Some new applications of P-P plots

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Abstract

The P-P plot is a powerful graphical tool to compare stochastically the magnitudes of two random variables. In this note, we introduce a new partial order, called P-P order based on P-P plots. For a pair of random variables (X_1, Y_1) and (X_2, Y_2) one can see the relative precedence of Y_2 over X_2 versus that of Y_1 over X_1 using P-P order. We show that several seemingly very technical and difficult concepts like convex transform order and super-additive ordering can be easily explained with the help of this new partial order. Several concepts of positive dependence can also be expressed in terms of P-P orders of the conditional distributions.

Keywords: Convex transform order, superadditive ordering, likelihood ratio order, hazard rate order, more regression dependence, more RTI, location model, PHR model.

1 Introduction

Consider an experiment in which k treatments are to be compared with their controls and we are interested in identifying the treatment which is most effective. A similar situation can also arise if we want to examine the effectiveness of a treatment relative to its control across different experimental conditions or different patient populations. The classical parametric approach would be to compare the k differences between the means of the treatment and the control effects or their standardized versions if the variances differ. As discussed in *Holmgren* (1995) such an approach may not be appropriate unless the underlying random variables belong to the same parametric family differing only in their location parameters. Here is an illustration of this fact.

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Let $X_1, \ldots X_4$ be independent random variables such that $X_i \sim N(\mu_i, \sigma_i^2)$, $i = 1, \ldots, 4$. Then

$$P[X_2 > X_1] = \Phi\left(\frac{\mu_2 - \mu_1}{\sqrt{(\sigma_1^2 + \sigma_2^2)}}\right)$$

and

$$P[X_4 > X_3] = \Phi\left(\frac{\mu_4 - \mu_3}{\sqrt{(\sigma_3^2 + \sigma_4^2)}}\right).$$

It can be seen that $\mu_4 - \mu_3 > \mu_2 - \mu_1$ may not imply $P[X_4 > X_3] \leq P[X_2 > X_1]$ for certain choices of σ_i^2's. Thus it does not always make sense to make such comparisons by looking at only the differences of the treatment means and the control means. The same is true if we make decisions based on differences of medians or percentiles of a certain order. Such comparisons become even less meaningful if the underlying distributions belong to different families.

Let X and Y be two continuous type random variables with cumulative distribution functions F and G, survival functions $\overline{F} = 1 - F$ and $\overline{G} = 1 - G$ and density functions f and g, respectively. We use the notation F^{-1} to denote the right continuous inverse of F. For the sake of convenience, let us think of F as the control distribution and G as the treatment distribution. The P-P plot is defined as the graph of G versus F and is thus expressed in the function form as GF^{-1}. Since the P-P plot can be expressed as

$$GF^{-1}(p) = P[\, Y < F^{-1}(p)\,] = P[\, F(Y) < p\,]$$

the P-P plot is the distribution function of $F(Y)$. Since $F(\dot)$ transforms X into a random variable with uniform distribution on $[0, 1]$, the P-P plot is a graph of the treatment distribution with the measurement scale transformed so that the control distribution is uniform on the unit interval.

In this paper we further study the stochastic properties of P-P plots and see how it can help in understanding the various concepts in probability and statistics. Let us first recall some notions of stochastic orders.

- X is said to be smaller than Y according to *likelihood ratio ordering*, denoted by $X \leq_{lr} Y$, if $g(x)/f(x)$ is nondecreasing in x, where f and g denote the densities of X and Y, respectively.

- X is said to be smaller than Y in the *hazard rate order*, denoted by $X \leq_{hr} Y$, if

$$P(X - t > x | X > t) \leq P(Y - t > x | Y > t), \quad \text{for all } x \geq 0 \text{ and all } t.$$

- X is said to be smaller than Y in the *reverse hazard rate order*, denoted by $X \leq_{rh} Y$, if

$$P(t - X > x | X \leq t) \geq P(t - Y > x | Y \leq t), \quad \text{for all } x \geq 0 \text{ and all } t.$$

- X is said to be *stochastically smaller* Y written as $X \leq_{st} Y$ if

$$G(x) \leq F(x), \forall x.$$

We have the following chain of implications among the above stochastic orders

$$
\begin{array}{ccc}
X \leq_{lr} Y & \Longrightarrow & X \leq_{hr} Y \\
\Downarrow & & \Downarrow \\
X \leq_{rh} Y & \Longrightarrow & X \leq_{st} Y.
\end{array}
$$

The next theorem shows the usefulness of P-P plots as a powerful graphical tool in detecting the above stochastic orders.

Theorem 1.1 (a) $X \leq_{lr} Y$ if and only if the P-P plot GF^{-1} is convex.

(b) $X \leq_{hr} Y$ if and only if the P-P plot GF^{-1} is convex at 1. That is,

$$GF^{-1}(p + (1-p)u) \leq p + (1-p) \, GF^{-1}(u) \text{ for all } p \in (0,1) \text{ and all } u \in [0,1], \qquad (1.1)$$

assuming that $GF^{-1}(1) = 1$.

(c) $X \leq_{rh} Y$ if and only if the P-P plot GF^{-1} is convex at 0. That is,

$$GF^{-1}(pu) \leq p \, GF^{-1}(u) \text{ for all } p \in (0,1) \text{ and all } u \in [0,1], \qquad (1.2)$$

assuming that $GF^{-1}(0) = 0$.

(d) $X \leq_{st} Y$ if and only if the P-P plot GF^{-1} is entirely below the 45 degree line line.

Proof :

(a) Since

$$\frac{d}{dx} GF^{-1}(u) = \frac{gF^{-1}(u)}{fF^{-1}(u)}$$

is nondecreasing in u if GF^{-1} is convex, it follows that $\frac{g(x)}{f(x)}$ is nondecreasing in x in that case , since $F^{-1}(u)$ is a nondecreasing function of u.

(b) Let $X_t \equiv \{X - t \mid X > t\}$ denote the residual life of X at time t. It is known that

$$X \leq_{hr} Y \quad \Leftrightarrow \quad X_t \leq_{st} Y_t \text{ for all } t \qquad (1.3)$$
$$\Leftrightarrow \quad F_t^{-1}(p) \leq G_t^{-1}(p) \text{ for all } p \in [0,1] \text{ and for all } t,$$

where F_t denotes the distribution function of X_t. It is easy to see that

$$F_t^{-1}(p) = F^{-1}[1 - (1-p)\bar{F}(t)] - t,$$

where $\bar{F} = 1 - F$, denotes the survival function of X.

Therefore, (1.3) holds if and only if

$$F^{-1}[1 - (1-p)\bar{F}(t)] \leq G^{-1}[1 - (1-p) \, \bar{G}(t)], \text{ for all } p \in [0,1] \text{ and for all } t,$$
$$\Leftrightarrow \quad GF^{-1}[p + (1-p)F(t)] \leq p + (1-p)G(t), \text{ for all } p \in [0,1] \text{ and for all } t,$$
$$\Leftrightarrow \quad GF^{-1}[p \cdot 1 + (1-p)u] \leq p \, GF^{-1}(1) + (1-p) \, GF^{-1}(u), \text{ for all } p \in [0,1] \text{ and for all } u \in [0,1],$$

assuming that $GF^{-1}(1) = 1$. This proves the desired result

(c) The proof is similar to the one in part (b).

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(d) The proof is obvious.

∎

Remarks:

(a) The condition (1.1) means that any chord from $(1, 1)$ to any point on the P-P plot is entirely above the P-P plot.

(b) The condition (1.2) means that any chord from $(0, 0)$ to any point on the P-P plot is entirely above the P-P plot. We call such a function as star-shaped.

In Figure 1 we plot the P-P plot of G versus F where F is the distribution function of a parallel system having 3 independent exponential lifetimes each with same failure rate 2 and G is the distribution function of a parallel system having 3 independent exponential lifetimes with failure rates $1/4, 4, 8$, respectively. We see that the P-P plot lies entirely below the 45 degree line thus showing that G is stochastically greater than F. In Figure 2 we see that the chord joining $(0, 0)$ to the point $(.6335, GF^{-1}(.6335))$ on this P-P plot does not lie above the P-P plot. This is an example where G is stochastically greater than F, but they are not ordered according to reverse hazard rate ordering. However, any chord from $(1, 1)$ to any point on the P-P plot lies entirely above the curve confirming that the hazard rate of G is smaller than that of F. A theoretical justification of this observation can be found in *Khaledi and Kochar (2000)*.

Figure 3 shows the P-P plot of G versus F where $F(x)$ and $G(x)$ are the distribution functions of two- component parallel systems with two independent exponential lifetimes with failure rates $(2, 15)$ and $(2, 5)$, respectively. Figure 4 shows that a chord from $(1, 1)$ to $(0, GF^{-1}(0, .45))$ does not lie above the P-P plot, indicating that hazard rate ordering does not hold. However, the stochastic ordering holds since the P-P plot is entirely below the 45 degree line. In fact, any chord from $(0, 0)$ to any point on the P-P plot lies above the chord indicating reverse hazard rate ordering between F and G.

The P-P order

As seen above, $X \leq_{st} Y$ if and only if the P-P plot of G versus F lies entirely below the 45 degree line. The same idea can be extended to compare two pairs of random pairs (X_1, Y_1) and (X_2, Y_2). We want to see whether the increase (or dominance) of Y_2 over X_2 is more than that of Y_1 over X_1. In the context of examining the efficacy of a Treatment as compared to the Control, we may like to see in which population Treatment is most effective.

Definition 1.2 *We say that a pair of random variables (X_2, Y_2) is preferred to another pair of random variables (X_1, Y_1) according to P-P order (written as $(X_2, Y_2) \succ_{PP} (X_1, Y_1)$) if and only if the P-P plot of (X_2, Y_2) is below that of (X_1, Y_1). That is, if and only if,*
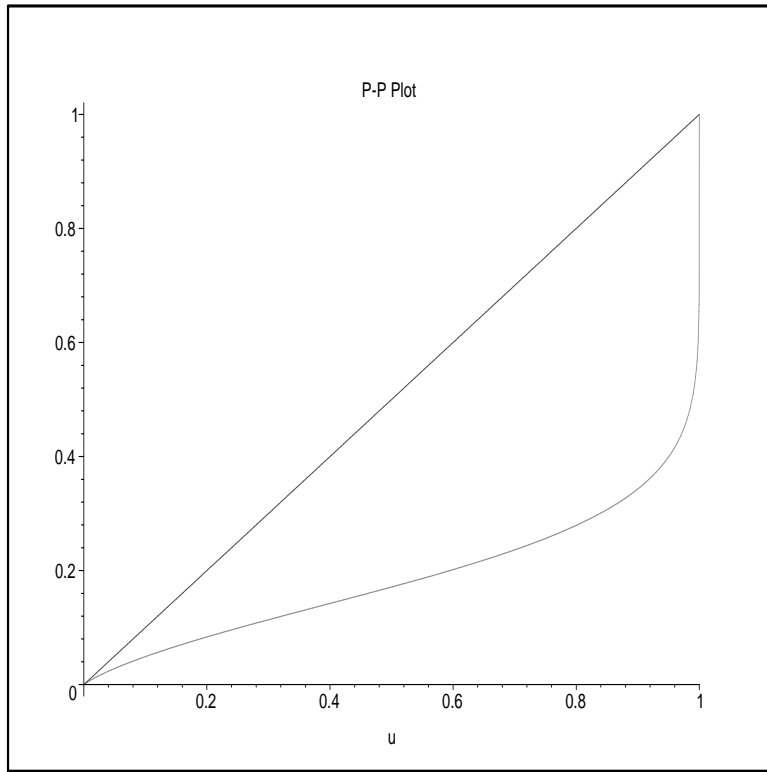
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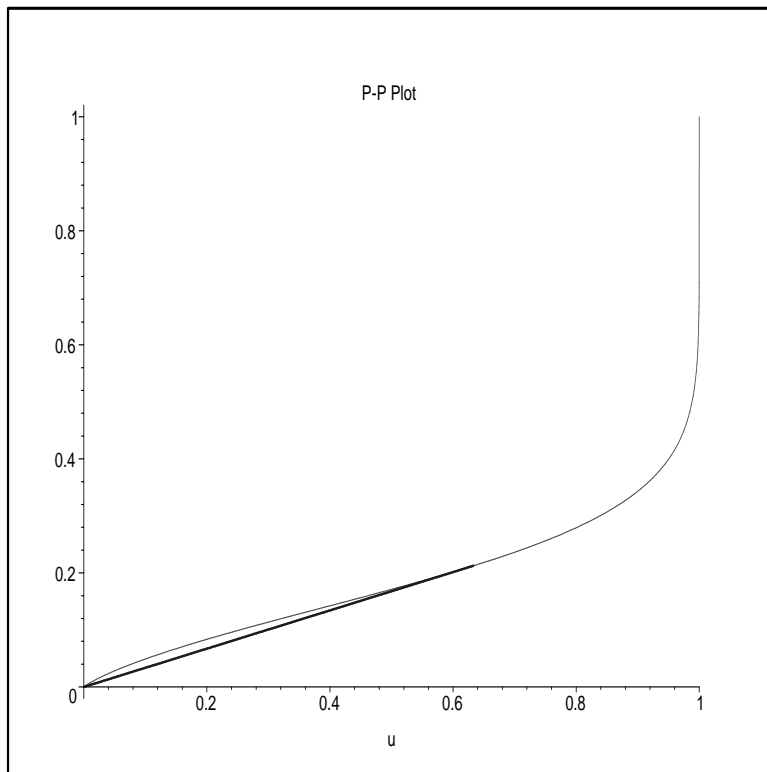Figure 1: P-P Plot showing stochastic ordering

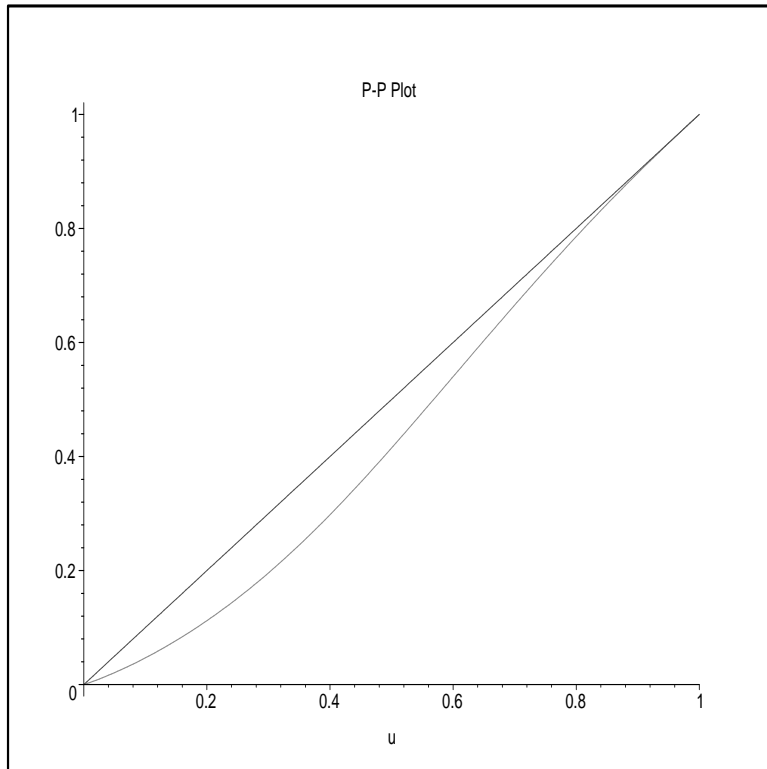Figure 2: P-P Plot showing stochastic ordering but not RH ordering
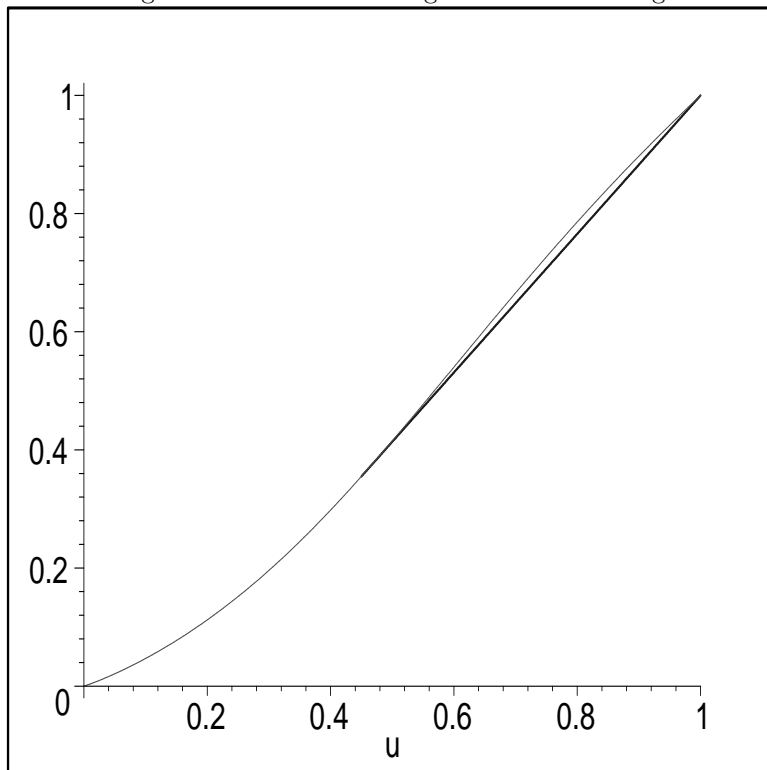
Figure 3: P-P Plot showing stochastic ordering

Figure 4: P-P Plot showing stochastic ordering but not HR ordering

$$G_2 F_2^{-1}(u) \le G_1 F_1^{-1}(u), \quad \forall \, u \quad in \ [0,1]. \tag{1.4}$$

Let $F(x) = 1 - exp(-(x/a_F)^{b_F})$ and $G(x) = 1 - exp(-(x/a_G)^{b_G})$ be two Weibull distribution functions. Then

$$GF^{-1}(p) = 1 - exp(-a^{b_G}[-log(1-p)]^b),$$

where $a = a_F/a_G$ and $b = b_G/b_F$.

Figure 5 shows the P-P plots of Weibull distributions for various choices of the scale and shape parameters. Let (F_1, G_1), (F_2, G_2) and (F_3, G_3) be three pairs of Weibull distributions with shape parameters $(2, 1.818)$, $(0.899, 0.818)$ and $(0.899, 0.428)$, respectively. The ratio of the scale parameters $a = .4545$ in all the three cases. Then, clearly $G_2 F_2^{-1}(u) \le G_1 F_1^{-1}(u)$, (green curve is below the violet curve) clearly indicating that $(X_2, Y_2) \succ_{PP} (X_1, Y_1)$. The brown curve crosses the other two curves, showing that this is a partial order. There is no P-P order between (X_1, Y_1) and (X_3, Y_3) and (X_2, Y_2) and (X_3, Y_3). The red line indicates the P-P plot with $F(x) = G(x)$.

For $0 \le p \le 1$, let $\xi_{p,1}$ and $\xi_{p,2}$ denote the quantiles of order p of F_1 and F_2, respectively. Then, $(X_2, Y_2) \succ_{PP} (X_1, Y_1)$ means that

$$G_2(\xi_{p,2}) \le G_1(\xi_{p,2}) \Leftrightarrow P[Y_2 \ge \xi_{p,2}] \ge P[Y_1 \ge \xi_{p,2}] \text{ for all } p \in (0,1).$$

As explained in *Holmgren* (1989), with the help of P-P plots, treatments and controls from different studies can be compared directly to see the relative magnitudes of different treatment effects as if the control distributions are homogeneous across studies.

Remarks: If in (1.4) above

(a) $F_1 = G_1$, then $G_2 F_2^{-1}(x) \le x$, that is, $X_2 <_{st} Y_2$,

(b) $F_2 = G_2$, then $G_1 F_1^{-1}(x) \ge x$, that is, $X_1 >_{st} Y_1$.

(c) It is easy to show that (1.4) is also equivalent to

$$F_2^{-1} F_1(x) \le G_2^{-1} G_1(x) \ \text{ for all } x \ . \tag{1.5}$$

Theorem 1.3 *Let X_i and Y_i be independent for $i = 1, 2$. Then*

$$(X_2, Y_2) \succ_{PP} (X_1, Y_1) \Rightarrow P(X_2 \le Y_2) \ge P(X_1 \le Y_1).$$

Proof : $G_2 F_2^{-1}(u) \le G_1 F_1^{-1}(u)$ for all $u \in (0,1)$ implies that

$$\int_0^1 G_2 F_2^{-1}(u) du \le \int_0^1 G_1 F_1^{-1}(u) du$$

which is equivalent to

$$\int_{-\infty}^{\infty} G_2(x) dF_2(x) \ \le \ \int_{-\infty}^{\infty} G_1(x) dF_1(x)$$
$$\Leftrightarrow P(Y_2 < X_2) \ \le \ P(Y_1 < X_1)$$
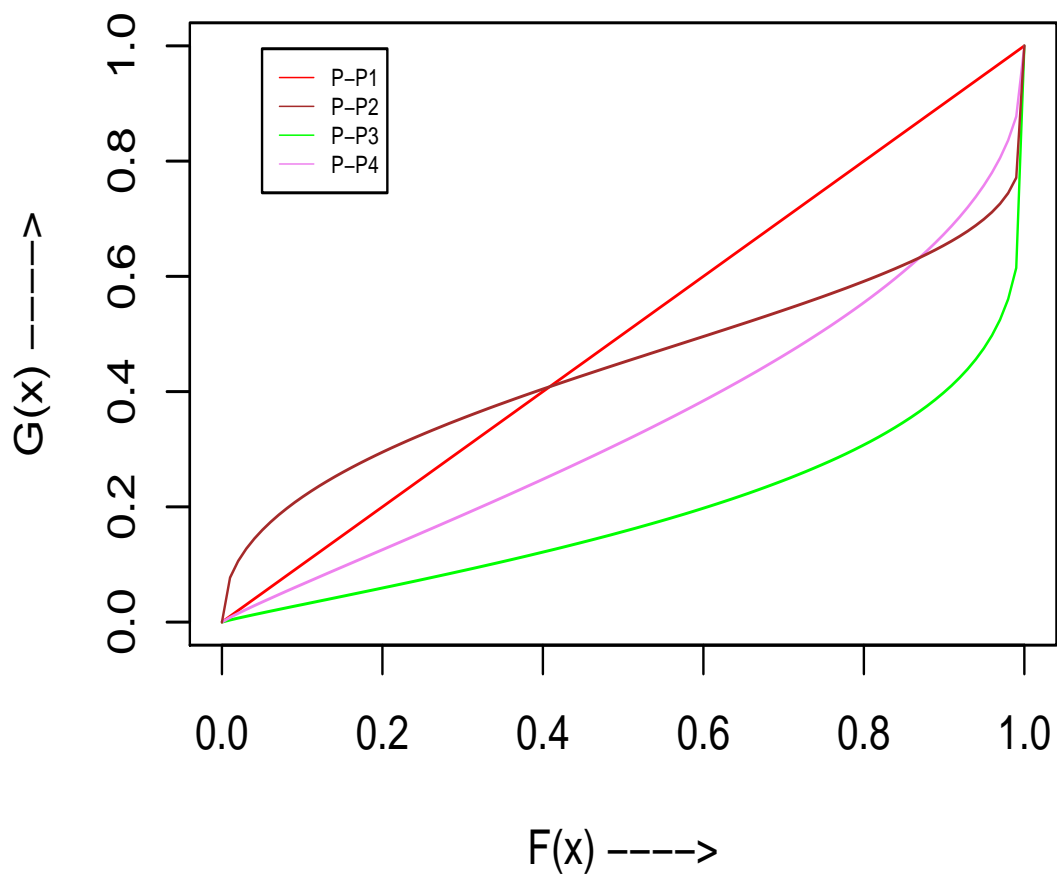$$\Leftrightarrow P(X_2 \le Y_2) \ \ge \ P(X_1 \le Y_1)$$

Figure 5: P-P plots of Weibull distributions

■

Now we look at some models which lead to P-P ordering between two pairs of independent random variables.

The Location Family

Let X_1 and X_2 be two continuous random variables belonging to a location family of distributions and let Y_1 and Y_2 belong to possibly another location family of distributions. It is shown in the next theorem that if the differences of location parameters in the two populations are ordered, then the pair of random variables are P-P ordered.

Theorem 1.4 *Suppose $F_1(x) = F(x - \mu_1), G_1(x) = G(x - \mu_2); F_2(x) = F(x - \mu_3), G_2(x) = G(x - \mu_4)$ for every x. Further assume that $\mu_2 - \mu_1 < \mu_4 - \mu_3$. Then $(X_2, Y_2) \succ_{\mathrm{PP}} (X_1, Y_1)$.*

Proof: It is easy to see that $F_1^{-1}(u) = \mu_1 + F^{-1}(u)$. Then

$$G_1 F_1^{-1}(u) = G_1[\mu_1 + F^{-1}(u)] = G[\mu_1 - \mu_2 + F^{-1}(u)].$$

Similarly

$$G_2 F_2^{-1}(u) = G_2[\mu_3 + F^{-1}(u)] = G[\mu_3 - \mu_4 + F^{-1}(u)].$$

Since $\mu_2 - \mu_1 < \mu_4 - \mu_3$, it follows that $G_2 F_2^{-1}(u) \leq G_1 F_1^{-1}(u)$ for all $u \in (0, 1)$. ■

The Scale Family

A similar result holds for the scale family of distributions for nonnegative random variables.

Theorem 1.5 *Suppose $F_1(x) = F(x/\sigma_1), G_1(x) = G(x/\sigma_2); F_2(x) = F(x/\sigma_3), G_2(x) = G(x/\sigma_4)$ for $x \geq 0$. Further assume that $\frac{\sigma_3}{\sigma_4} < \frac{\sigma_1}{\sigma_2}$. Then $(X_2, Y_2) \succ_{\mathrm{PP}} (X_1, Y_1)$.*

Proof: It is easy to see that $F_1^{-1}(x) = \sigma_1 F^{-1}(x)$. Then

$$G_1 F_1^{-1}(x) = G_1[\sigma_1 F^{-1}(x)] = G[\frac{\sigma_1}{\sigma_2} F^{-1}(x)].$$

Similarly

$$G_2 F_2^{-1}(x) = G_2[\sigma_3 F^{-1}(x)] = G[\frac{\sigma_3}{\sigma_4} F^{-1}(x)].$$

Since $\frac{\sigma_3}{\sigma_4} < \frac{\sigma_1}{\sigma_2}$, the result follows.

■

In particular, results in Theorems 1.4 and 1.5 hold if all the four random variables belong to the same location and scale family, respectively.

The Proportional Hazard Rate Family

Theorem 1.6 *Suppose* $\bar{F}_1(x) = \bar{F}^{\alpha_1}(x), \bar{G}_1(x) = \bar{F}^{\alpha_2}(x)$ *and* $\bar{F}_2(x) = \bar{G}^{\alpha_3}(x), \bar{G}_2(x) = \bar{G}^{\alpha_4}(x)$. *Then* $\frac{\alpha_2}{\alpha_1} > \frac{\alpha_4}{\alpha_3}$ *implies* $(X_2, Y_2) \succ_{PP} (X_1, Y_1)$.

Proof : The proof follows once we note that $G_1 F_1^{-1}(u) = 1 - (1-u)^{\frac{\alpha_2}{\alpha_1}}$ and $G_2 F_2^{-1}(u) = 1 - (1-u)^{\frac{\alpha_4}{\alpha_3}}$ and since $0 \leq (1-u) \leq 1$. ∎

The Proportional Reverse Hazard Rate Family

Theorem 1.7 *Suppose* $F_1(x) = F^{\alpha_1}(x), G_1(x) = F^{\alpha_2}(x)$ *and* $F_2(x) = G^{\alpha_3}(x), G_2(x) = G^{\alpha_4}(x)$. *Then* $\frac{\alpha_2}{\alpha_1} < \frac{\alpha_4}{\alpha_3}$ *implies* $(X_2, Y_2) \succ_{PP} (X_1, Y_1)$.

Proof : The proof follows once we note that $G_1 F_1^{-1}(u) = u^{\frac{\alpha_2}{\alpha_1}}$ and $G_2 F_2^{-1}(u) = u^{\frac{\alpha_4}{\alpha_3}}$ and since $0 \leq u \leq 1$. ∎

In this note we show that several partial orderings and concepts on stochastic orderings can be conveniently interpreted in terms of P-P order. In Section 2, we show that various aging orders like convex transform ordering and super-additive ordering can be easily explained by comparing the P-P plots of the residual life times. The last section is devoted to the study of various positive dependence orders where we show that these can be expressed in terms of the P-P orders between appropriate conditional distributions.

2 Aging comparisons

In this section we show that several *aging* orderings of life distributions can be easily explained with the help of P-P orders between the residual life times.

2.1 More IFR comparisons

The residual life of a random variable X at time t is given by $X_t = X - t | X > t$. A random variable X with cdf F is said to have increasing failure rate (IFR) distribution if X_t is stochastically decreasing in t. That is, if and only if for all $s < t$, $X_t \leq_{st} X_s$ or equivalently if

$$X_{\xi_q} \leq_{st} X_{\xi_p} \text{ for all } 0 \leq p < q \leq 1, \tag{2.1}$$

where $\xi_p = F^{-1}(p)$ denotes the *pth* quantile of X.

van Zwet (1970) introduced the following concept of convex ordering to compare the shapes of two probability distributions.

Definition 2.1 X is said to be smaller than Y in *the convex transform order*, denoted by $X \leq_c Y$ if and only if, $G^{-1}F(x)$ is convex in x on the support of X.

If $X \leq_c Y$, then Y is more skewed than X as explained in *van Zwet* (1970) and *Marshall and Olkin* (2007). The convex transform order is also known as more IFR (increasing failure rate) order in

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reliability theory, since when f and g exist, the convexity of $G^{-1}F(x)$ is equivalent to

$$\frac{f\left(F^{-1}(u)\right)}{g\left(G^{-1}(u)\right)} = \frac{r_X\left(F^{-1}(u)\right)}{r_Y\left(G^{-1}(u)\right)} \quad \text{being increasing in } u \in [0,1],$$

where r_X and r_Y denote the failure (or hazard) rates of X and Y, respectively. Thus $X \leq_c Y$ can be interpreted to mean that X ages faster than Y in some sense. This partial order is scale invariant. Note that neither F nor G need to be IFR for this definition to hold. It can be seen that a random variable X has an IFR distribution if and only if it is convex ordered with respect to exponential distribution, which has a constant failure rate. It is known that if X_{α_i} has Gamma distribution with shape parameter α_i, $i = 1,2$, then $X_{\alpha_1} \leq_c X_{\alpha_2}$ for $\alpha_2 \leq \alpha_1$. *Kochar and Xu (2009)* used this order to compare two parallel systems with exponential components.

The above definition of *more IFR* ordering does not explicitly relate to the residual lifetimes of random variables. In the next theorem we show that convex ordering can also be expressed in terms of the more P-P ordering of the residual lifetimes and thus giving us a better insight into this seemingly technical concept.

Theorem 2.2 $X \leq_c Y$ if and only if for all $0 \leq p < q \leq 1$,

$$(X_{\xi_q}, X_{\xi_p}) \succ_{\mathrm{PP}} (Y_{\xi_q^*}, Y_{\xi_p^*}), \tag{2.2}$$

where $\xi_p^* = G^{-1}(p)$ denotes the *pth* quantile of Y.

Proof : As noted earlier the survival function of X_t is

$$\bar{F}_t(x) = \frac{\bar{F}(t+x)}{\bar{F}(t)} \quad \text{and} \quad F_t^{-1}(x) = F^{-1}[1 - (1-x)\bar{F}(t)] - t.$$

Suppose that (2.2) holds. In the light of the equivalence of (1.4) and (1.5), (2.2) is equivalent to

$$F_{\xi_q}^{-1}G_{\xi_q^*}(x) \leq F_{\xi_p}^{-1}G_{\xi_p^*}(x) \quad \text{for all } x \text{ and } 0 \leq p < q \leq 1, \tag{2.3}$$

or equivalently

$$F_{\xi_p}^{-1}G_{\xi_p^*}(x) \quad \text{decreasing in } p \text{ for every } x. \tag{2.4}$$

Now

$$\begin{aligned} F_{\xi_p}^{-1}G_{\xi_p^*}(x) &= F^{-1}\left[1 - \overline{G}(x + \xi_p^*)\right] - \xi_p \\ &= F^{-1}G(x + G^{-1}(p)) - F^{-1}(p) \end{aligned}$$

is decreasing in p for every x if and only if

$$F^{-1}G(x+y) - F^{-1}G(y) \text{ is decreasing in } y \text{ for every } x \geq 0,$$

which is equivalent to $F^{-1}G(x)$ being concave or equivalently $G^{-1}F(x)$ being convex. Hence the proof.

∎

The relation expressed by (2.2) gives a natural and intuitive way of comparing the relative aging of two units. As a consequence of Theorem 2.2, it follows that $X \leq_c Y$ implies $P(X_{\xi_q} \leq X_{\xi_p}) \geq P(Y_{\xi_q^*} \leq Y_{\xi_p^*})$ for all $0 \leq p < q \leq 1$. If Y has exponential distribution, then the distribution of $Y_{\xi_p^*}$ does not depend on p and $P(Y_{\xi_q^*} \leq Y_{\xi_p^*}) = 1/2$.

2.2 More NBU comparisons

A random variable X with cdf F is said to be *new better than used* (NBU) if the residual life of a new component is better than that of a used component which has been working for a time t for all $t > 0$. This is equivalent to saying that

$$X_t \leq_{st} X \text{ for all } t > 0.$$

To compare the relative aging of two arbitrary nonnegative random variables X and Y with distribution functions F and G, respectively, the following definition of *super-additive ordering* is well known in the literature (cf. *Barlow and Proschan* (1981)).

Definition 2.3 X is said to be smaller than Y in *super-additive order*, denoted by $X \leq_{su} Y$ if and only if, $G^{-1}F(x)$ is super-additive in x on the support of X. That is, if and only if

$$G^{-1}F(x+y) \geq G^{-1}F(x) + G^{-1}F(y) \text{ for all } x, y \geq 0. \tag{2.5}$$

If $X \leq_{su} Y$ holds we also say that X *is more NBU than* Y. Like convex ordering, superadditive ordering is also scale invariant. It is known that convex ordering implies superadditive ordering, but the converse is not true. It is easy to show that if in the above definition, G is exponential, then F is NBU. Again the above definition of super-additive ordering is very technical and does not help in understanding the concept of one distribution aging faster than another. The next theorem gives an easy interpretation of this concept in terms of the P-P plots of the residual lifetimes.

Theorem 2.4 $X \leq_{su} Y$ if and only if for all $0 \leq p \leq 1$,

$$(X_{\xi_p}, X) \succ_{\text{PP}} (Y_{\xi_p^*}, Y), \tag{2.6}$$

where $\xi_p^* = G^{-1}(p)$ denotes the *pth* quantile of Y.

Proof : Note that (2.6) is equivalent to

$$
\begin{aligned}
F_{\xi_p}^{-1} G_{\xi_p^*}(x) &\leq F^{-1}G(x) \text{ for all } x \geq 0 \text{ and } p \in (0,1) \\
\Leftrightarrow F^{-1}G(x + G^{-1}(p)) - F^{-1}(p) &\leq F^{-1}G(x) \text{ for all } x \geq 0 \text{ and } p \in (0,1) \\
\Leftrightarrow F^{-1}G(x + G^{-1}(p)) &\leq F^{-1}G(x) + F^{-1}(p) \text{ for all } x \geq 0 \text{ and } p \in (0,1) \\
\Leftrightarrow F^{-1}G(x+y) &\leq F^{-1}G(x) + F^{-1}G(y) \text{ for all } x, y \geq 0.
\end{aligned}
$$

The last equation is the definition of $F^{-1}G$ being sub-additive or equivalently $G^{-1}F$ being super-additive. Hence the proof. ∎

3 Dependence orderings

In this section we use the P-P plots of the conditional distributions to describe various dependence orderings.

3.1　More monotone regression dependence

The following definition of *monotone regression dependence* or *stochastic increasingness* was introduced by *Lehmann* (1966) and can also be found in Chapter 5 of *Barlow and Proschan* (1981).

Definition 3.1 Given two random variables X and Y, we say that Y is stochastically increasing in X, denoted by $\mathrm{SI}(Y|X)$, if $P(Y > y \mid X = x)$ is increasing in x for all y; or equivalently,

$$P(Y \leq y|X = x) \geq P(Y \leq y|X = x^*), \quad x \leq x^*. \tag{3.1}$$

Observing that when X and Y are continuous, inequality (3.1) can be written as

$$H_{[\xi_q]}H_{[\xi_p]}^{-1}(u) \leq u \quad \text{for all } 0 < u < 1 \text{ and for } 0 < p \leq q < 1,$$

where $\xi_p = F^{-1}(p)$ stands for the pth quantile of the marginal distribution of X, and $H_{[s]}$ denotes the conditional distribution of Y given $X = s$.

The concept of more regression dependence or more SI was originally considered by *Yanagimoto and Okamoto* (1969) and later extended and further investigated by *Schriever* (1987), *Capéraà and Genest* (1990) and *Fang and Joe* (1992. The following formulation of *more monotone regression dependence* as given in *Avérous, et al.* (2005) is copula based.

Definition 3.2 Y_1 is said to be less *stochastic increasing* (or monotone regression dependent) in X_1 than Y_2 is in X_2, denoted by $(Y_1|X_1) \preceq_{\mathrm{SI}} (Y_2|X_2)$, if and only if, for $0 \leq u \leq 1$ and $0 < p \leq q < 1$,

$$H_{2,[\xi_{2q}]}H_{2,[\xi_{2p}]}^{-1}(u) \leq H_{1,[\xi_{1q}]}H_{1,[\xi_{1p}]}^{-1}(u), \tag{3.2}$$

where $\xi_{ip} = F_i^{-1}(p)$ stands for the pth quantile of the marginal distribution of X_i, and $H_{i,[s]}$ denotes the conditional distribution of Y_i given $X_i = s$, for $i = 1, 2$.

Let $Y_{i,[\xi_{ip}]}$ denote the random variable with distribution function $H_{i,[\xi_{ip}]}$ for $i = 1, 2$. Then (3.2) is equivalent to

$$0 < p \leq q < 1 \Longrightarrow (Y_{2,[\xi_{2p}]}, Y_{2,[\xi_{2q}]}) \succ_{\mathrm{PP}} (Y_{1,[\xi_{1p}]}, Y_{1,[\xi_{1q}]}) \tag{3.3}$$

Avérous, et al. (2005) used this order to compare the relative degree of dependence among order statistics.

3.2　More right tail increasing dependence

Let (X, Y) be a bivariate random vector with joint cdf H and with marginal cdfs F and G, respectively. Y is said to be *right tail increasing* in X if

$$x < x' \Rightarrow \{Y|X > x\} \leq_{st} \{Y|X > x'\},$$

or equivalently if

$$0 < p \leq q < 1 \Rightarrow \{Y|X > \xi_p\} \leq_{st} \{Y|X > \xi_q\}. \tag{3.4}$$

Let H_x denote the cdf of the conditional distribution of Y given $X > x$. Then (3.4) is equivalent to

$$H_{\xi_q}H_{\xi_p}^{-1}(u) \leq u \text{ for all } 0 < u < 1 \text{ and } 0 < p \leq q < 1. \tag{3.5}$$

Now for $i = 1, 2$, let (X_i, Y_i) be a pair of continuous random variables with joint cumulative distribution function H_i and marginals F_i and G_i. *Dolati, Genest and Kochar* (2008) introduced the following concept of *more right tail increasing* dependence to compare the relative degree of dependence between two pairs of random variables. This version based on copulas is a modification of the one originally proposed by *Avérous and Dortet-Bernadet* (2000).

Definition 3.3 *Y_2 is said to be more right tail increasing in X_2 than Y_1 is in X_1, denoted by $(Y_1 | X_1) \prec_{\mathrm{RTI}}$ $(Y_2 | X_2)$ or $H_1 \prec_{\mathrm{RTI}} H_2$, if and only if*

$$0 < p \le q < 1 \implies H_{2,\xi_{2q}} H_{2,\xi_{2p}}^{-1}(u) \le H_{1,\xi_{1q}} H_{1,\xi_{1p}}^{-1}(u), \tag{3.6}$$

for all $u \in (0, 1)$, where for $i = 1, 2$, $H_{i,s}$ denotes the conditional distribution of Y_i given $X_i > s$, and $\xi_{ip} = F_i^{-1}(p)$ stands for the pth quantile of the marginal distribution of X_i.

Let $Y_{i,\xi_{ip}}$ denote the random variable with distribution function $H_{i,\xi_{ip}}$ for $i = 1, 2$. Then (3.6) is equivalent to

$$0 < p \le q < 1 \implies (Y_{2,\xi_{2p}}, Y_{2,\xi_{2q}}) \succ_{\mathrm{PP}} (Y_{1,\xi_{1p}}, Y_{1,\xi_{1q}}) \tag{3.7}$$

Obviously, (3.6) implies that Y_2 is right tail increasing in X_2 if X_1 and Y_1 are independent. It also implies that if Y_1 is stochastically increasing in X_1, then so is Y_2 in X_2; and conversely, if Y_2 is stochastically decreasing in X_2, then so is Y_1 in X_1. *Dolati, Genest and Kochar* (2008) used this concept to compare the degree of dependence of the largest order statistic on the smallest order statistic in a proportional hazard rate model as a function of the proportionality parameters.

3.3   More positive quadrant dependence

Let (X, Y) be a bivariate random vector with joint cdf H and with marginal cdfs F and G, respectively. Then the pair (X, Y) is said to be positive quadrant dependent (PQD) (*Barlow and Proschan (1981)*) if

$$H(x, y) \ge F(x)G(y), \quad \forall x, y,$$

or equivalently if

$$H(\xi_p, \xi_q) \ge pq, \quad \forall \ 0 < p, q < 1, \tag{3.8}$$

where $F(\xi_p) = p$ and $G(\xi_q) = q$, which can be rewritten as

$$C(u, v) \ge uv \ \ \forall \ 0 < u, v < 1,$$

where $C(u, v) = H(\xi_u, \xi_v)$ denotes the copula of the pair (X, Y). Note that $C(u, v) = uv$ if X and Y are independent.

For $i = 1, 2$, let (X_i, Y_i), be a pair of continuous random variables with joint cumulative distribution function H_i and marginals F_i and G_i. The following concept of *more positive quadrant dependence* has been widely discussed in the literature.

Definition 3.4 Y_2 *is said to be more positive quadrant dependent in X_2 than Y_1 is in X_1, denoted by* $(Y_1|X_1) \prec_{\mathrm{PQD}} (Y_2|X_2)$ *or* $H_1 \prec_{\mathrm{PQD}} H_2$ *if and only if*

$$C_1(u,v) \leq C_2(u,v) \ \text{ for all } 0 \leq u, \ v \leq 1, \tag{3.9}$$

where C_i is the copula of H_i for $i = 1, 2$.

It is easy to see that (3.9) is equivalent to

$$0 < p < 1 \implies (Y_2, Y_{2,\xi_{2p}}) \succ_{\mathrm{PP}} (Y_1, Y_{1,\xi_{1p}}) \tag{3.10}$$

where $Y_{i,\xi_{ip}}, \ \ i = 1, 2$ is as defined in Section 3.2.

We have the following chain of implications among the above partial orderings

$$(Y_1|X_1) \preceq_{\mathrm{SI}} (Y_2|X_2) \Rightarrow (Y_1|X_1) \preceq_{\mathrm{RTI}} (Y_2|X_2) \Rightarrow (Y_1|X_1) \preceq_{\mathrm{PQD}} (Y_2|X_2).$$

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