \in \mathbb{R}$

$$\begin{aligned}
g'(t) &= 4ue^{4ut} + 2ue^{-2ut} + 18we^{-2wt} \\
&\quad -2e^{ut}\left(u\cos(\sqrt{3}vt) - \sqrt{3}v\sin(\sqrt{3}vt)\right) \\
&\quad +2e^{-2ut}\left(-2u\cos(2\sqrt{3}vt) - 2\sqrt{3}v\sin(2\sqrt{3}vt)\right) \\
&= 2u\left[2e^{4ut} + e^{-2ut} - e^{ut} - 2e^{-2ut}\right] + 18we^{-2wt} \\
&\quad +4ue^{ut}\sin^2(\frac{\sqrt{3}}{2}vt) + 2\sqrt{3}ve^{ut}\sin(\sqrt{3}vt) \\
&\quad +8ue^{-2ut}\sin^2(\sqrt{3}vt) - 4\sqrt{3}ve^{-2ut}\sin(2\sqrt{3}vt).
\end{aligned}$$

We note that $g'(0) = 18w \geq 0$. Now for $t \geq 0$,

$$\begin{aligned} g'(t) &\geq 2u\left[2e^{4ut} - e^{ut} - e^{-2ut}\right] - 2\sqrt{3}ve^{ut}(\sqrt{3}vt) - 4\sqrt{3}ve^{-2ut}(2\sqrt{3}vt) \\ &= 2\left[u(2e^{4ut} - e^{ut} - e^{-2ut}) - 3v^2e^{ut}t - 4 \times 3v^2e^{-2ut}t\right] \\ &= 2e^{-2ut}\left[u(2e^{6ut} - e^{3ut} - 1) - 3v^2e^{3ut}t - 12v^2t\right] \\ &= 2e^{-2ut}\left[u(e^{3ut} - 1)e^{3ut} + u(e^{6ut} - 1) - 3v^2te^{3ut} - 12v^2t\right] \\ &\geq 2e^{-2ut}\left[3u^2te^{3ut} + 6u^2t - 3v^2te^{3ut} - 12v^2t\right] \\ &= 2e^{-2ut}\left[\frac{3}{2}u^2te^{3ut} + (u^2 - 2v^2)\left(\frac{3}{2}te^{3ut} + 6t\right)\right]. \end{aligned}$$

Because $u > 0$, we have for $t > 0$, $g'(t) > 0$ in case $u^2 \geq 2v^2$.

One can obtain $g'(t) > 0$ for $t > 0$ for less restricted cases but we prefer to confine our attention to this simple case and go on with the case $u \geq \sqrt{2}|v|$. Then g is strictly increasing on $[0, \infty)$. So there exists a unique to $\in (0, \infty)$ such that $g(t_0) = 0$, $g(t) < 0$ for $0 \leq t < t_0$ and $g(t) > 0$ for $t_0 < t < \infty$. As a consequence, there exists a unique $t_0 \in (0, \infty)$ such that $h(t_0) = 0$, $h(t) < 0$ for $0 \leq t < t_0$ and $h(t) > 0$ for $t_0 < t < \infty$. So by Remark 3.3 (iii) (a), $\boldsymbol{\rho}(t)$ is not completely co-positive for $t < t_0$ but is completely co-positive for $t \geq t_0$.

(g) Hence for $(b, c) \neq (0, 0)$, $a \geq d$, $b + c \geq \sqrt{2}|b - c|$, there exists a unique $t_0 \in (0, \infty)$ that satifies
(α) for $0 \leq t < t_0$, $\boldsymbol{\rho}(t)$ is not PPT, and
(β) for $t \geq t_0$, $\boldsymbol{\rho}(t)$ is PPT.

This illustrates the condition (ii) of Trichotomy in Remark 5.5 (ii)(a) in a concrete manner.

(xii) Let $\boldsymbol{\tau}(t) = e^{t\tau[a,b,c,d]}$, $t \geq 0$. Then $\boldsymbol{\tau}(t) = D(\boldsymbol{a}(t), \boldsymbol{b}(t), \boldsymbol{c}(t)) \oplus (\cosh(td)I_{F_n} + \sinh(td)\tau_{F_n})$. Its Choi matrix in expanded form is $C_{\boldsymbol{\tau}(t)} =$

$$\begin{bmatrix} \boldsymbol{a}(t) & 0 & 0 & 0 & \cosh(td) & 0 & 0 & 0 & \cosh(td) \\ 0 & \boldsymbol{c}(t) & 0 & \sinh(td) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \boldsymbol{b}(t) & 0 & 0 & 0 & \sinh(td) & 0 & 0 \\ 0 & \sinh(td) & 0 & \boldsymbol{b}(t) & 0 & 0 & 0 & 0 & 0 \\ \cosh(td) & 0 & 0 & 0 & \boldsymbol{a}(t) & 0 & 0 & 0 & \cosh(td) \\ 0 & 0 & 0 & 0 & 0 & \boldsymbol{c}(t) & 0 & \sinh(td) & 0 \\ 0 & 0 & \sinh(td) & 0 & 0 & 0 & \boldsymbol{c}(t) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sinh(td) & 0 & \boldsymbol{b}(t) & 0 \\ \cosh(td) & 0 & 0 & 0 & \cosh(td) & 0 & 0 & 0 & \boldsymbol{a}(t) \end{bmatrix}.$$

It has trace $\mu(t) = 3\ (\boldsymbol{a}(t)+\boldsymbol{b}(t)+\boldsymbol{c}(t)) > 0$. Further, for $t > 0$, $C_{\boldsymbol{\tau}(t)}$ is a positive matrix if and only if $\boldsymbol{a}(t) \geq \cosh(td)$, $\boldsymbol{b}(t) \geq 0$, $\boldsymbol{c}(t) \geq 0$ and $\boldsymbol{b}(t)\boldsymbol{c}(t) \geq \sinh^2(td)$. Computations of the type done in this example give that this happens for all $t \geq 0$ if $\frac{2}{3}b = \frac{2}{3}c \geq a \geq |d|$; and, in fact, for less restricted cases as well. We may consider the non-trivial sub-block

$$\begin{bmatrix} \boldsymbol{a}(t) & \cosh(td) & \cosh(td) \\ \cosh(td) & \boldsymbol{a}(t) & \cosh(td) \\ \cosh(td) & \cosh(td) & \boldsymbol{a}(t) \end{bmatrix}$$

as in 3.9(i) and conclude that $\frac{1}{\mu(t)}C_{\boldsymbol{\tau}(t)}$ has Schmidt number 3.

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