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Trivolutions

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TRIVOLUTIONS

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Abstract. We define a trivolution on a complex algebra A as a conjugate-linear, anti-homomorphism τ on A, which is its own generalized inverse, that is, $\tau^3 = \tau$. We give several characterizations of trivolutions and show with examples that they appear naturally on many Banach algebras, particularly those arising from group algebras. We give several results on the existence or non-existence of involutions on the dual of a topologically introverted space. We investigate conditions under which the dual of a topologically introverted space admit trivolutions.

1. Introduction and preliminaries

By a well-known result of Civin and Yood [8, Theorem 6.2], if A is a Banach algebra with an involution $\rho\colon A \longrightarrow A$, then the second (conjugate-linear) adjoint $\rho^{**}\colon A^{**} \longrightarrow A^{**}$ is an involution on A^{**} (with respect to either of the Arens products) if and only if A is Arens regular; when this is the case, ρ^{**} is called the canonical extension of ρ. Grosser [22, Theorem 1] has shown that if A has a bounded right [left] approximate identity, then a necessary condition for the existence of an involution on A^{**} with respect to the first [second] Arens product is that $A^* \cdot A = A^*$ [$A \cdot A^* = A^*$]. The above results applied to the group algebra $L^1(G)$ imply that a necessary condition for $L^1(G)^{**}$ to have an involution (with respect to either of the Arens products) is that G is discrete (Grosser [22, Theorem 2]), and the natural involution of $L^1(G)$ has a canonical extension to $L^1(G)^{**}$ if and only if G is finite (Young [33]).

In general, since A is not norm dense in A^{**}, an involution on A may have extensions to A^{**} which are different from the canonical extension. A necessary and sufficient condition for the existence of such extensions does not seem to be known. However, for the special case of the group algebra $L^1(G)$, Farhadi and Ghahramani [15, Theorem 3.2(a)] have shown that if a locally compact group G has an infinite amenable subgroup, then $L^1(G)^{**}$ does not have any involution extending the natural involution of $L^1(G)$. (See also the related paper by Neufang [30], answering a question raised in [15].)

In Singh [32], the third author introduced the concept of α-amenability for a locally compact group G. Given a cardinal α, a group G is called α-amenable if there exists a subset $\mathcal{F} \subset L^1(G)^{**}$ containing a mean M (not necessarily left invariant) such that

$|\mathcal{F}| \leq \alpha$ and the linear span of $\mathcal{F}$ is a left ideal of $L^1(G)^{**}$. The group G is called subamenable if G is α-amenable for some cardinal $1 \leq \alpha < 2^{2^{\kappa(G)}}$, where $\kappa(G)$ denotes the compact covering number of G (that is, the least cardinality of a compact covering of G). It follows that 1-amenability of G is equivalent to the amenability of G, and α-amenability implies β-amenability for every $\beta \geq \alpha$. If G is a non-compact locally compact group, then every non-trivial right ideal in $L^1(G)^{**}$ or in $LUC(G)^*$ has a (vector space) dimension of at least $2^{2^{\kappa(G)}}$ (Filali–Pym [18, Theorem 5], and, Filali –Salmi [19, Theorem 6]). It follows from this lower bound that if G is a subamenable, non-compact, locally compact group, then $L^1(G)^{**}$ has no involution (Singh [32, Theorem 2.2]). Singh [32, Theorem 2.9(i)] also showed that every discrete group G is a subgroup of a subamenable discrete group G_σ with $|G_\sigma| \leq 2^{|G|}$. It is not known whether there exists any non-subamenable group, and in particular, it remains an open question whether the free group on 2 generators is subamenable.

All the above results show that the existence of involutions on second dual Banach algebras impose strong conditions on A. So it seems natural to consider involution-like operators on Banach algebras and their second duals. In this paper, we relax the condition of bijectivity on an involution ρ and of ρ being its own inverse to that of ρ being a generalized inverse of itself, namely, $\rho^3 = \rho$, and call them trivolutions (Definition 2.1) following the terminology of J. W. Degen [12]. It follows from the definition that every involution is a trivolution, but as we shall show later there are many naturally arising trivolutions which are not involutions.

In section 2 we start with a general study of trivolutions on algebras and we give several characterizations of trivolutions in Theorem 2.3. In Theorem 2.9 we show that, unlike involutions, a trivolution can have various extensions to the unitized algebra $A^\sharp$, and we give a complete characterization of all such extensions. We show concepts such as hermitian, normal, and unitary elements, usually associated with involutions, can be naturally defined in the context of trivoluted algebras. In section 3, we study involutions on the dual of topologically introverted spaces. In Theorem 3.1 we extend the result of Civin and Yood (discussed above) to the dual of a topologically introverted space, and obtain the result of Farhadi and Ghahramani [15, Theorem 3.2(a)] as a corollary. In Theorem 3.3 we investigate the relationship between the existence of topologically invariant elements and the existence of involutions on the dual of a topologically introverted space. As a corollary we show that under fairly general conditions, neither of the Banach algebras $PM_p(G)^*$ and $UC_p(G)^*$, $1 < p < \infty$, have involutions (for the definition of these spaces see below as well as the discussion prior to Corollary 3.4). In section 4 we give some sufficient conditions under which a second dual Banach algebra A^{**} admits trivolutions (Theorem 4.1). In Theorem 4.3 we show that for G non-discrete, $L^1(G)^{**}$ does not admit any trivolutions with range $L_0^\infty(G)^*$. However, the space $L_0^\infty(G)^*$ itself always admits trivolutions (Theorem 4.6).

We close this section with a few preliminary definitions and notation. Given a Banach algebra A, the dual space A^* can be viewed as a Banach A-bimodule with the canonical operations:

$$\langle \lambda \cdot a, b\rangle = \langle \lambda, ab\rangle, \quad \langle a \cdot \lambda, b\rangle = \langle \lambda, ba\rangle,$$

where $\lambda \in A^*$ and $a, b \in A$. Let X be a norm closed A-submodule of A^*. Then given $\Psi \in X^*$, $\lambda \in X$, we may define $\Psi \cdot \lambda \in A^*$ by $\langle \Psi \cdot \lambda, a\rangle = \langle \Psi, \lambda \cdot a\rangle$. If $\Psi \cdot \lambda \in X$ for all choices of $\Psi \in X^*$ and $\lambda \in X$, then X is called a left topologically introverted subspace of A^*. The dual of a left topologically introverted subspace X can be turned into a Banach algebra if, for all $\Phi, \Psi \in X^*$, we define $\Phi \Box \Psi \in X^*$ by $\langle \Phi \Box \Psi, \lambda\rangle = \langle \Phi, \Psi \cdot \lambda\rangle$. In particular, by taking $X = A^*$, we obtain the first (or the left) Arens product on A^{**}, defined by Arens [1, 2]. The space X^* can be identified with the quotient algebra A^{**}/X°, where $X^\circ = \{\Phi \in A^{**} \colon \Phi|_X = 0\}$. If X is faithful (that is, $a = 0$ whenever $\lambda(a) = 0$ for all $\lambda \in X$), then the natural map of A into X^* is an embedding, and we will regard A as a subalgebra of $(X^*, \Box)$. The space X^* has a canonical A-bimodule structure defined by $\langle a \cdot \Phi, \lambda\rangle = \langle \Phi, \lambda \cdot a\rangle$, $\langle \Phi \cdot a, \lambda\rangle = \langle \Phi, a \cdot \lambda\rangle$ ($\Phi \in X^*, \lambda \in X, a \in A$). One can then verify that $a \cdot \Phi = a \Box \Phi$, $\Phi \cdot a = \Phi \Box a$ for each $a \in A$ and $\Phi \in X^*$. We assume X^* is equipped with the w^*-topology $\sigma(X^*, X)$. In this topology, for each $\Phi \in X^*$, the map $\Psi \mapsto \Psi \Box \Phi$, $X^* \longrightarrow X^*$, is w^*-continuous.

Right topologically introverted subspaces of A^* are defined similarly; for these spaces the second (or the right) Arens product on X^* will be denoted by $\Phi \Diamond \Psi$. A space which is both left and right topologically introverted is called topologically introverted. When A is commutative there will be no distinction between left and right topologically introverted spaces.

Let A be a Banach algebra. The space of [weakly] almost periodic functionals on A, denoted by $[WAP(A)]$ $AP(A)$, is defined as the set of all $\lambda \in A^*$ such that the linear map $A \longrightarrow A^*$, $a \mapsto a \cdot \lambda$, is [weakly] compact. The spaces A^*, $WAP(A)$, and $AP(A)$ are examples of topologically introverted spaces. The space of left uniformly continuous functionals on A defined by $LUC(A) = \overline{\mathrm{lin}}\,(A^* \cdot A)$ (the closure is in norm topology), is an example of a left topologically introverted space. If G is a locally compact group, then $LUC(L^1(G))$ coincides with $LUC(G)$, the space of left uniformly continuous functions on G (cf. Lau [27]). For more information and additional examples one may consult [10, 11, 13, 20, 26].

If $f\colon X \longrightarrow X$ is a map on a set X, for simplicity and when no confusion arises, we write f^n to denote the n-times composition of f with itself, that is, $f^n := f \circ \cdots \circ f$ (n-times). Also when no confusion arises, we write $\Phi\Psi$ to denote the first Arens product $\Phi \Box \Psi$.

2. Trivolutions

Definition 2.1. A *trivolution* on a complex algebra A is a conjugate linear, anti-homomorphism $\tau\colon A \longrightarrow A$, such that $\tau^3 = \tau$. When A is a normed algebra, we shall assume that $\|\tau\| \leq 1$. The pair (A, τ) is called a *trivoluted algebra*.

Remarks 2.2. (i) It follows from the definition that every involution is a trivolution. If (A, τ) is a trivoluted normed algebra, then $\|\tau(x)\| \leq \|x\|$ for every $x \in A$; in particular, when τ is an involution, we have $\|\tau(x)\| = \|x\|$ for every x.

(ii) If (A, τ) is a trivoluted normed algebra, then $\tau(A)$ is closed subalgebra of A: if (x_n) is a sequence such that $\tau(x_n) \to x$ for some $x \in A$, then $\tau^3(x_n) = \tau(x_n) \to \tau^2(x)$, and hence $x = \tau^2(x) \in \tau(A)$.

The next result gives several characterizations of trivolutions.

Theorem 2.3. *Let A be a complex algebra, $\tau\colon A \longrightarrow A$ a map, and $B := \tau(A)$. Let $\mathscr{I}_B$ denote the set of all involutions on B. Then the following statements are equivalent.*

(i) *τ is a trivolution.*
(ii) *τ is a conjugate-linear, anti-homomorphism, and $\tau|_B \in \mathscr{I}_B$.*
(iii) *B is a subalgebra of A and there exists a surjective homomorphism $p\colon A \longrightarrow B$, with $p^2 = p$, and an involution $\rho \in \mathscr{I}_B$ such that $\tau = \rho \circ p$.*
(iv) *B is a sublagebra of A, $\mathscr{I}_B$ is non-empty, and for each $\rho_2 \in \mathscr{I}_B$, there is a surjective homomorphism $\rho_1\colon A \longrightarrow B$ that satisfies $\tau = \rho_2 \circ \rho_1$ and $\rho_1 \circ \rho_2 \in \mathscr{I}_B$.*

The statements (i)–(iv) *remain equivalent if A is a normed algebra provided that B is assumed to be closed in A and the maps in* (i)–(iv) *are assumed to be contractive.*

Proof. (i) $\Longrightarrow$ (ii): Since τ is a conjugate-linear, anti-homomorphism, it follows that B is a subalgebra of A; moreover, the identity $\tau^3 = \tau$ implies that $(\tau|_B)^2$ is the identity map on B, and therefore $\tau|_B \in \mathscr{I}_B$.

(ii) $\Longrightarrow$ (iii): B is a subalgebra of A since τ is a conjugate-linear anti-homomorphism. Let $\rho := \tau|_B$ and $p := \tau \circ \tau\colon A \longrightarrow B$. We leave it for the reader to verify the easy facts that $\tau = \rho \circ p$, and p and ρ satisfy the requirements in (iii).

(iii) $\Longrightarrow$ (iv): The first two statements of (iv) are immediate consequences of (iii). Let $\rho_2 \in \mathscr{I}_B$ be given and define $\rho_1 := \rho_2 \circ \tau$. Then ρ_1 is a surjective homomorphism from A

to B and furthermore, $\rho_2 \circ \rho_1 = \rho_2 \circ \rho_2 \circ \tau = I_B \circ \tau = \tau$. Since $\rho_1 \circ \rho_2$ is a conjugate-linear, anti-homomorphism on B, it remains to show that $(\rho_1 \circ \rho_2)^2 = I_B$. To this end, we note that from (iii) we have, for a in A,

$$\tau^3(a) = \rho \circ (p \circ \rho) \circ (p \circ \rho) \circ p(a) = \rho \circ I_B \circ p(a) = \tau(a), \tag{1}$$

and therefore, $(\tau|_B)^2 = I_B$. From the last identity and the fact that $\rho_2 \in \mathscr{I}_B$, it follows that

$$\begin{aligned}(\rho_1 \circ \rho_2)^2 &= \rho_1 \circ \rho_2 \circ \rho_1 \circ \rho_2 \\ &= \rho_2 \circ \tau \circ \rho_2 \circ \rho_2 \circ \tau \circ \rho_2 \\ &= \rho_2 \circ \tau \circ \tau \circ \rho_2 \\ &= \rho_2 \circ \rho_2 = I_B,\end{aligned}$$

completing the proof that $\rho_1 \circ \rho_2 \in \mathscr{I}_B$.

(iv) $\Longrightarrow$ (i): Since $\tau = \rho_2 \circ \rho_1$, with ρ_1, ρ_2 as in (iv), it follows that τ is a surjective, conjugate-linear anti-homomorphism. An argument similar to the proof of (1), shows that $\tau^3 = \tau$; hence τ is a trivolution. □

The following corollary follows from the definition of a trivolution and the equivalence of (i) and (ii) in the above theorem.

Corollary 2.4. *Let A be an algebra and τ be a trivolution on A. Then the following are equivalent.*

(i) *τ is an involution;*
(ii) *τ is injective;*
(iii) *τ is surjective.*

Corollary 2.5. *Let A be a [normed] algebra and $\tau\colon A \longrightarrow A$ a map with $B = \tau(A)$. Then τ is a trivolution if and only if there exist a [continuous] projection p of A onto B, with p an algebra homomorphism, a [closed] two-sided ideal I of A, and an involution ρ on B such that*

$$A = I \oplus B, \quad \tau = \rho \circ p. \tag{2}$$

The vector space direct sum $A = I \oplus B$ is a topological direct sum if A is a Banach algebra.

Proof. The corollary follows immediately from the equivalence of (i) and (iii) in Theorem 2.3 if we let $I = \ker p$. The assertion that the direct sum $A = I \oplus B$ is topological if A is a Banach algebra follows from the fact that if p is a continuous projection then $B = p(A)$ is a closed subspace of A (cf. Conway [9, Section III.13]). □

Next, we give a few examples of trivolutions.

Examples 2.6. (a) Let (X, μ) be a measurable space and $K \subset X$ be a measurable subset of X. Let $L^\infty_K(X, \mu)$ be the subalgebra of $L^\infty(X, \mu)$ consisting of all those functions which vanish locally almost everywhere on $X - K$. Let χ_K be the characteristic function of K and $p\colon L^\infty(X, \mu) \longrightarrow L^\infty_K(X, \mu)$ be the homomorphism $f \mapsto \chi_K f$. If ρ is the usual complex conjugation on $L^\infty_K(X, \mu)$, then $p \circ \rho = \rho$, and hence by Corollary 2.5, the map $\tau(f) := \rho \circ p(f) = \chi_K \overline{f}$ defines a trivolution on $L^\infty(X, \mu)$.

(b) Let H be a Hilbert space, X a closed subspace of H, and P be the orthogonal projection on X. Let M be a von Neumann algebra on H such that $P \in M'$, where M' is the commutant of M. Let N be the von Neumann subalgebra of M defined by $N = \{PT\colon T \in M\}$. Let $p : M \longrightarrow N$, be defined by $p(T) = PT$, and let ρ be the natural adjoint map on N. In that case, $p \circ \rho = \rho$, and by Corollary 2.5, $\tau(T) := \rho \circ p(T) = PT^*$ defines a trivolution on M.

(c) Let A be a complex [normed] algebra, B a [closed] subalgebra of A, and I a [closed] two-sided ideal of A, such that A is the vector space direct sum $A = I \oplus B$. Let $p\colon A \longrightarrow B$ be the natural projection on B, and let $\tau'\colon B \longrightarrow B$ be a trivolution. In that case, $\tau := \tau' \circ p$ is a trivolution on A, since:

$$\tau^3 = \tau' \circ (p \circ \tau')^2 \circ p = \tau' \circ \tau'^2 \circ p = \tau' \circ p = \tau.$$

(d) Let τ be a trivolution on a Banach algebra A, X be a closed subalgebra of A^{**}. Let $\tau^*\colon A^* \longrightarrow A^*$, be the conjugate-linear adjoint of τ defined by $\langle \tau^*(f), a\rangle = \overline{\langle f, \tau(a)\rangle}$. If X is invariant under τ^{**} and the two Arens products of A^{**} agree on X, then $\tau^{**}|_X$ is a trivolution on X. In particular, if A is Arens regular, then τ^{**} is a trivolution on A^{**}.

(e) The quotient of a trivoluted [normed] algebra by a two-sided [closed] ideal which is invariant under the trivolution, is a trivoluted [normed] algebra. Finite products of trivoluted [normed] algebras, the completion of a trivoluted normed algebra, and the opposite of a trivoluted [normed] algebra, are all trivoluted [normed] algebras in canonical ways.

Theorem 2.7. *Let τ be an anti-homomorphism on an algebra A and let $B = \tau(A)$.*

(i) *If $e \in B$ is a right identity of A, then $\tau(e) = e$ and e is the identity of B.*
(ii) *The set B can contain at most one right identity of A.*
(iii) *Let A be a subalgebra of an algebra C of the form eC with e being a right identity of C. If τ is a trivolution on A, and if ℓ_e denotes the left multiplication map by e on C, then $\tau_1 := \tau \circ \ell_e$ is a trivolution on C and $\tau_1(C) = \tau(A) = B$.*

Proof. (i) If $a \in A$, then $a = ae$ and hence $\tau(a) = \tau(e)\tau(a)$, which shows that $\tau(e)$ is a left identity for B. Since $e \in B$, $e = \tau(e)e = \tau(e)$, proving that $e = \tau(e)$ and e is the identity for B.

(ii) This is an immediate consequence of (i).

(iii) Since e is a right identity of C and $B \subset A = eC$, it follows that $\ell_e|_B$ is the identity map on B: in fact, given $b \in B$, we can write $b = ec$ for some $c \in C$, and hence $\ell_e(b) = e(ec) = ec = b$. It follows that $\ell_e \circ \tau = \tau$, and hence

$$\tau_1^3 = (\tau \circ \ell_e)^3 = \tau \circ (\ell_e \circ \tau) \circ (\ell_e \circ \tau) \circ \ell_e = \tau^3 \circ \ell_e = \tau \circ \ell_e = \tau_1.$$

In addition, since e is a right identity of C, we have $e(c_1c_2) = (ec_1)(ec_2)$, for all $c_1, c_2 \in C$. Hence ℓ_e is a homomorphism on C which implies that $\tau_1 = \tau \circ \ell_e$ is a conjugate-linear, anti-homomorphism, completing the proof that τ_1 is a trivolution on C. The fact that $\tau_1(C) = B$ is now immediate. □

Remarks 2.8. (i) Similar results hold if a right identity is replaced by a left identity in Theorem 2.7; we leave the formulation of the results and their proofs for the readers.

(ii) Let τ be a trivolution on A and let $B = \tau(A)$. If A has the identity e, then $\tau(e)$ is the identity of B, which we may denote by e_B. Clearly $e = e_B$ if and only if $e \in B$. This however may not always be the case: let $A = \mathbf{C}^2$, $B = \mathbf{C} \times \{0\}$, and $\tau(z_1, z_2) = (\overline{z}_1, 0)$; then $e = (1, 1)$ but $e_B = (1, 0)$.

Next we consider the problem of extending a trivolution to the unitized algebra $A^\sharp = \mathbf{C} \times A$. Let (A, τ) be a trivoluted algebra and $\tau^\sharp \colon A^\sharp \longrightarrow A^\sharp$ be a trivolution extending τ, namely, $\tau^\sharp(0, x) = (0, \tau(x))$, for all $x \in A$. If $\tau^\sharp(1, 0) = (\lambda_0, x_0)$, then, from conjugate linearity of $\tau^\sharp$ we obtain:

$$\tau^\sharp(\lambda, x) = \tau^\sharp(\lambda(1, 0) + (0, x)) = (\overline{\lambda}\lambda_0, \overline{\lambda}x_0 + \tau(x)). \tag{3}$$

If $(\lambda_0, x_0) = (1, 0)$, then $\tau^\sharp(\lambda, x) = (\overline{\lambda}, \tau(x))$. We call this map the canonical extension of τ to $A^\sharp$. We note that by Theorem 2.7(i), the condition $\tau^\sharp(1, 0) = (1, 0)$ is equivalent to $(1, 0)$ being in the range of $\tau^\sharp$, and the latter condition can be shown to be equivalent to $\lambda_0 \neq 0$ and $x_0 \in \tau(A)$.

While every involution has only the canonical extension to an involution on the unitized algebra, the situation is different for trivolutions, as the following theorem shows.

Theorem 2.9. *Let τ be a trivolution on a complex algebra A. The map $\tau^\sharp$ in (3) is a trivolution extending τ if and only if either of the following conditions hold:*

(i) $\tau^\sharp(\lambda, x) = (\overline{\lambda}, \overline{\lambda}x_0 + \tau(x))$, *where $x_0 \in A$ is such that*

$$x_0^2 = -x_0, \quad x_0\tau(A) = \tau(A)x_0 = \{0\}, \quad \tau(x_0) = 0. \tag{4}$$

(ii) $\tau^\sharp(\lambda, x) = (0, \overline{\lambda}x_0 + \tau(x))$, *where* $x_0 \in \tau(A)$ *is the identity of* $\tau(A)$.

Proof. The proof, that both (i) and (ii) define trivolutions on $A^\sharp$ extending τ, is routine and is left for the reader. We prove the necessity part of the theorem. Using the idempotence of $(1,0)$ we get

$$(\lambda_0, x_0) = \tau^\sharp(1,0) = \tau^\sharp(1,0)^2 = (\lambda_0^2, 2\lambda_0 x_0 + x_0^2),$$

which implies that either $\lambda_0 = 1$ and $x_0^2 = -x_0$; or $\lambda_0 = 0$ and $x_0^2 = x_0$. We consider these two cases.

Case I: $\lambda_0 = 1$ and $x_0^2 = -x_0$. In this case $\tau^\sharp(\lambda, x) = (\overline{\lambda}, \overline{\lambda}x_0 + \tau(x))$. Applying $\tau^\sharp$ to the identity $(1,0)(0,x) = (0,x)$, we obtain

$$(0, \tau(x) + \tau(x)x_0) = (0, \tau(x)),$$

which implies that $\tau(x)x_0 = 0$ for all $x \in A$. Similarly, starting from the identity $(0,x)(1,0) = (0,x)$ we can show that $x_0\tau(x) = 0$ for all $x \in A$. Moreover it follows from $(\tau^\sharp)^3(1,0) = \tau^\sharp(1,0)$, that

$$(1, x_0 + \tau(x_0) + \tau^2(x_0)) = (1, x_0),$$

which is equivalent to $\tau(x_0) + \tau^2(x_0) = 0$. Therefore

$$0 = x_0\tau(x_0) = \tau^2(x_0)\tau(x_0) = -\tau(x_0)^2 = -\tau(x_0^2) = \tau(x_0).$$

Thus x_0 satisfies all the conditions in (4).

Case II: $\lambda_0 = 0$. In this case $\tau^\sharp(\lambda, x) = (0, \overline{\lambda}x_0 + \tau(x))$. Applying $\tau^\sharp$ to the identities $(1,0)(0,x) = (0,x)$ and $(0,x)(1,0) = (0,x)$, we obtain respectively, $\tau(x)x_0 = \tau(x)$, $x_0\tau(x) = \tau(x)$, for all $x \in A$. Moreover, from $(\tau^\sharp)^3(1,0) = \tau^\sharp(1,0)$, it follows that $x_0 = \tau^2(x_0) \in \tau(A)$. Thus x_0 is the identity of $\tau(A)$. ☐

Corollary 2.10. *Let* (A, τ) *be a trivoluted normed algebra and* $A^\sharp$ *be the unitized algebra with the norm* $\|(\lambda, x)\| = |\lambda| + \|x\|$. *A map* $\tau^\sharp \colon A^\sharp \longrightarrow A^\sharp$, *is a trivolution extending* τ *if and only if either of the following conditions hold:*

(i) $\tau^\sharp(\lambda, x) = (\overline{\lambda}, \tau(x))$;
(ii) $\tau^\sharp(\lambda, x) = (0, \overline{\lambda}x_0 + \tau(x))$, *where* $x_0 \in \tau(A)$ *is the identity of* $\tau(A)$ *with* $\|x_0\| = 1$.

Proof. If $\tau^\sharp$ is of the form given in Theorem 2.9(i), then $\tau^\sharp(\lambda, 0) = (\overline{\lambda}, \overline{\lambda}x_0)$ for all $\lambda \in \mathbf{C}$. Since we must have $\|\tau^\sharp\| \le 1$, we obtain $|\lambda| + |\lambda|\|x_0\| \le |\lambda|$, implying that $x_0 = 0$.

If however $\tau^\sharp$ is of the form given in Theorem 2.9(ii), then $\tau^\sharp(\lambda, 0) = (0, \overline{\lambda}x_0)$, for all $\lambda \in \mathbf{C}$. Hence the condition $\|\tau^\sharp\| \le 1$, implies that $|\lambda|\|x_0\| \le |\lambda|$, and therefore $\|x_0\| \le 1$. ☐

Remark 2.11. Let τ be an involution on a complex algebra A. Then $\tau(A) = A$, and hence any extension of τ of the form given in Theorem 2.9(i), is necessarily equal to $\tau^\sharp(\lambda, \tau(x)) = (\overline{\lambda}, \tau(x))$, since $x_0\tau(A) = 0$ implies that $x_0^2 = 0$ and hence $x_0 = 0$ (as $x_0^2 = -x_0$). It should be noted that if A has no identity, then τ has no extension of the form given in Theorem 2.9(ii).

We can define the concepts of normality, hermiticity, and positivity for elements of trivoluted algebras.

Definition 2.12. Let (A, τ) be a trivoluted algebra and let $x \in A$. Then x is called

(i) *hermitian* if $\tau(x) = x$;
(ii) *normal* if $x\tau(x) = \tau(x)x$ and $x\tau^2(x) = \tau^2(x)x$;
(iii) *projection* if x is hermitian and $x^2 = x$;
(iv) *unitary* if A is unital with identity e and $x\tau(x) = \tau(x)x = e$;
(v) *positive* if x is hermitian and $x = \tau(y)y$ for some $y \in A$.

We denote the set of all hermitian (respectively, unitary, positive) elements of A, by A_h (respectively, A_u, A^+). It follows that A_h is a real vector subspace of A, and A_u, is a group under multiplication (the unitary group of A). It follows from the definition that if x is hermitian, then $x \in A^+$ if and only if $x = z\tau(z)$ for some $z \in A$. It should be noted that for trivoluted algebras in general, A^+ may not form a positive cone. Definition of normality is designed to have the τ-invariant algebra generated by x (and therefore, containing both $\tau(x)$ and $\tau^2(x)$) commutative (since $\tau(x)\tau^2(x) = \tau(\tau(x)x) = \tau(x\tau(x)) = \tau^2(x)\tau(x)$). If x is unitary in A, then by letting $B = \tau(A)$ and $e_B = \tau(e)$, we see that $\tau^2(x)$ is the inverse of $\tau(x)$ in B, and $x \in B$ implies that $e \in B$ (and $e = e_B$). Thus, $e \in B$ if and only if B contains at least one unitary element.

Let (A, τ) be a trivoluted algebra and τ^* be the conjugate-linear adjoint of τ defined in Example 2.6(d). If $f\colon A \longrightarrow \mathbf{C}$ is a linear functional on A, then $f^\tau := \tau^*(f)$, is also a linear functional on A. One can easily check that the map $f \longrightarrow f^\tau$, is conjugate-linear and in general $f^{\tau\tau\tau} = f^\tau$. If (A, τ) is normed, then $\|f^\tau\| \leq \|f\|$. We call f hermitian if $f^\tau = f$. Clearly if χ is a character on A, then χ^τ is also a character on A.

We close this section by stating the following two results whose straightforward proofs are omitted for briefness.

Theorem 2.13. *Let (A, τ) be a unital trivoluted algebra and $B = \tau(A)$. Let $x \in A$.*

(i) *If x is invertible in A, then $\tau(x)$ is invertible in B and $\tau(x)^{-1} = \tau(x^{-1})$.*
(ii) *If $\tau(x)$ is invertible in A, then $\tau(x)$ is invertible in B.*
(iii) $\mathrm{Sp}_B(\tau(x)) \subset \overline{\mathrm{Sp}_A(x)}$ *(where the bar denotes the complex conjugate).*

Theorem 2.14. *Let (A, τ) be a trivoluted algebra.*

(i) *$x \in A$ can be written uniquely in the form $x = x_1 + ix_2$, with x_1, x_2 hermitian, if and only if $x \in \tau(A)$.*
(ii) *$f \in A^*$ can be written uniquely in the form $f = f_1 + if_2$, with f_1, f_2 hermitian, if and only if $f \in \tau^*(A^*)$.*
(iii) *A linear functional f is hermitian if and only if f is real valued on A_h and it vanishes on $\ker \tau$.*
(iv) *The map $f \longrightarrow f|_{A_h}$, is an isomorphism between the real vector space of all hermitian linear functionals and the dual vector space of the real space A_h.*

3. Involutions on the dual of a topologically introverted space

The following theorem is an extension of a result of Civin and Yood [8, Theorem 6.2] to the dual of a topologically introverted space.

Theorem 3.1. *Let A be a Banach algebra and X, a faithful, topologically (left and right) introverted subspace of A^*.*

(i) *If there is a w^*-continuous, injective, anti-homomorphism (with respect to either of the Arens products) $\Theta\colon X^* \longrightarrow X^*$ such that $\Theta(A) \subset A$, then the two Arens products coincide on X^*.*
(ii) *Let $\theta\colon A \longrightarrow A$ be an involution on A and let $\theta^*\colon A^* \longrightarrow A^*$ be its conjugate-linear adjoint. If $\theta^*(X) \subset X$ and if the two Arens products coincide on X^*, then $\Theta = (\theta^*|_X)^*\colon X^* \longrightarrow X^*$, is an involution on X^*, extending θ.*

Proof. (i) Let $\mu, \nu \in X^*$, and let $(a_\alpha), (b_\beta)$ be two nets in A such that $a_\alpha \to \mu$, $b_\beta \to \nu$, in the w^*-topology. Let us assume Θ is an anti-homomorphism with respect to the first Arens product. Then $\Theta(\alpha_\alpha) \to \Theta(\mu)$, and $\Theta(b_\beta) \to \Theta(\nu)$. Hence

$$\begin{aligned}
\Theta(\mu\Box\nu) &= \Theta(\nu)\Box\Theta(\mu)\\
&= w^*\text{-}\lim_\beta \Theta(b_\beta)\Box\Theta(\mu)\\
&= w^*\text{-}\lim_\beta(w^*\text{-}\lim_\alpha \Theta(b_\beta)\Box\Theta(a_\alpha))\\
&= w^*\text{-}\lim_\beta(w^*\text{-}\lim_\alpha \Theta(b_\beta)\Theta(a_\alpha))\\
&= w^*\text{-}\lim_\beta(w^*\text{-}\lim_\alpha \Theta(a_\alpha b_\beta))\\
&= w^*\text{-}\lim_\beta \Theta(\mu\Diamond b_\beta)\\
&= \Theta(\mu\Diamond\nu).
\end{aligned}$$

Since Θ is injective, $\mu\Box\nu = \mu\Diamond\nu$, which is we wanted to show.

The claim in (ii) follows by a similar argument as in (i). $\Box$

It is well known that the two Arens products coincide on $WAP(A^*)^*$ and $AP(A^*)^*$ (Dales–Lau [11, Proposition 3.11]). It is also straight forward to check that both of these spaces are invariant under the conjugate-linear adjoint of any involution of A. Therefore if either of these spaces is faithful (which is the case, for example, if the spectrum of A separates the points of A; see Dales–Lau [11, p. 32]), then its dual has an involution extending that of A. Hence as a corollary of the above theorem we obtain the following result due to Farhadi and Ghaharamani ([15, Theorem 3.5]).

Corollary 3.2. *Suppose that A is an involutive Banach algebra and X is either of the topologically introverted spaces $AP(A^*), WAP(A^*)$. If X is faithful, then X^* has an involution extending the involution of A.*

Let $X \subset A^*$ be a faithful, topologically left introverted subspace of A^*. Let $\sigma(A)$ denote the spectrum of A and let $\varphi \in \sigma(A) \cap X$. We call an element $m \in X^*$ a φ-topological invariant mean (φ-TIM) if $\langle m, \varphi\rangle = 1$ and $a \cdot m = m \cdot a = \varphi(a)m$ for all $a \in A$.

The following theorem is an extension of a result of Farhadi and Ghahramani [15, Theorem 3.2(a)] (see the introduction) to the dual of topologically left introverted spaces.

Theorem 3.3. *Let A be a Banach algebra and $X \subset A^*$ be a faithful, topologically left introverted subspace of A^*. Let $\varphi \in \sigma(A) \cap X$. If X^* contains at least two φ-TIMs, then X^* cannot have an involution $*$ such that $\varphi(a^*) = \overline{\varphi(a)}$ for every $a \in A, \varphi \in \sigma(A)$.*

Proof. Let us suppose that X^* has an involution as in the statement of the theorem. Let $m \in X^*$ be an arbitrary φ-TIM. Then we have

$$a \cdot m^* = m^* \cdot a = \varphi(a)m^*, \qquad (a \in A). \tag{5}$$

To prove the above identities, we note that for every $a \in A$:

$$a \cdot m^* = (m \cdot a^*)^* = (\varphi(a^*)m)^* = (\overline{\varphi(a)}m)^* = \varphi(a)m^*.$$

The other identity in (5) is proved similarly. Using the w^*-continuity of the product $n\Box m$ on the variable n, it follows from (5) that

$$n\Box m^* = \langle n, \varphi\rangle m^*, \qquad (n \in X^*). \tag{6}$$

Thus using (6) and the fact that $\langle m, \varphi\rangle = 1$, we get

$$m = (m^*)^* = (m\Box m^*)^* = m\Box m^* = \langle m, \varphi\rangle m^* = m^*. \tag{7}$$

Now if m_1, m_2 are two distinct φ-TIMs, then using (6) and (7), we have

$$m_1 = m_2 \Box m_1 = (m_2 \Box m_1)^* = m_1^* \Box m_2^* = m_1 \Box m_2 = m_2;$$

which is a contradiction. □

To state a corollary of the above theorem, we first recall a few definitions. Let G be a locally compact group, $1 < p < \infty$, and $\mathscr{L}(L^p(G))$ be the space of continuous linear operators on $L^p(G)$. Let $\lambda_p \colon M(G) \longrightarrow \mathscr{L}(L^p(G))$, $\lambda_p(\mu)(g) = \mu * g$, where $\mu * g(x) = \int_G g(y^{-1}x)\, d\mu(y)$, be the left regular representation of $M(G)$ on $L^p(G)$. The space $PM_p(G)$ is the w^*-closure of $\lambda_p(M(G))$ in $\mathscr{L}(L^p(G))$. This space is the dual of the Herz–Figà-Talamanca algebra $A_p(G)$, consisting of all functions $u \in C_0(G)$, such that $u = \sum_{i=1}^{\infty} g_i * \check{f}_i$, where $f_i \in L^p(G), g_i \in L^q(G)$, $1/p + 1/q = 1$, and $\sum_{i=1}^{\infty} \|f_i\|_p \|g_i\|_q < \infty$ (Herz [24]). When $p = 2$, $A_2(G)$ and $PM_2(G)$ coincide respectively, with the Fourier algebra $A(G)$ and the group von Neumann algebra $VN(G)$ studied by Eymard in [14]. In the following, for simplicity of notation, we denote $UC(A_p(G))$ by $UC_p(G)$; when $p = 2$, this space is also denoted by $UC(\widehat{G})$ in the literature.

Corollary 3.4. *Let $1 < p < \infty$, G be a non-discrete locally compact group, and $X = PM_p(G)$ or $X = UC_p(G)$. Then X^* does not have any involution $*$ such that $\varphi(u^*) = \overline{\varphi(u)}$ for every $u \in A_p(G), \varphi \in \sigma(A_p(G))$.*

Proof. Let $e \in G$ be the identity of G, and $\varphi_e \in \sigma(A_p(G)) \cap X$ be the evaluation functional at e. Let $TIM(X^*)$ denote the set of all φ_e-TIMs on X^*. Granirer [21, Theorem, p. 3400] has shown that if G is non-discrete, then $|TIM(X^*)| \geq 2^{\mathfrak{c}}$, where $\mathfrak{c}$ is the cardinality of real numbers. Therefore our result follows from Theorem 3.3. □

Remark 3.5. For $p = 2$, the cardinality of $TIM(PM_2(G))$ was determined for second countable groups by Chou [7], and in full generality by Hu [25].

Let G be a locally compact group and $LUC(G)$ the space of all left uniformly continuous functions on G. It is known, and easy to verify, that the natural restriction map $\pi \colon L^\infty(G)^* \longrightarrow LUC(G)^*$ is a continuous algebra homomorphism (with respect to the first Arens product). Using this fact we can prove the following analogue of Singh's result [32, Theorem 2.2] for the non-existence of involutions on $LUC(G)^*$. Our result extends Farhadi–Ghahramani [15, Theorem 3.2(b)], from amenable to subamenable groups.

Theorem 3.6. *If G is a non-compact subamenable group, then $LUC(G)^*$ has no involutions.*

Proof. Since G is subamenable, there exists a subset $\mathcal{F}$ of $L^\infty(G)^*$ containing a mean m such that $|\mathcal{F}| < 2^{2^{\kappa(G)}}$, and the linear span of $\mathcal{F}$ is a left ideal J of $L^\infty(G)^*$. Since $\pi\colon L^\infty(G)^* \longrightarrow LUC(G)^*$ (defined above) is a continuous homomorphism, it follows that $\pi(J)$ is a non-trivial left ideal in $LUC(G)^*$ ($\pi(J)$ is non-trivial since $\pi(m) \neq 0$). Since the dimension of $\pi(J)$ is less than or equal to the dimension of J, it follows that $LUC(G)^*$ has a non-trivial left ideal of dimension less that $2^{2^{\kappa(G)}}$. By Filali–Pym [18, Theorem 5], if G is a non-compact locally compact group, every non-trivial right ideal of $LUC(G)^*$ has dimension at least $2^{2^{\kappa(G)}}$. It follows that for non-compact subamenable groups G, $LUC(G)^*$ cannot have any involutions. $\square$

4. Trivolutions on the duals of introverted spaces

In Corollary 3.4 and Theorem 3.6 we saw several examples of topologically left introverted spaces X for which there can be no involution on X^*. Our objective in this section is to consider some cases for which A^{**} or X^* admits trivolutions.

Theorem 4.1. *Let A be Banach algebra with an involution θ. Then under each of the following conditions, A^{**} admits a trivolution.*

(i) *There exists a topologically introverted, faithful subspace $X \subset A^*$, such that the two Arens products coincide on X^*, $\theta^*(X) \subset X$, and $A^{**} = X^\circ \oplus X^*$.*
(ii) *A is a dual Banach algebra.*
(iii) *A has a bounded two-sided approximate identity and is a right ideal in $(A^{**}, \square)$.*

Proof. (i) This follows from Theorem 3.1(ii) and Corollary 2.5.

(ii) Let A_* be a predual of A, and consider the canonical Banach space decomposition $A^{**} = (A_*)^\circ \oplus A$ (cf. Dales [10, p. 241]). It is easy to verify that A_* is a topologically introverted subspace of A^*, and clearly the two Arens products coincide on $(A_*)^* = A$. Hence (ii) follows from Corollary 2.5.

(iii) Since A has a bounded two-sided approximate identity we have the decomposition $A^{**} \cong (A^* \cdot A)^\circ \oplus (A^* \cdot A)^*$; and since A is a right ideal in its second dual, we have $WAP(A^*) = A^* \cdot A$ (cf. [3, Corollary 1.2, Theorem 1.5]). Therefore, $A^{**} \cong WAP(A^*)^\circ \oplus WAP(A^*)^*$. It is easy to check that under these conditions, $WAP(A^*)$ is faithful, and therefore our result follows by Corollaries 3.2 and 2.5 $\square$

Next we study trivolutions on the Banach algebra $L^\infty(G)^*$, equipped with its first Arens product $\square$. For simplicity of notation, in the following we shall denote $E\square F$ by EF, whenever $E, F \in L^\infty(G)^*$. Let G be a locally compact group. If $K \subset G$ is measurable and $f \in L^\infty(G)$, let

$$\|f\|_K = \operatorname{ess\,sup}\{|f(x)|\colon x \in K\},$$

and let $L_0^\infty(G)$ be the closed ideal of $L^\infty(G)$ consisting of all $f \in L^\infty(G)$ such that for given $\epsilon > 0$, there exists a compact set $K \subset G$ such that $\|f\|_{G\setminus K} < \epsilon$.

In [29, Theorems 2.7 and 2.8], Lau and Pym showed that $L_0^\infty(G)$ is a faithful, topologically introverted subspace of $L^\infty(G)$ and $L^\infty(G)^*$ is the Banach space direct sum

$$L^\infty(G)^* = L_0^\infty(G)^\circ \oplus L_0^\infty(G)^*. \tag{8}$$

In this decomposition $L_0^\infty(G)^*$ is identified with the closed subalgebra of $L^\infty(G)^*$ defined as the norm closure of elements in $L^\infty(G)^*$ with compact carriers ($F \in L^\infty(G)^*$ has compact carrier if for some compact set K, $F(f) = F(\chi_K f)$ for every $f \in L^\infty(G)$). In addition, Lau and Pym showed that if $\pi\colon L^\infty(G)^* \longrightarrow LUC(G)^*$ is the natural restriction map, then $\pi(L_0^\infty(G)^*) = M(G)$. Lau and Pym [29] make a case for the study of $L_0^\infty(G)^*$ for general G (in place of $L^1(G)^{**}$). In [31], the third named author has expressed $L_0^\infty(G)^*$ as the second dual of $L^1(G)$ with a locally convex topology similar to the strict topology (see also [23]). Let $\mathscr{E}(G)$ denote the set of all right identities of $L^\infty(G)^*$, and $\mathscr{E}_1(G)$ the set of those with norm one. In $L^\infty(G)^*$ when G is not discrete, there is an abundance of such right identities, a fact noted and well-utilized in ([22], [29], [31], [15]), for instance.

For the convenience of our readers, we shall now state the following result of Lau and Pym ([29], Theorems 2.3 and 2.11) which will be needed repeatedly in what follows. In the following results all products are with respect to the first Arens product.

Theorem 4.2 (Lau–Pym)**.** *Let G be a locally compact group and let the map $\pi\colon L^\infty(G)^* \longrightarrow LUC(G)^*$ be the natural restriction map. Then*

(i) $\mathscr{E}_1(G) \subset L_0^\infty(G)^*$.
(ii) *For each $E \in \mathscr{E}(G)$, $\pi\big|_{EL^\infty(G)^*}$ is a continuous isomorphism from $EL^\infty(G)^*$ to $LUC(G)^*$, and if $\|E\| = 1$, the isomorphism is an isometry.*
(iii) *For each $E \in \mathscr{E}_1(G)$, $L_0^\infty(G)^* = EL_0^\infty(G)^* + (\ker\pi \cap L_0^\infty(G)^*)$, and the algebra $EL_0^\infty(G)^*$ is isometrically isomorphic with $M(G)$ via π.*

Theorem 4.3. *Let G be a non-discrete locally compact group and X and Y be subalgebras of $L^\infty(G)^*$ with $L_0^\infty(G)^* \subset Y \subset X$. Then there are no trivolutions of X onto Y. In particular, $L^\infty(G)^*$ has no trivolutions with range $L_0^\infty(G)^*$.*

Proof. If G is compact, then $X = Y = L^\infty(G)^*$, and hence any trivolution of X onto Y is an involution on $L^\infty(G)^*$ (Corollary 2.4). Such an involution does not exist if G is non-discrete by Grosser [22, Theorem 2].

Let G be non-compact. To obtain a contradiction, let ρ be a trivolution from X onto Y. By Theorem 4.2(i), $\mathscr{E}_1(G) \subset Y$. By Theorem 2.7(i), each E in $\mathscr{E}_1(G)$ is the identity for Y, and therefore, also the identity for $L_0^\infty(G)^*$. But the identity for $L_0^\infty(G)^*$ is clearly in the topological centre of $L_0^\infty(G)^*$ and so it belongs to $L^1(G)$ (cf. Budak–Işık–Pym[4, Proposition 5.4]), which is not possible since G is not discrete. □

Theorem 4.4. *The algebra $L_0^\infty(G)^*$ has an involution if and only if G is discrete. Further, if G is discrete, $L^\infty(G)^*$ has a trivolution with range $L^1(G)$, extending the natural involution on $L^1(G)$.*

Proof. If G is discrete, then $L_0^\infty(G)^* = C_0(G)^* = L^1(G)$ has a natural involution, and hence by (8) and by Corollary 2.5, $L^\infty(G)^*$ has a trivolution with range $L^1(G)$, extending the involution of $L^1(G)$

If G is not discrete, then the result follows from Theorem 4.3 upon taking $X = Y = L_0^\infty(G)^*$. □

Theorem 4.5. *If G is compact, then for each $E \in \mathcal{E}(G)$, there are trivolutions of $L^\infty(G)^*$ onto $EL^\infty(G)^*$.*

Proof. Let $E \in \mathscr{E}(G)$. The compactness of G implies that $L_0^\infty(G) = L^\infty(G)$ and $LUC(G)^* = M(G)$. Let ρ be any involution on $LUC(G)^*$ and let $\pi' = \pi\big|_{EL^\infty(G)^*}$. It follows from Theorem 4.2(iii) that $\rho' := (\pi')^{-1} \circ \rho \circ \pi'$ is an involution on $EL^\infty(G)^*$. Let $\ell_E \colon L^\infty(G)^* \longrightarrow EL^\infty(G)^*$ be the left multiplication by E. Then by Theorem 2.7(iii), $\tau := \rho' \circ \ell_E$ is a trivolution of $L^\infty(G)^*$ onto $EL^\infty(G)^*$, as required. □

Theorem 4.6. *Let G be a locally compact group. For each $E \in \mathscr{E}_1(G)$, there exists a trivolution of $L_0^\infty(G)^*$ onto $EL_0^\infty(G)^*$.*

Proof. By Theorem 4.2, $\mathscr{E}_1(G) \subset L_0^\infty(G)^*$, and for each $E \in \mathscr{E}_1(G)$, $EL_0^\infty(G)^* \cong M(G)$. If ρ is an involution on $M(G)$, then it is easily checked that $\rho' := (\pi\big|_{EL_0^\infty(G)^*})^{-1} \circ \rho \circ (\pi\big|_{EL_0^\infty(G)^*})$ is an involution on $EL_0^\infty(G)^*$, and hence by Theorem 2.7(iii), $\tau := \rho' \circ \ell_E$ is a trivolution of $L_0^\infty(G)^*$ onto $EL_0^\infty(G)^*$. □

Remark 4.7. If ρ in the proofs of Theorems 4.5 or 4.6 restricts to an involution ρ_0 on $L^1(G)$, then in view of the fact that π is the identity on $L^1(G)$, the trivolution τ constructed in the respective proofs will be an extension of ρ_0.

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