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On Weighted Generalized Entropy

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Abstract

In the literature of information theory, the concept of generalized entropy has been proposed. Recently, the length based shift-dependent information measure has been studied by Di crescenzo and Longobardi (2006). In this paper, the concept of weighted generalized entropy has been introduced. The properties of weighted generalized residual entropy and weighted generalized past entropy are also discussed.

Key Words and Phrases: generalized entropy, generalized residual entropy, generalized past entropy.
AMS 2010 Classifications: Primary 62N05; Secondary 90B25.

1 Introduction

In the area of information theory as well as engineering sciences, the Shannon entropy and its applications is a very important and well known concept. Information theory includes the study of uncertainty measures and various practical and economical methods of coding information for transmission. Let X be an absolutely continuous nonnegative random variable having probability density function $f_X(t)$. Then Shannon's entropy is defined as

$$H(X) = -\int_0^\infty f_X(x)\ln f_X(x)dx = -E\left[\ln f_X(X)\right]. \tag{1.1}$$

One of the main drawback of $H(X)$ is that for some probability distribution it may be negative and then it is no longer an uncertainty measure. This drawback is removed in the generalized entropy. By choosing a convex function ϕ such that $\phi(1) = 0$, Khinchin (1957) generalized (1.1) and defined the measure as

$$H^\phi(X) = \int f_X(x)\phi(f_X(x))dx. \tag{1.2}$$

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For two particular choice of ϕ, (1.2) can be written as

$$H_1^{\beta}(X) = \frac{1}{\beta - 1}\left[1 - \int_0^{\infty} f_X^{\beta}(x)dx\right], \tag{1.3}$$

and

$$H_2^{\beta}(X) = \frac{1}{1-\beta}\ln\int_0^{\infty} f_X^{\beta}(x)dx, \tag{1.4}$$

for some fixed $\beta > 0$ and $\beta \neq 1$. As $\beta \to 1$ in (1.3) or in (1.4), then they tends to (1.1). It can be seen that by choosing appropriate value of β, one can always find nonnegative $H_1^{\beta}(X)$ and $H_2^{\beta}(X)$ but (1.1) may be negative for some distribution. Sometime it is very important to study about the system that survived up to an age t, then Shannon's entropy function is not useful in measuring the uncertainty about the residual lifetime of the system. Ebrahimi (1996) have introduced residual entropy and defined as

$$H(X;t) = -\int_t^{\infty} \frac{f_X(x)}{\bar{F}_X(t)} \ln\left(\frac{f_X(x)}{\bar{F}_X(t)}\right) dx, \tag{1.5}$$

where $\bar{F}_X(t)$ be the survival function of the random variable X. Nanda and Paul (1996) have introduced generalized residual entropy and they have redefined (1.3) and (1.4) for a unit surviving up to age t as

$$H_1^{\beta}(X;t) = \frac{1}{\beta - 1}\left[1 - \int_t^{\infty} \left(\frac{f_X(x)}{\bar{F}_X(t)}\right)^{\beta} dx\right], \tag{1.6}$$

and

$$H_2^{\beta}(X;t) = \frac{1}{1-\beta}\ln\int_0^{\infty} \left(\frac{f_X(x)}{\bar{F}_X(t)}\right)^{\beta} dx, \tag{1.7}$$

respectively. As $\beta \to 1$ in (1.6) or in (1.7), then they tends to (1.5). In some practical situation sometime it is important to study the uncertainty related to the past rather than the future. The past entropy over $(0, t)$ have been introduced by Di Crescenzo and Longobardi (2002). If X be the lifetime of a system then the past entropy of the system is defined as

$$\bar{H}(X;t) = -\int_0^{t} \frac{f_X(x)}{F_X(t)} \ln\left(\frac{f_X(x)}{F_X(t)}\right) dx, \tag{1.8}$$

where $F_X(t)$ be the distribution function of the random variable X. Gupta and Nanda (2002) have been defined generalized past entropies given by

$$\bar{H}_1^{\beta}(X;t) = \frac{1}{\beta - 1}\left[1 - \int_0^{t} \left(\frac{f_X(x)}{F_X(t)}\right)^{\beta} dx\right], \tag{1.9}$$

and

$$\bar{H}_2^{\beta}(X;t) = \frac{1}{1-\beta}\ln\int_0^{\infty} \left(\frac{f_X(x)}{F_X(t)}\right)^{\beta} dx, \tag{1.10}$$

respectively. When $\beta \to 1$ in (1.9) or in (1.10), then they tends to (1.8).

When an investigator collects a sample of observations produced by nature, according to the certain model, the original distribution may not be reproduced due to various reasons (cf. C.R. Rao (1965)). In many practical circumstances for modeling statistical data, sometime the standard distributions are not found appropriate. For this reason, it is important to consider the concept of weighted distributions. Guiasu (1986) has shown that weighted entropy has been used to balance the amount of information and the degree of homogeneity associated to a partition of data in classes. When the weight function depends on the lengths of the component, the resulting distribution is called length biased weighted function. Di Crescenzo and Longobardi (2006) have considered a length based shift dependent information measure, related to the differential entropy and also introduced the concept of weighted residual entropy and weighted past entropy. Misagh and Yari (2011) have studied the weighted differential information measure for two-sided truncated random variable. Motivated with the usefulness of the generalized entropy and the weighted entropy, the concept of weighted generalized entropy has been introduced. Further, weighted generalized residual entropy and the weighted generalized past entropy have been discussed in this paper.

This paper is organized as follows. In Section 2 of this paper some basic notation and properties of weighted generalized entropy have been studied. Section 3 discusses the properties of weighted generalized residual entropy, while Section 4 deals with some properties of weighted past generalized entropy. Finally, Section 5 presents some concluding remarks.

Throughout the paper, *increasing* and *decreasing* properties of a function are not used in the strict sense. For any twice differentiable function $g(t)$, we write $g'(t)$ and $g''(t)$ to denote the first and the second derivatives of $g(t)$ with respect to t, respectively, and $a \stackrel{\text{def}}{=} b$ means that a is defined by b.

2 Weighted generalized entropy

If X is an absolutely continuous non-negative random variable with probability density function $f_X(t)$ and survival function $\bar{F}_X(t)$, then the probability density function of length based weighted random variable X_ω associated to the random variable X is

$$f^\omega(t) = \frac{t}{E(X)} f_X(t),$$

and the survival function is

$$\bar{F}^\omega(t) = \frac{E(X|X > t)}{E(X)} \bar{F}_X(t).$$

Then the weighted entropy is given by

$$\begin{aligned} H^\omega(X) &= -\int_0^\infty f^\omega(x) \ln f^\omega(x) dx \\ &= -\frac{E\left(X \ln X\right)}{E(X)} + \frac{E(X) \ln E(X)}{E(X)} - \frac{1}{E(X)} \int_0^\infty x f_X(x) \ln f_X(x) dx. \end{aligned} \tag{2.1}$$

The weighted generalized entropy is given by

$$
\begin{aligned}
H_1^{\omega^\beta}(X) &= \frac{1}{\beta-1}\left[1-\int_0^\infty f^{\omega^\beta}(x)dx\right] \\
&= \frac{1}{\beta-1}\left[1-\frac{1}{(E(X))^\beta}\int_0^\infty x^\beta f_X^\beta(x)dx\right], \qquad (2.2)
\end{aligned}
$$

and

$$
H_2^{\omega^\beta}(X) = \frac{1}{1-\beta}\left[\ln\left(\frac{1}{(E(X))^\beta}\int_0^\infty x^\beta f_X^\beta(x)dx\right)\right]. \qquad (2.3)
$$

It can be noted that as $\beta \to 1$ in (2.2) or in (2.3), they reduces to (2.1). $H_1^{\omega^\beta}(X)$ and $H_2^{\omega^\beta}(X)$ are called as first kind weighted entropy of order β and second kind weighted entropy of order β respectively.

The following example shows that although two distributions have same generalized entropies but they have different weighted generalized entropies.

Example 2.1 *Let X and Y be random variables with density functions*

$$
f_X(t) = \begin{cases} \frac{1+t}{4}, & 0 \leqslant t < 2, \\ 0, & \text{otherwise}, \end{cases}
$$

and

$$
f_Y(t) = \begin{cases} 1-\frac{1+t}{4}, & 0 \leqslant t < 2, \\ 0, & \text{otherwise}. \end{cases}
$$

Take $\beta = 2$. Then, we can see that $H_1^\beta(X) = H_1^\beta(Y) = \frac{11}{24}$, where $H_1^\beta(X)$ and $H_1^\beta(Y)$ are the first kind generalized entropies of X and Y. The first kind weighted generalized entropies of the random variables X and Y are given by $H_1^{\omega^\beta}(X) = \frac{53}{245}$ and $H_1^{\omega^\beta}(Y) = \frac{53}{125}$ respectively. Therefore, $H_1^{\omega^\beta}(X) \neq H_1^{\omega^\beta}(Y)$.

Again, the second kind generalized entropies of the random variables X and Y are given by $H_2^\beta(X) = H_2^\beta(Y) = \ln\frac{24}{13}$. But we can see that $H_2^{\omega^\beta}(X) = \ln\frac{245}{192}$ and $H_2^{\omega^\beta}(Y) = \ln\frac{125}{72}$ are not equal.

Hence, even though $H_1^\beta(X) = H_1^\beta(Y)$ and $H_2^\beta(X) = H_2^\beta(Y)$, the weighted generalized entropy about the predictability of X by the density function $f_X(t)$ is smaller than the predictability of Y by the density function $f_Y(t)$. □

The following propositions gives the properties of $H_1^{\omega^\beta}(X)$ and $H_2^{\omega^\beta}(X)$.

Proposition 2.1 *Let Z be a random variable defined by $Z = aX + b$. Then*

$$
\begin{aligned}
(i)\ H_1^{\omega^\beta}(Z) &= \frac{1}{\beta-1}\left[1-\frac{(E(X))^\beta}{a^{\beta-1}(aE(X)+b)^\beta}\left(1-(\beta-1)H_1^{\omega^\beta}(X)\right)\right] - \frac{\beta}{(\beta-1)a^{\beta-1}(aE(X)+b)^\beta} \\
&\int_{t=0}^\infty t^{\beta-1}\left(\bar{F}_X^\beta\left(\frac{t-b}{a}\right)\left[1-(\beta-1)H_1^\beta\left(X;\frac{t-b}{a}\right)\right] - \bar{F}_X^\beta(t)\left[1-(\beta-1)H_1^\beta(X;t)\right]\right)dt;
\end{aligned}
$$

$$(ii)\ H_2^{\omega^\beta}(Z) = \frac{1}{1-\beta}\ln\left[\frac{(E(X))^\beta}{a^{\beta-1}\left[aE(X)+b\right]^\beta}\exp\Big((1-\beta)H_2^\beta(X)\Big)\right] + \frac{\beta}{(1-\beta)a^{\beta-1}\left[aE(X)+b\right]^\beta}$$
$$\int_{t=0}^{\infty} t^{\beta-1}\left[\bar{F}_X^\beta\left(\frac{t-b}{a}\right)\exp\left((1-\beta)H_2^\beta\left(X;\frac{t-b}{a}\right)\right) - \bar{F}_X^\beta(t)\exp\Big((1-\beta)H_2^\beta(X;t)\Big)\right]dt,$$

where $a > 0$, $b \geqslant 0$ and X be any absolutely continuous random variable. □

Proposition 2.2 *If X and Y are independent, then*

(i) $H_1^{\omega^\beta}(X,Y) = H_1^{\omega^\beta}(X) + H_1^{\omega^\beta}(Y) - (\beta-1)H_1^{\omega^\beta}(X)H_1^{\omega^\beta}(Y)$

(ii) $H_2^{\omega^\beta}(X,Y) = H_2^{\omega^\beta}(X) + H_2^{\omega^\beta}(Y)$. □

3 Weighted generalized residual entropy

Di Crescenzo and Longobardi (2006) have introduced the concept of weighted residual entropy which can be defined as

$$\begin{aligned} H^\omega(X;t) &= -\int_t^\infty \frac{f^\omega(x)}{\bar{F}^\omega(t)}\ln\left(\frac{f^\omega(x)}{\bar{F}^\omega(t)}\right)dx \\ &= -\frac{1}{E\left[X|X>t\right]}\int_t^\infty x\frac{f_X(x)}{\bar{F}_X(t)}\ln\left(\frac{xf_X(x)}{E\left[X|X>t\right]\bar{F}_X(t)}\right)dx. \end{aligned} \tag{3.1}$$

In this section the concept of weighted generalized residual entropy functions have been introduced and some properties of weighted generalized entropy have been discussed. The weighted generalized residual entropy functions are defined by

$$\begin{aligned} H_1^{\omega^\beta}(X;t) &= \frac{1}{\beta-1}\left[1-\int_t^\infty\left(\frac{f^\omega(x)}{\bar{F}^\omega(t)}\right)^\beta dx\right] \\ &= \frac{1}{\beta-1}\left[1-\frac{1}{(E(X|X>t))^\beta}\int_t^\infty x^\beta\left(\frac{f_X(x)}{\bar{F}_X(t)}\right)^\beta dx\right], \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} H_2^{\omega^\beta}(X;t) &= \frac{1}{1-\beta}\ln\int_t^\infty\left(\frac{f^\omega(x)}{\bar{F}^\omega(t)}\right)^\beta dx \\ &= \frac{1}{1-\beta}\ln\int_t^\infty x^\beta\left(\frac{f_X(x)}{\bar{F}_X(t)}\right)^\beta dx - \frac{\beta}{1-\beta}\ln E(X|X>t). \end{aligned} \tag{3.3}$$

As $\beta \to 1$ in (3.2) and (3.3), we can see that they reduce to $H^\omega(X;t)$ as defined in (3.1). Now, we can see that

$$\begin{aligned} \int_t^\infty x^\beta\left(\frac{f_X(x)}{\bar{F}_X(t)}\right)^\beta dx &= \int_t^\infty\left(\int_0^x \beta y^{\beta-1}dy\right)\left(\frac{f_X(x)}{\bar{F}_X(t)}\right)^\beta dx \\ &= \beta\int_t^\infty\left(\int_0^t \beta y^{\beta-1}dy + \int_t^x \beta y^{\beta-1}dy\right)\left(\frac{f_X(x)}{\bar{F}_X(t)}\right)^\beta dx \end{aligned}$$

$$
\begin{aligned}
&= \beta\left[\int_t^\infty\left(\int_0^t y^{\beta-1}dy\right)\left(\frac{f_X(x)}{\bar{F}_X(t)}\right)^\beta dx + \int_t^\infty\left(\int_t^x y^{\beta-1}dy\right)\left(\frac{f_X(x)}{\bar{F}_X(t)}\right)^\beta dx\right] \\
&= \beta\left[\frac{t^\beta}{\beta}\int_t^\infty\left(\frac{f_X(x)}{\bar{F}_X(t)}\right)^\beta dx + \frac{1}{\bar{F}_X^\beta(t)}\int_{y=t}^\infty\left(y^{\beta-1}\int_{t=y}^\infty f_X^\beta(x)dx\right)dy\right].
\end{aligned}
$$

Again,

$$
H_1^\beta(X;t) = \frac{1}{\beta-1}\left[1-\int_t^\infty\left(\frac{f_X(x)}{\bar{F}_X(t)}\right)^\beta dx\right],
$$

and

$$
H_2^\beta(X;t) = \frac{1}{1-\beta}\ln\int_t^\infty\left(\frac{f_X(x)}{\bar{F}_X(t)}\right)^\beta dx,
$$

which are equivalent to

$$
\int_t^\infty\left(\frac{f_X(x)}{\bar{F}_X(t)}\right)^\beta dx = 1-(\beta-1)H_1^\beta(X;t),
$$

$$
\int_t^\infty f_X^\beta(x)dx = \bar{F}_X^\beta(t)\left[1-(\beta-1)H_1^\beta(X;t)\right],
$$

and

$$
\int_t^\infty\left(\frac{f_X(x)}{\bar{F}_X(t)}\right)^\beta dx = \exp\left[1-(1-\beta)H_2^\beta(X;t)\right],
$$

$$
\int_t^\infty f_X^\beta(x)dx = \bar{F}_X^\beta(t)\exp\left[1-(1-\beta)H_2^\beta(X;t)\right].
$$

Therefore, (3.2) can be rewritten as

$$
\begin{aligned}
H_1^{\omega^\beta}(X;t) &= \frac{1}{\beta-1}\Bigg[1-\frac{1}{(E(X|X>t))^\beta}\Bigg(t^\beta\left[1-(\beta-1)H_1^\beta(X;t)\right]+ \\
&\qquad \beta\int_t^\infty y^{\beta-1}\left(\frac{\bar{F}_X(y)}{\bar{F}_X(t)}\right)^\beta\left[1-(\beta-1)H_1^\beta(X;y)\right]dy\Bigg)\Bigg], \qquad (3.4)
\end{aligned}
$$

and (3.3) can be rewritten as

$$
\begin{aligned}
H_2^{\omega^\beta}(X;t) &= \frac{1}{1-\beta}\ln\Bigg[t^\beta\exp\left[(1-\beta)H_2^\beta(X;t)\right]+ \\
&\qquad \beta\int_t^\infty y^{\beta-1}\left(\frac{\bar{F}_X(y)}{\bar{F}_X(t)}\right)^\beta\exp\left[(1-\beta)H_2^\beta(X;y)\right]dy\Bigg] \\
&\qquad -\frac{\beta}{1-\beta}\ln E(X|X>t). \qquad (3.5)
\end{aligned}
$$

The following theorem characterizes the weighted generalized residual entropy in the sense that under certain condition the weighted generalized residual entropy uniquely determine the distribution function.

Theorem 3.1 *Let X be a nonnegative absolutely continuous random variable having probability density function $f_X(t)$ and the survival function $\bar{F}_X(t)$. If*

(i) $H_1^{\beta}(X;t)$ is increasing in t, then $H_1^{\omega^\beta}(X;t)$ uniquely determine $\bar{F}_X(t)$;

(ii) $H_2^{\beta}(X;t)$ is increasing in t, then $H_2^{\omega^\beta}(X;t)$ uniquely determine $\bar{F}_X(t)$.

Proof: (i) From (3.4), we have

$$\begin{aligned} H_1^{\omega^\beta}(X;t) &= \frac{1}{\beta-1}\left[1-\frac{1}{(E(X|X>t))^{\beta}}\left(t^{\beta}\left[1-(\beta-1)H_1^{\beta}(X;t)\right]+\right.\right. \\ &\quad \left.\left.\beta\int_t^{\infty} y^{\beta-1}\left(\frac{\bar{F}_X(y)}{\bar{F}_X(t)}\right)^{\beta}\left[1-(\beta-1)H_1^{\beta}(X;y)\right]dy\right)\right], \end{aligned}$$

which is equivalent to

$$\begin{aligned} 1-(\beta-1)H_1^{\omega^\beta}(X;t) &= \frac{1}{(E(X|X>t))^{\beta}}\left(t^{\beta}\left[1-(\beta-1)H_1^{\beta}(X;t)\right]+\right. \\ &\quad \left.\beta\int_t^{\infty} y^{\beta-1}\left(\frac{\bar{F}_X(y)}{\bar{F}_X(t)}\right)^{\beta}\left[1-(\beta-1)H_1^{\beta}(X;y)\right]dy\right) \\ &= \frac{I(t)}{[g(t)]^{\beta}}, \end{aligned} \tag{3.6}$$

where $g(t)=E(X|X>t)$ and

$$I(t)=t^{\beta}\left[1-(\beta-1)H_1^{\beta}(X;t)\right]+\beta\int_t^{\infty} y^{\beta-1}\left(\frac{\bar{F}_X(y)}{\bar{F}_X(t)}\right)^{\beta}\left[1-(\beta-1)H_1^{\beta}(X;y)\right]dy.$$

Differentiating $I(t)$ with respect to t, we get

$$I'(t) = -(\beta-1)t^{\beta}\frac{d}{dt}H_1^{\beta}(X;t)+\beta^2 r_X(t)\int_t^{\infty} y^{\beta-1}\left(\frac{\bar{F}_X(y)}{\bar{F}_X(t)}\right)^{\beta}\left[1-(\beta-1)H_1^{\beta}(X;y)\right]dy.$$

Differentiating (3.6) with respect to t, we get

$$-(\beta-1)\frac{d}{dt}H_1^{\omega^\beta}(X;t)=-\beta\frac{g'(t)}{g^{\beta+1}(t)}I(t)+\frac{1}{g^{\beta}(t)}I'(t). \tag{3.7}$$

Again, from (3.2), we have

$$1-(\beta-1)H_1^{\omega^\beta}(X;t) = \frac{1}{g^{\beta}(t)}\int_t^{\infty} x^{\beta}\left(\frac{f_X(x)}{\bar{F}_X(t)}\right)^{\beta}dx. \tag{3.8}$$

Differentiating (3.8) with respect to t, we get

$$\begin{aligned} -(\beta-1)\frac{d}{dt}H_1^{\omega^\beta}(X;t) &= -\beta\frac{g'(t)}{g^{\beta+1}(t)}\int_t^{\infty} x^{\beta}\left(\frac{f_X(x)}{\bar{F}_X(t)}\right)^{\beta}dx+\frac{\beta}{g^{\beta}(t)}r_X(t)\int_t^{\infty} x^{\beta}\left(\frac{f_X(x)}{\bar{F}_X(t)}\right)^{\beta}dx \\ &\quad -\frac{t^{\beta}}{g^{\beta}(t)}r_X^{\beta}(t). \end{aligned} \tag{3.9}$$

Therefore, from (3.7) and (3.9), we have

$$\begin{aligned}-\beta\frac{g'(t)}{g^{\beta+1}(t)}I(t)+\frac{1}{g^{\beta}(t)}I'(t) &= -\beta\frac{g'(t)}{g^{\beta+1}(t)}\int_t^{\infty}x^{\beta}\left(\frac{f_X(x)}{\bar{F}_X(t)}\right)^{\beta}dx\\ &\quad+\frac{\beta}{g^{\beta}(t)}r_X(t)\int_t^{\infty}x^{\beta}\left(\frac{f_X(x)}{\bar{F}_X(t)}\right)^{\beta}dx-\frac{t^{\beta}}{g^{\beta}(t)}r_X^{\beta}(t),\end{aligned}$$

which is equivalent to

$$\beta\frac{g'(t)}{g^{\beta+1}(t)}\left[I(t)-\int_t^{\infty}y^{\beta}\left(\frac{f_X(y)}{\bar{F}_X(t)}\right)^{\beta}dy\right]+\frac{\beta-1}{g^{\beta}(t)}t^{\beta}\frac{d}{dt}H_1^{\beta}(X;t)-\frac{\beta}{g^{\beta}(t)}r_X(t)$$
$$\left[\beta\int_t^{\infty}y^{\beta-1}\left(\frac{\bar{F}_X(y)}{\bar{F}_X(t)}\right)^{\beta}\left[1-(\beta-1)H_1^{\beta}(X;y)\right]dy-\int_t^{\infty}y^{\beta}\left(\frac{f_X(y)}{\bar{F}_X(t)}\right)^{\beta}dy\right]-\frac{t^{\beta}}{g^{\beta}(t)}r_X^{\beta}(t)=0.$$

For a fixed $t>0$, $r_X(t)$ is a solution of $A(x)=0$, where

$$A(x)=\beta\frac{g'(t)}{g^{\beta+1}(t)}\left[I(t)-\int_t^{\infty}y^{\beta}\left(\frac{f_X(y)}{\bar{F}_X(t)}\right)^{\beta}dy\right]+\frac{\beta-1}{g^{\beta}(t)}t^{\beta}\frac{d}{dt}H_1^{\beta}(X;t)-\frac{\beta}{g^{\beta}(t)}x$$
$$\left[\beta\int_t^{\infty}y^{\beta-1}\left(\frac{\bar{F}_X(y)}{\bar{F}_X(t)}\right)^{\beta}\left[1-(\beta-1)H_1^{\beta}(X;y)\right]dy-\int_t^{\infty}y^{\beta}\left(\frac{f_X(y)}{\bar{F}_X(t)}\right)^{\beta}dy\right]-\frac{t^{\beta}}{g^{\beta}(t)}x^{\beta}.$$

Differentiating $A(x)$ with respect to x, we get

$$\begin{aligned}A'(x) &= \frac{\beta}{g^{\beta}(t)}\left[\int_t^{\infty}y^{\beta}\left(\frac{f_X(y)}{\bar{F}_X(t)}\right)^{\beta}dy-\beta\int_t^{\infty}y^{\beta-1}\left(\frac{\bar{F}_X(y)}{\bar{F}_X(t)}\right)^{\beta}\left[1-(\beta-1)H_1^{\beta}(X;y)\right]dy\right]\\ &\quad-\frac{\beta t^{\beta}}{g^{\beta}(t)}x^{\beta-1}.\end{aligned}$$

Now, $A'(x)=0$ gives

$$\begin{aligned}x &= \left[\frac{1}{t^{\beta}}\left(\int_t^{\infty}y^{\beta}\left(\frac{f_X(y)}{\bar{F}_X(t)}\right)^{\beta}dy-\beta\int_t^{\infty}y^{\beta-1}\left(\frac{\bar{F}_X(y)}{\bar{F}_X(t)}\right)^{\beta}\left[1-(\beta-1)H_1^{\beta}(X;y)\right]dy\right)\right]^{\frac{1}{\beta-1}}\\ &= t_0,\ \textit{say}.\end{aligned}$$

Again, we see that

$$\begin{aligned}A(0) &= \beta\frac{g'(t)}{g^{\beta+1}(t)}\left[I(t)-\int_t^{\infty}y^{\beta}\left(\frac{f_X(y)}{\bar{F}_X(t)}\right)^{\beta}dy\right]+\frac{\beta-1}{g^{\beta}(t)}t^{\beta}\frac{d}{dt}H_1^{\beta}(X;t)\\ &= \frac{\beta-1}{g^{\beta}(t)}t^{\beta}\frac{d}{dt}H_1^{\beta}(X;t).\end{aligned}$$

Case I : For $\beta>1$, $A(0)>0$, if $H_1^{\beta}(X;t)$ is increasing in t and $A(\infty)=-\infty$. Further it can be seen that

$$\begin{aligned}A''(x) &= -\frac{\beta(\beta-1)t^{\beta}}{g^{\beta}(t)}x^{\beta-2}\\ &\leqslant 0.\end{aligned}$$

Therefore, $A'(x)$ is decreasing in x, and $A'(t_0) = 0$, $A'(\infty) = -\infty$. Thus, we see that

$$\begin{aligned} A'(x) &\geqslant 0, \qquad 0 < x \leqslant t_0, \\ &\leqslant 0, \qquad x \geqslant t_0. \end{aligned}$$

Therefore, $A(x) = 0$ has a unique solution. But we have seen that $r_X(t)$ is a solution. Hence, $x = r_X(t)$ is a unique solution of $A(x) = 0$.

Case II : For $\beta < 1$, $A(0) < 0$, if $H_1^\beta(X;t)$ is increasing in t and $A(\infty) = -\infty$. Further it can be seen that $A'(x)$ is increasing in t, and $A'(t_0) = 0$, $A'(\infty) = \infty$. Thus, we see that

$$\begin{aligned} A'(x) &\leqslant 0, \qquad 0 < x \leqslant t_0, \\ &\geqslant 0, \qquad x \geqslant t_0. \end{aligned}$$

Therefore, $A(x) = 0$ has a unique solution and $x = r_X(t)$ is the unique solution of $A(x) = 0$.

Therefore, from the two cases it can be conclude that if $H_1^\beta(X;t)$ is increasing in $t > 0$ and $A(t_0) = 0$, then $r_X(t)$ is the unique solution of $A(x) = 0$. Thus, $H_2(X;t)$ determines $r_X(t)$ uniquely. Again, $r_X(t)$ uniquely determine $\bar{F}_X(t)$. Hence, the result follows.

To prove (ii), we have, from (3.3)

$$\exp\left[(1-\beta)H_2^{\omega^\beta}(X;t)\right] = \frac{1}{(E(X|X>t))^\beta}\int_t^\infty x^\beta \left(\frac{f_X(x)}{\bar{F}_X(t)}\right)^\beta dx. \tag{3.10}$$

Differentiating (3.10) with respect to t, we have

$$\begin{aligned} \frac{d}{dt}\left(\exp\left[(1-\beta)H_2^{\omega^\beta}(X;t)\right]\right) &= -\beta\frac{g'(t)}{g^{\beta+1}(t)}\int_t^\infty y^\beta\left(\frac{f_X(y)}{\bar{F}_X(t)}\right)^\beta dy + \frac{\beta}{g^\beta(t)}r_X(t) \\ &\quad \int_t^\infty y^\beta\left(\frac{f_X(y)}{\bar{F}_X(t)}\right)^\beta dy - \frac{t^\beta}{g^\beta(t)}r_X^\beta(t). \end{aligned} \tag{3.11}$$

Again, from (3.5), we have

$$\begin{aligned} \exp\left[(1-\beta)H_2^{\omega^\beta}(X;t)\right] &= \frac{1}{(E(X|X>t))^\beta}\Bigg(t^\beta\exp\left[(1-\beta)H_2^\beta(X;t)\right] + \\ &\quad \beta\int_t^\infty y^{\beta-1}\left(\frac{\bar{F}_X(y)}{\bar{F}_X(t)}\right)^\beta \exp\left[(1-\beta)H_2^\beta(X;y)\right]dy\Bigg) \\ &= \frac{I_1(t)}{[g(t)]^\beta}, \end{aligned} \tag{3.12}$$

where $I_1(t) = t^\beta\exp\left[(1-\beta)H_2^\beta(X;t)\right] + \beta\int_t^\infty y^{\beta-1}\left(\frac{\bar{F}_X(y)}{\bar{F}_X(t)}\right)^\beta \exp\left[(1-\beta)H_2^\beta(X;y)\right]dy$. Differentiating $I_1(t)$ with respect to t, we get

$$\begin{aligned} I_1'(t) &= t^\beta(1-\beta)\frac{d}{dt}H_2^\beta(X;t)\exp\left[(1-\beta)H_2^\beta(X;t)\right] + \beta^2 r_X(t) \\ &\quad \int_t^\infty y^{\beta-1}\left(\frac{\bar{F}_X(y)}{\bar{F}_X(t)}\right)^\beta \exp\left[(1-\beta)H_2^\beta(X;y)\right]dy. \end{aligned}$$

Differentiating (3.12) with respect to t, we get

$$\frac{d}{dt}\left(\exp\left[(1-\beta)H_2^{\omega^\beta}(X;t)\right]\right) = -\beta\frac{g'(t)}{g^{\beta+1}(t)}I_1(t) + \frac{I_1'(t)}{g^\beta(t)}$$

$$
\begin{aligned}
&= \frac{1}{g^\beta(t)}\Bigg(t^\beta(1-\beta)\frac{d}{dt}H_2^\beta\,(X;t)\exp\Big[(1-\beta)H_2^\beta\,(X;t)\Big]+\beta^2 r_X(t) \\
&\quad \int_t^\infty y^{\beta-1}\left(\frac{\bar F_X(y)}{\bar F_X(t)}\right)^\beta \exp\Big[(1-\beta)H_2^\beta\,(X;y)\Big]\,dy\Bigg)-\beta\frac{g'(t)}{g^{\beta+1}(t)}I_1(t). \qquad (3.13)
\end{aligned}
$$

From (3.11) and (3.13), we get

$$
\begin{aligned}
&\beta\frac{g'(t)}{g^{\beta+1}(t)}\left[\int_t^\infty y^\beta\left(\frac{f_X(y)}{\bar F_X(t)}\right)^\beta dy-I_1(t)\right]+\frac{t^\beta}{g^\beta(t)}(1-\beta)\frac{d}{dt}H_2^\beta\,(X;t)+\frac{\beta r_X(t)}{g^\beta(t)} \\
&\left(\beta\int_t^\infty y^{\beta-1}\left(\frac{\bar F_X(y)}{\bar F_X(t)}\right)^\beta \exp\Big[(1-\beta)H_2^\beta\,(X;y)\Big]dy-\int_t^\infty y^\beta\left(\frac{f_X(y)}{\bar F_X(t)}\right)^\beta dy\right)+\frac{t^\beta}{g^\beta(t)}r_X^\beta(t)=0.
\end{aligned}
$$

For a fixed $t>0$, $r_X(t)$ is a solution of $A_1(x)=0$, where

$$
\begin{aligned}
A_1(x) &= \frac{t^\beta}{g^\beta(t)}(1-\beta)\frac{d}{dt}H_2^\beta\,(X;t)-\frac{\beta}{g^\beta(t)} \\
&\quad \left(\int_t^\infty y^\beta\left(\frac{f_X(y)}{\bar F_X(t)}\right)^\beta dy-\beta\int_t^\infty y^{\beta-1}\left(\frac{\bar F_X(y)}{\bar F_X(t)}\right)^\beta \exp\Big[(1-\beta)H_2^\beta\,(X;y)\Big]dy\right)x+\frac{t^\beta}{g^\beta(t)}x^\beta.
\end{aligned}
$$

Now, $A_1'(x)=0$ gives

$$
\beta\frac{t^\beta}{g^\beta(t)}x^{\beta-1}-\frac{\beta}{g^\beta(t)}\left(\int_t^\infty y^\beta\left(\frac{f_X(y)}{\bar F_X(t)}\right)^\beta dy-\beta\int_t^\infty y^{\beta-1}\left(\frac{\bar F_X(y)}{\bar F_X(t)}\right)^\beta \exp\Big[(1-\beta)H_2^\beta\,(X;y)\Big]dy\right)=0,
$$

which is equivalent to

$$
\begin{aligned}
x &= \left[\frac{1}{t^\beta}\left(\int_t^\infty y^\beta\left(\frac{f_X(y)}{\bar F_X(t)}\right)^\beta dy-\beta\int_t^\infty y^{\beta-1}\left(\frac{\bar F_X(y)}{\bar F_X(t)}\right)^\beta \exp\Big[(1-\beta)H_2^\beta\,(X;y)\Big]dy\right)\right]^{\frac{1}{\beta-1}} \\
&= t_1,\ \textit{say}.
\end{aligned}
$$

Again, $A_1(0)=\frac{t^\beta}{g^\beta(t)}(1-\beta)\frac{d}{dt}H_2^\beta\,(X;t)$.
CaseI : $\beta>1$, $A_1(0)<0$, if $H_2^\beta(X;t)$ is increasing in t and $A_1(\infty)=\infty$. By similar way one can say that $A_1(x)=0$ has a unique solution. But we have seen that $r_X(t)$ is a solution. Hence, $x=r_X(t)$ is a unique solution of $A_1(x)=0$.
CaseII : $\beta<1$, $A_1(0)>0$, if $H_2^\beta(X;t)$ is increasing in t. By similar way one can say that $r_X(t)$ is a solution. Hence, $x=r_X(t)$ is a unique solution of $A_1(x)=0$.

Therefore, from the two cases it can be conclude that if $H_2^\beta(X;t)$ is increasing in $t>0$ and $A_1(t_1)=0$, then $r_X(t)$ is the unique solution of $A_1(x)=0$. Thus, $H_2(X;t)$ determines $r_X(t)$ uniquely. Again, $r_X(t)$ uniquely determine $\bar F_X(t)$. Hence, the result follows. □

Di Crescenzo and Longobardi (2006) have been defined two nonparametric classes of distributions based on the monotonicity properties of weighted entropy are given below.

Definition 3.1 *A random variable X is said to have decreasing (resp. increasing) weighted uncertainty residual life (DWURL (resp. IWURL)) if $H^\omega(X;t)$ is decreasing (resp. increasing) in $t\geqslant 0$.* □

Here two nonparametric classes of distributions based on the monotonicity properties of weighted generalized residual entropy have been introduced.

Definition 3.2 *A nonnegative random variable X is said to have* *(i) decreasing (resp. increasing) weighted uncertainty residual life of first kind of order β [DWURLF(β) (resp. IWURLF(β))] if $H_1^{\omega^\beta}(X;t)$ is decreasing (resp. increasing) in $t \geqslant 0$;*

(i) decreasing (resp. increasing) weighted uncertainty residual life of second kind of order β [DWURLS(β) (resp. IWURLS(β))] if $H_2^{\omega^\beta}(X;t)$ is decreasing (resp. increasing) in $t \geqslant 0$. □

The following counterexample shows that there exist distributions which are not monotone in terms of $H_1^{\omega^\beta}(X;t)$ or $H_2^{\omega^\beta}(X;t)$.

Counterexample 3.1 *Let X be a random variable having probability density function $f_X(t) = \frac{2}{(1+t)^3}$, $t \geqslant 0$. Then the corresponding survival function is given by $\bar{F}_X(t) = \frac{1}{(1+t)^2}$, $t \geqslant 0$. Take $\beta = 2$. Then, we see that for all $t \geqslant 0$*

$$\begin{aligned} H_1^{\omega^\beta}(X;t) &= 1 - \frac{(1+t)^4}{(1+2t)^2}\left(\frac{2(1+5t+10t^2)}{15(1+t)^5}\right) \\ &= a_1(t), say, \end{aligned}$$

which is not monotone in $0 \leqslant t \leqslant 0.5$ as shown in Figure 1. Again, we see that for all $t \geqslant 0$

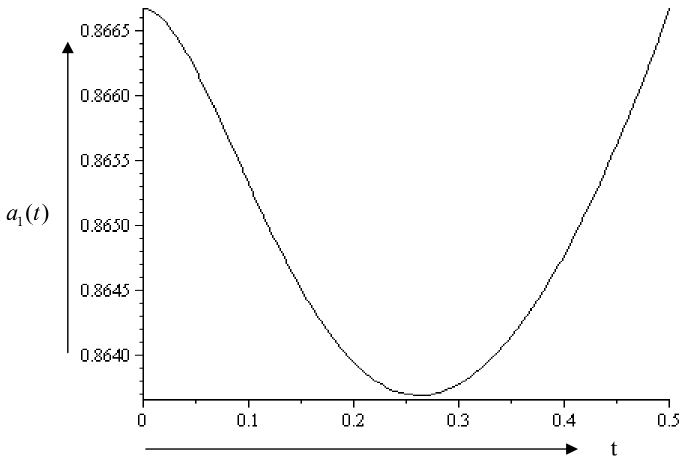

Figure 1: Plot of $a_1(t)$ for and $t \in [0, 0.5]$ (Counterexample 3.1)

$$\begin{aligned} H_2^{\omega^\beta}(X;t) &= -\ln\left[\frac{(1+t)^4}{(1+2t)^2}\left(\frac{2(1+5t+10t^2)}{15(1+t)^5}\right)\right] \\ &= a_2(t), say, \end{aligned}$$

which is also not monotone in $0 \leqslant t \leqslant 0.5$ as shown in Figure 2. Hence, $H_1^{\omega^\beta}(X;t)$ and $H_2^{\omega^\beta}(X;t)$ are not monotone. □

The following theorem gives the upper (resp. lower) bound to the failure rate function $r_X(t)$, in terms of $H_1^{\omega^\beta}(X;t)$ and $H_2^{\omega^\beta}(X;t)$. The proof is simple and hence omitted.

Theorem 3.2 *(i) If X is IWURLF(β) (resp. DWURLF(β)), then*

$$r_X(t) \geqslant (resp. \leqslant) \frac{E(X|X>t)}{t}\left[\beta\left(1-(\beta-1)H_1^{\omega^\beta}(X;t)\right)\right]^{\frac{1}{\beta-1}}.$$

(ii) If X is IWURLS(β) (resp. DWURLS(β)), then

$$r_X(t) \geqslant (resp. \leqslant) \frac{E(X|X>t)}{t}\left[\beta\exp\left((1-\beta)H_2^{\omega^\beta}(X;t)\right)\right]^{\frac{1}{\beta-1}},$$

for all $t > 0$. □

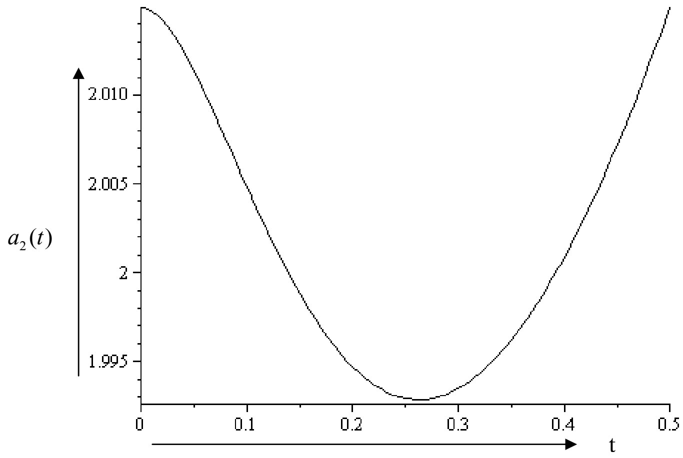

Figure 2: Plot of $a_2(t)$ for $t \in [0, 0.5]$ (Counterexample 3.1)

By taking $\beta \to 1$, we have the following corollary. The proof is omitted.

Corollary 3.1 *If X is IWURL (resp. DWURL), then*

$$r_X(t) \geqslant (resp. \leqslant) \frac{E\left(X|X>t\right)}{t} \exp\left[1 - H^{\omega}\left(X;t\right)\right],$$

for all $t > 0$. □

The following theorem provides a lower bounds for weighted generalized residual entropy. The proof is omitted.

Theorem 3.3 *(i) If the failure rate function $r_X(t)$ is decreasing in $t \geqslant 0$, then*

$$H_1^{\omega^\beta}\left(X;t\right) \geqslant \frac{1}{\beta-1}\left[1 - \frac{E(X^\beta|X>t)}{\left(E(X|X>t)\right)^\beta} r_X^{\beta-1}(t)\right],$$

for all $\beta > 0$, $t \geqslant 0$.

(ii) If the failure rate function $r_X(t)$ is decreasing in $t \geqslant 0$, then

$$H_2^{\omega^\beta}\left(X;t\right) \geqslant \frac{1}{1-\beta} \ln\left[\frac{E(X^\beta|X>t)}{\left(E(X|X>t)\right)^\beta} r_X^{\beta-1}(t)\right],$$

for all $\beta > 0$, $t \geqslant 0$. □

By taking $\beta \to 1$, we have the following corollary. The proof is omitted.

Corollary 3.2 *If the failure rate function $r_X(t)$ is decreasing in $t \geqslant 0$, then*

$$H^{\omega}\left(X;t\right) \geqslant -\frac{E\left[X \ln X|X>t\right]}{E\left[X|X>t\right]} - \ln\left(\frac{r_X(t)}{E\left[X|X>t\right]}\right),$$

for all $\beta > 0$, $t \geqslant 0$. □

4 Weighted generalized past entropy

Di Crescenzo and Longobardi (2006) have defined weighted past entropy. The weighted past entropy at time t of a random lifetime X is defined by

$$\begin{aligned}\bar{H}^{\omega}(X;t) &= -\int_0^t \frac{f^{\omega}(x)}{F^{\omega}(t)} \ln\left(\frac{f^{\omega}(x)}{F^{\omega}(t)}\right) dx \\ &= -\int_0^t \frac{x f_X(x)}{E(X|X<t)F_X(t)} \ln\left(\frac{x f_X(x)}{E(X|X<t)F_X(t)}\right) dx. \qquad (4.1)\end{aligned}$$

Here the weighted version of generalized past entropies of lifetime distribution have been defined.

$$\bar{H}_1^{\omega^\beta}(X;t) = \frac{1}{\beta-1}\left[1 - \frac{1}{[E(X|X<t)]^\beta}\int_0^t x^\beta \left(\frac{f_X(x)}{F_X(t)}\right)^\beta dx\right], \qquad (4.2)$$

and

$$\bar{H}_2^{\omega^\beta}(X;t) = \frac{1}{1-\beta}\ln\left[\frac{1}{[E(X|X<t)]^\beta}\int_0^t x^\beta \left(\frac{f_X(x)}{F_X(t)}\right)^\beta dx\right]. \qquad (4.3)$$

As $\beta \to 1$ in (4.2) and (4.3), we can see that they reduce to $\bar{H}^{\omega}(X;t)$ as defined in (4.1). Alternatively, (4.2) and (4.3) can be written as

$$\begin{aligned}\bar{H}_1^{\omega^\beta}(X;t) &= \frac{1}{\beta-1}\left[1 - \frac{1}{[E(X|X<t)]^\beta}\left(t^\beta(1-(\beta-1)\bar{H}_1^\beta(X;t))\right.\right. \\ &\left.\left. -\beta\int_{y=0}^t y^{\beta-1}\left(\frac{F(y)}{F(t)}\right)^\beta \left[1-(\beta-1)\bar{H}_1^\beta(X;y)\right] dy\right)\right], \qquad (4.4)\end{aligned}$$

and

$$\begin{aligned}\bar{H}_2^{\omega^\beta}(X;t) &= \frac{1}{1-\beta}\ln\left[\frac{1}{[E(X|X<t)]^\beta}\left(t^\beta \exp\left((1-\beta)\bar{H}_2^\beta(X;t)\right)\right.\right. \\ &\left.\left. -\beta\int_{y=0}^t y^{\beta-1}\left(\frac{F(y)}{F(t)}\right)^\beta \exp\left((1-\beta)\bar{H}_2^\beta(X;y)\right) dy\right)\right]. \qquad (4.5)\end{aligned}$$

The following definitions gives two partial orders based on past entropy.

Definition 4.1 *A random variable X is said to be larger than another random variable Y in weighted past entropy order (written as $X \geqslant_{WPE} Y$) if $\bar{H}^{\omega}(X;t) \leqslant \bar{H}^{\omega}(Y;t)$.* □

Definition 4.2 *A random variable X is said to be larger than another random variable Y in weighted generalized past entropy of order β (written as $X \geqslant_{WGPE} Y$) if $\bar{H}_1^{\omega^\beta}(X;t) \leqslant \bar{H}_1^{\omega^\beta}(Y;t)$ (or, equivalently, $\bar{H}_2^{\omega^\beta}(X;t) \leqslant \bar{H}_2^{\omega^\beta}(Y;t)$).* □

The following counterexample show that weighted past entropy order is the subclass of the weighted generalized past entropy of order β.

Counterexample 4.1 *Let X be a random variable having probability density function given by*

$$f_X(t) = \begin{cases} \frac{t}{2}, & 0 < t < 2, \\ 0, & t > 2. \end{cases}$$

Again, let Y be another random variable having probability density function given by

$$f_X(t) = \begin{cases} \frac{3e^{-\frac{3}{2}t}}{5\left(1-e^{-\frac{3}{2}}\right)}, & 0 \leqslant t \leqslant 1, \\ \frac{3te^{-2+\frac{t^2}{2}}}{5\left(1-e^{-\frac{3}{2}}\right)}, & 1 \leqslant t < 2, \\ 0, & t > 2. \end{cases}$$

Now, for $1 \leqslant t \leqslant 2$

$$\begin{aligned} \bar{H}^{\omega}(X;t) - \bar{H}^{\omega}(Y;t) &= \int_0^1 \frac{xe^{-\frac{3x}{2}} \ln\left(\frac{3xe^{-\frac{3x}{2}}}{\int_0^1 3xe^{-\frac{3x}{2}}dx + \int_1^t 3x^2e^{-2+\frac{x^2}{2}}dx}\right)}{\int_0^1 xe^{-\frac{3x}{2}}dx + \int_1^t x^2e^{-2+\frac{x^2}{2}}dx}dx \\ &\quad + \int_1^t \frac{3x^2e^{-2+\frac{x^2}{2}} \ln\left(\frac{3x^2e^{-2+\frac{x^2}{2}}}{\int_0^1 3xe^{-\frac{3x}{2}}dx + \int_1^t 3x^2e^{-2+\frac{x^2}{2}}dx}\right)}{\int_0^1 3xe^{-\frac{3x}{2}}dx + \int_1^t 3x^2e^{-2+\frac{x^2}{2}}dx}dx \\ &\quad - \int_0^t \frac{3x^2}{t^3} \ln\left(\frac{3x^2}{t^3}\right)dx \\ &= \alpha(t), \text{ say,} \end{aligned}$$

which is not always negative as shown in Figure 3. Therefore, $X \ngeq_{WPE} Y$. Take $\beta = \frac{1}{3}$. Then,

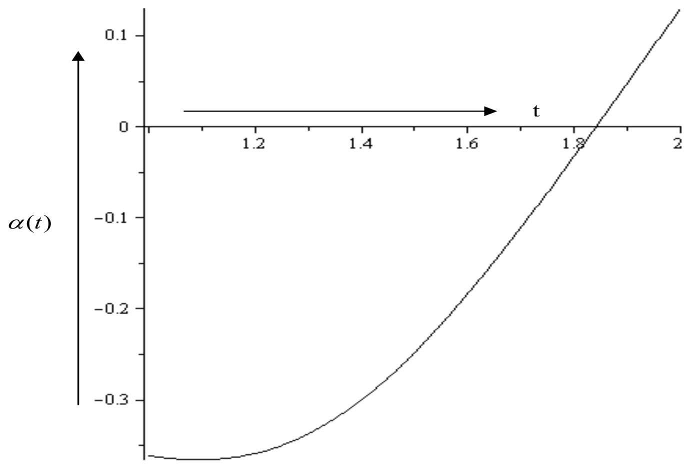

Figure 3: Plot of $\alpha(t)$ for and $t \in [1,2]$ (Counterexample 4.1)

$$
\begin{aligned}
\alpha_1(t) &\stackrel{\text{def}}{=} \bar{H}_1^{\omega^\beta}(X;t) - \bar{H}_1^{\omega^\beta}(Y;t)\\
&= \begin{cases}
-\frac{3}{2}\left[\dfrac{\displaystyle\int_0^t \left(xe^{-\frac{3}{2}x}\right)^{\frac{1}{3}} dx}{\left(\displaystyle\int_0^t xe^{-\frac{3}{2}x}dx\right)^{\frac{1}{3}}} - \frac{3^{\frac{4}{3}}}{5}t^{\frac{2}{3}}\right], & 0 \leqslant t \leqslant 1,\\
-\frac{3}{2}\left[\dfrac{\displaystyle\int_0^1 \left(xe^{-\frac{3}{2}x}\right)^{\frac{1}{3}} dx + \int_1^t \left(x^2e^{-2+\frac{x^2}{2}}\right)^{\frac{1}{3}} dx}{\left(\displaystyle\int_0^1 xe^{-\frac{3}{2}x}dx + \int_1^t x^2e^{-2+\frac{x^2}{2}}dx\right)^{\frac{1}{3}}} - \frac{3^{\frac{4}{3}}}{5}t^{\frac{2}{3}}\right], & 1 \leqslant t \leqslant 2,\\
-0.0207, & t \geqslant 2.
\end{cases}
\end{aligned}
$$

From Figure 4, we can see that $\bar{H}_1^{\omega^\beta}(X;t) \leqslant \bar{H}_1^{\omega^\beta}(Y;t)$*. Again, we see that*

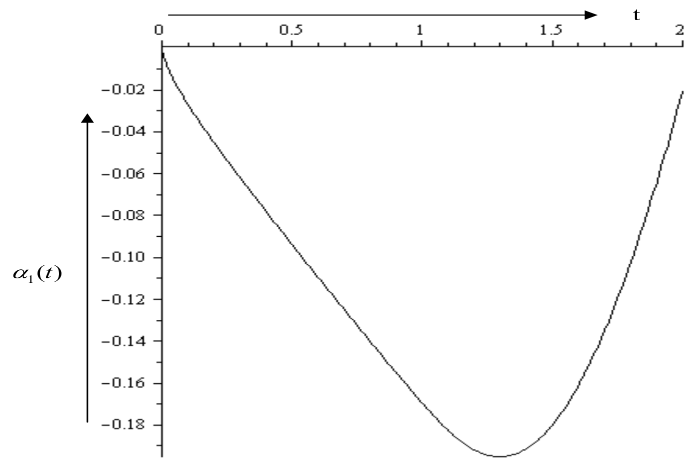

Figure 4: Plot of $\alpha_1(t)$ for and $t \in [0,2]$ (Counterexample 4.1)

$$
\begin{aligned}
\kappa_1(t) &\stackrel{\text{def}}{=} \bar{H}_2^{\omega^\beta}(X;t) - \bar{H}_2^{\omega^\beta}(Y;t)\\
&= \begin{cases}
\frac{3}{2}\left[\ln\left(\frac{3^{\frac{4}{3}}}{5}t^{\frac{2}{3}}\right) - \ln\left(\dfrac{\displaystyle\int_0^t \left(xe^{-\frac{3}{2}x}\right)^{\frac{1}{3}} dx}{\left(\displaystyle\int_0^t xe^{-\frac{3}{2}x}dx\right)^{\frac{1}{3}}}\right)\right], & 0 \leqslant t \leqslant 1,\\
\frac{3}{2}\left[\ln\left(\frac{3^{\frac{4}{3}}}{5}t^{\frac{2}{3}}\right) - \ln\left(\dfrac{\displaystyle\int_0^1 \left(xe^{-\frac{3}{2}x}\right)^{\frac{1}{3}} dx + \int_1^t \left(x^2e^{-2+\frac{x^2}{2}}\right)^{\frac{1}{3}} dx}{\left(\displaystyle\int_0^1 xe^{-\frac{3}{2}x}dx + \int_1^t x^2e^{-2+\frac{x^2}{2}}dx\right)^{\frac{1}{3}}}\right)\right], & 1 \leqslant t \leqslant 2,\\
-0.01498, & t \geqslant 2.
\end{cases}
\end{aligned}
$$

From Figure 5, we can see that $\bar{H}_2^{\omega^\beta}(X;t) \leqslant \bar{H}_2^{\omega^\beta}(Y;t)$*. Therefore,* $X \geqslant_{WGPE} Y$*. Hence, weighted past entropy order is the subclass of the weighted generalized past entropy of order* β*.* □

The following example shows that $X \geqslant_{ST} Y$, but $X \ngeqslant_{WGPE} Y$. A nonnegative random variable X is said to be greater than another nonnegative random variable Y in stochastic order (written as $X \geqslant_{ST} Y$) if $F_X(t) \leqslant F_Y(t)$, for all $t > 0$, where $F_X(t)$ and $F_Y(t)$ are the distribution functions of X and Y, respectively.

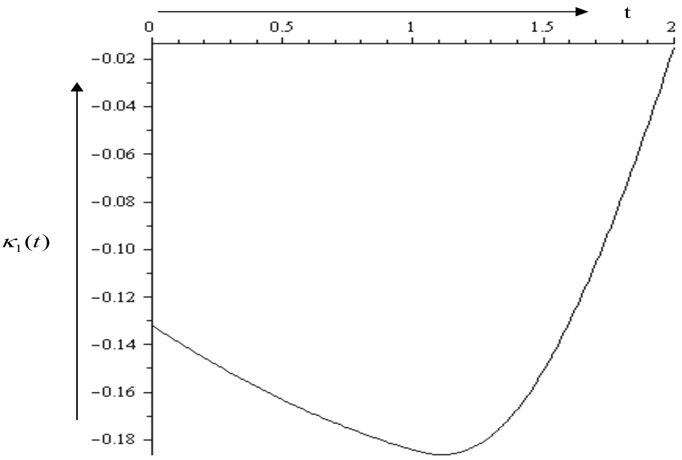

Figure 5: Plot of $\kappa_1(t)$ for and $t \in [0, 2]$ (Counterexample 4.1)

Example 4.1 *Let X be a random variable having probability density function $f_X(t) = \frac{1}{(1+t)^2}$, $t \geqslant 0$ and the corresponding distribution function is given by $F_X(t) = 1 - \frac{1}{1+t}$, $t \geqslant 0$. Suppose that Y be a random variable having probability density function $f_Y(t) = \frac{2}{(1+t)^3}$, $t \geqslant 0$ and the corresponding distribution function is given by $F_Y(t) = 1 - \frac{1}{(1+t)^2}$, $t \geqslant 0$. It can be shown that $F_X(t) \leqslant F_Y(t)$, for all $t \geqslant 0$. Thus, $X \geqslant_{ST} Y$. Take $\beta = 2$. Now, we see that for all $t \geqslant 0$*

$$\begin{aligned} \bar{H}_1^{\omega^\beta}(X;t) - \bar{H}_1^{\omega^\beta}(Y;t) &= \frac{\frac{1}{2(1+t)^4} - \frac{1}{3(1+t)^3} - \frac{1}{5(1+t)^5} + \frac{1}{30}}{\left(\frac{1}{2(1+t)^2} - \frac{1}{1+t} + \frac{1}{2}\right)^2} - \frac{\frac{1}{(1+t)^2} - \frac{1}{3(1+t)^3} - \frac{1}{1+t} + \frac{1}{3}}{\left(\frac{1}{1+t} + \ln(1+t) - 1\right)^2} \\ &= \alpha_2(t), \ \textit{say}. \end{aligned}$$

We see that $\alpha_2(0.5) = -0.1061$ and $\alpha_2(3) = 0.03046$. Again, we see that for all $t \geqslant 0$

$$\begin{aligned} \bar{H}_2^{\omega^\beta}(X;t) - \bar{H}_2^{\omega^\beta}(Y;t) &= \ln\left(\frac{\frac{1}{2(1+t)^4} - \frac{1}{3(1+t)^3} - \frac{1}{5(1+t)^5} + \frac{1}{30}}{\left(\frac{1}{2(1+t)^2} - \frac{1}{1+t} + \frac{1}{2}\right)^2}\right) - \ln\left(\frac{\frac{1}{(1+t)^2} - \frac{1}{3(1+t)^3} - \frac{1}{1+t} + \frac{1}{3}}{\left(\frac{1}{1+t} + \ln(1+t) - 1\right)^2}\right) \\ &= \kappa_2(t), \ \textit{say}. \end{aligned}$$

We see that $\kappa_2(0.5) = -0.0458$ and $\kappa_2(3) = 0.08406$. Hence, $X \ngeqslant_{WGPE} Y$. $\square$

The following example shows that $X \ngeqslant_{ST} Y$, but $X \geqslant_{WGPE} Y$.

Example 4.2 *Let X be a random variable having probability density function given by*

$$f_X(t) = \begin{cases} \frac{2(1-t)}{1+4t^2} + \frac{8t(1-t)^2}{(1+4t^2)^2}, & 0 \leqslant t \leqslant 1, \\ 0, & t \geqslant 1. \end{cases}$$

The corresponding distribution function is given by

$$F_X(t) = \begin{cases} 1 - \frac{(1-t)^2}{1+4t^2}, & 0 \leqslant t \leqslant 1, \\ 1, & t \geqslant 1. \end{cases}$$

Again, let Y be another random variable having probability density function given by

$$f_Y(t) = \begin{cases} 2(1-t)e^{-t} + (1-t)^2 e^{-t}, & 0 \leqslant t \leqslant 1, \\ 0, & t \geqslant 1. \end{cases}$$

The corresponding distribution function is given by

$$F_Y(t)=\begin{cases}1-(1-t)^2e^{-t}, & 0\leqslant t\leqslant 1,\\ 1, & t\geqslant 1.\end{cases}$$

Now, we can see that for $0\leqslant t\leqslant 1$

$$\begin{aligned}F_X(t)-F_Y(t) &= (1-t)^2e^{-t}-\frac{(1-t)^2}{1+4t^2}\\ &= \begin{cases}-0.04593, & t=0.1,\\ \ \ 0.02663, & t=0.5.\end{cases}\end{aligned}$$

Therefore, $X\ngeqslant_{ST} Y$*. By taking* $\beta=2$*, we can see that*

$$\begin{aligned}\alpha_3(t) &\stackrel{\text{def}}{=} \bar{H}_1^{\omega^\beta}(X;t)-\bar{H}_1^{\omega^\beta}(Y;t)\\ &= \begin{cases}\frac{\frac{3}{8}(1-e^{-2t})+\frac{5}{2}t^3(1+t^2)e^{-2t}-\frac{3}{4}t(1+t)e^{-2t}-\frac{19}{4}t^4(1+2t^2)e^{-2t}}{1-e^{-t}(1+t-t^2+t^3)}\\ -\frac{(75+900t^2+3600t^4+4800t^6)\tan^{-1}2t-t(3075t^5+3744t^4-2304t^3+1088t^2+150)}{64(1+4t^2)^2[2t(8t-3)-2(1+4t^2)\ln(1+4t^2)+3(1+4t^2)\tan^{-1}2t]}, & 0\leqslant t\leqslant 1,\\ -0.14175, & t\geqslant 1,\end{cases}\end{aligned}$$

which is negative as shown in Figure 6. Again, we see that

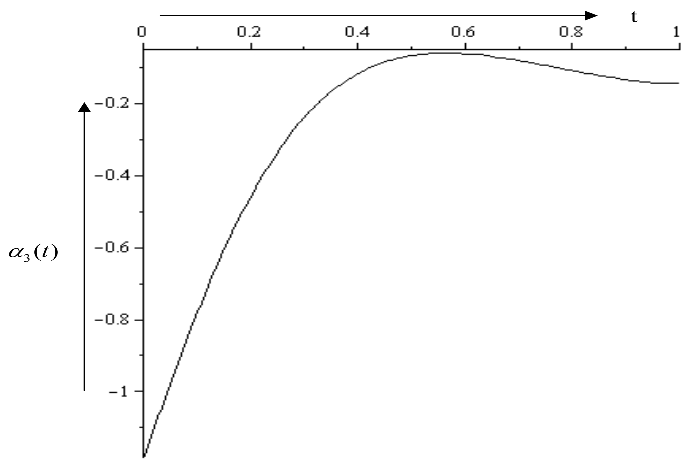

Figure 6: Plot of $\alpha_3(t)$ for and $t\in[0,1]$ (Example 4.2)

$$\begin{aligned}\kappa_3(t) &\stackrel{\text{def}}{=} \bar{H}_2^{\omega^\beta}(X;t)-\bar{H}_2^{\omega^\beta}(Y;t)\\ &= \begin{cases}\ln\left(\frac{\frac{3}{8}(1-e^{-2t})+\frac{5}{2}t^3(1+t^2)e^{-2t}-\frac{3}{4}t(1+t)e^{-2t}-\frac{19}{4}t^4(1+2t^2)e^{-2t}}{1-e^{-t}(1+t-t^2+t^3)}\right)\\ -\ln\left(\frac{(75+900t^2+3600t^4+4800t^6)\tan^{-1}2t-t(3075t^5+3744t^4-2304t^3+1088t^2+150)}{64(1+4t^2)^2[2t(8t-3)-2(1+4t^2)\ln(1+4t^2)+3(1+4t^2)\tan^{-1}2t]}\right), & 0\leqslant t\leqslant 1,\\ -0.10933, & t\geqslant 1,\end{cases}\end{aligned}$$

which is negative as shown in Figure 7. Hence, $X\geqslant_{WGPE} Y$*.* □

Di Crescenzo and Longobardi (2006) have been defined a nonparametric classes of distribution based on the monotonicity property of weighted past entropy is given below.

Definition 4.3 *A random variable* X *is said to have increasing weighted uncertainty past life IWUPL if* $\bar{H}^\omega(X;t)$ *is increasing in* $t\geqslant 0$*.* □

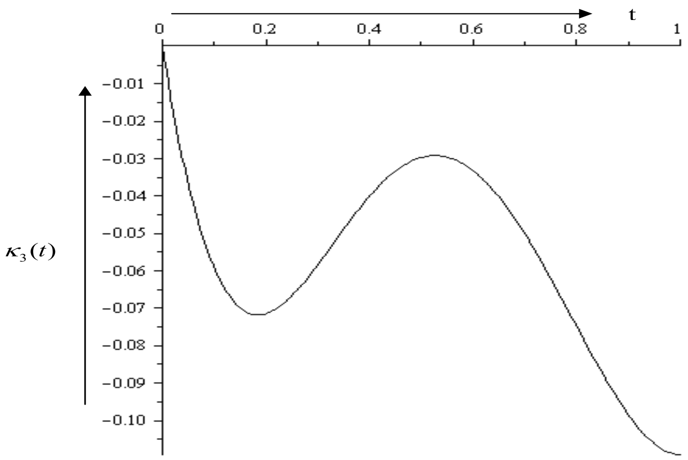

Figure 7: Plot of $\kappa_3(t)$ for and $t \in [0, 1]$ (Example 4.2)

Here one nonparametric class of distribution based on the monotonicity property of weighted generalized past entropy has been introduced.

Definition 4.4 *A nonnegative random variable X is said to have*

(i) increasing weighted uncertainty past life of first kind of order β [IWUPLF(β)] if $\bar{H}_1^{\omega^\beta}(X;t)$ is increasing in $t \geqslant 0$;

(ii) increasing weighted uncertainty past life of second kind of order β [IWUPLS(β))] if $\bar{H}_2^{\omega^\beta}(X;t)$ is increasing in $t \geqslant 0$. □

The following counterexample shows that the class IWUPLF(β) or IWUPLS(β) does not coincide, in general, with IWURL class.

Counterexample 4.2 *Let X be a random variable having probability density function given by*

$$f_X(t) = \begin{cases} \frac{3e^{-\frac{3}{2}t}}{5\left(1-e^{-\frac{3}{2}}\right)}, & 0 \leqslant t \leqslant 1, \\ \frac{3te^{-2+\frac{t^2}{2}}}{5\left(1-e^{-\frac{3}{2}}\right)}, & 1 \leqslant t < 2, \\ 0, & t > 2. \end{cases}$$

Now, for $1 \leqslant t \leqslant 2$, the weighted past entropy of the random variable X is given by

$$\begin{aligned} \bar{H}^{\omega}(X;t) &= -\frac{1}{\int_0^1 \frac{3xe^{-\frac{3}{2}x}}{5\left(1-e^{-\frac{3}{2}}\right)}dx + \int_1^t \frac{3x^2e^{-2+\frac{x^2}{2}}}{5\left(1-e^{-\frac{3}{2}}\right)}dx} \left[\int_0^1 \frac{3xe^{-\frac{3}{2}x}}{5\left(1-e^{-\frac{3}{2}}\right)} \ln\left(\frac{3xe^{-\frac{3}{2}x}}{5\left(1-e^{-\frac{3}{2}}\right)}\right)dx + \right. \\ & \int_1^t \frac{3x^2e^{-2+\frac{x^2}{2}}}{5\left(1-e^{-\frac{3}{2}}\right)} \ln\left(\frac{3x^2e^{-2+\frac{x^2}{2}}}{5\left(1-e^{-\frac{3}{2}}\right)}\right)dx - \left(\int_0^1 \frac{3xe^{-\frac{3}{2}x}}{5\left(1-e^{-\frac{3}{2}}\right)}dx + \int_1^t \frac{3x^2e^{-2+\frac{x^2}{2}}}{5\left(1-e^{-\frac{3}{2}}\right)}dx\right) \\ & \left. \ln\left(\int_0^1 \frac{3xe^{-\frac{3}{2}x}}{5\left(1-e^{-\frac{3}{2}}\right)}dx + \int_1^t \frac{3x^2e^{-2+\frac{x^2}{2}}}{5\left(1-e^{-\frac{3}{2}}\right)}dx\right)\right] \\ &= \alpha_4(t), \ \textit{say}. \end{aligned}$$

We see that $\alpha_4(1.2) = 0.10925$*,* $\alpha_4(0.5) = 0.22122$ *and* $\alpha_4(2) = 0.13153$*. Therefore,* X *is not IWURL. By taking* $\beta = 0.5$*, the weighted generalized past entropy of first kind of the random variable* X *is given by*

$$\alpha_5(t) \overset{\text{def}}{=} \bar{H}_1^{\omega^\beta}(X;t) = \begin{cases} -2\left[1 - \dfrac{\displaystyle\int_0^t \left(3xe^{-\frac{3}{2}x}\right)^{0.5} dx}{\left(\displaystyle\int_0^t 3xe^{-\frac{3}{2}x} dx\right)^{0.5}}\right], & 0 \leqslant t \leqslant 1, \\ -2\left[1 - \dfrac{\displaystyle\int_0^1 \left(3xe^{-\frac{3}{2}x}\right)^{0.5} dx + \int_1^t \left(3x^2 e^{-2+\frac{x^2}{2}}\right)^{0.5} dx}{\left(\displaystyle\int_0^t 3xe^{-\frac{3}{2}x} dx + \int_1^t 3x^2 e^{-2+\frac{x^2}{2}}\right)^{0.5}}\right], & 1 \leqslant t \leqslant 2, \\ 0.4350, & t \geqslant 2, \end{cases}$$

which is increasing in $t \geqslant 0$ *as shown in Figure 8. Again, the weighted generalized past entropy of*

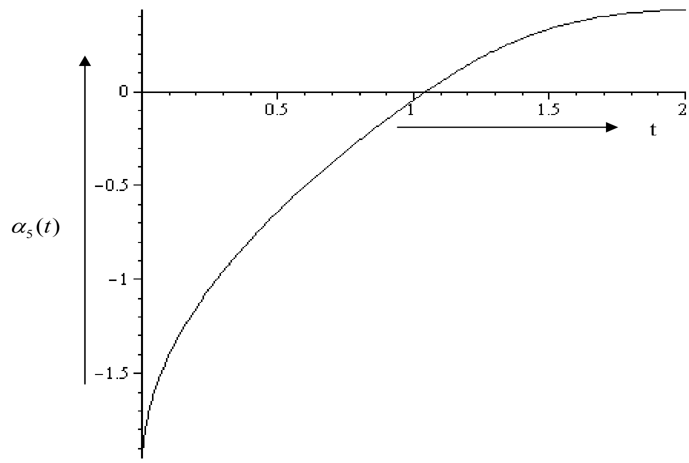

Figure 8: Plot of $\alpha_5(t)$ for and $t \in [0, 2]$ (Counterexample 4.2)

second kind of the random variable X *is given by*

$$\kappa_4(t) \overset{\text{def}}{=} \bar{H}_2^{\omega^\beta}(X;t) = \begin{cases} 2\ln\left(\dfrac{\displaystyle\int_0^t \left(3xe^{-\frac{3}{2}x}\right)^{0.5} dx}{\left(\displaystyle\int_0^t 3xe^{-\frac{3}{2}x} dx\right)^{0.5}}\right), & 0 \leqslant t \leqslant 1, \\ 2\ln\left(\dfrac{\displaystyle\int_0^1 \left(3xe^{-\frac{3}{2}x}\right)^{0.5} dx + \int_1^t \left(3x^2 e^{-2+\frac{x^2}{2}}\right)^{0.5} dx}{\left(\displaystyle\int_0^t 3xe^{-\frac{3}{2}x} dx + \int_1^t 3x^2 e^{-2+\frac{x^2}{2}}\right)^{0.5}}\right), & 1 \leqslant t \leqslant 2, \\ 0.3936, & t \geqslant 2, \end{cases}$$

which is increasing in $t \geqslant 0$ *as shown in Figure 9. Hence,* X *is not IWURL but it is IWUPLF(*β*), for* $\beta = 0.5$. □

The following theorem characterizes the weighted generalized past entropy in the sense that under certain condition the weighted generalized residual entropy uniquely determine the distribution function.

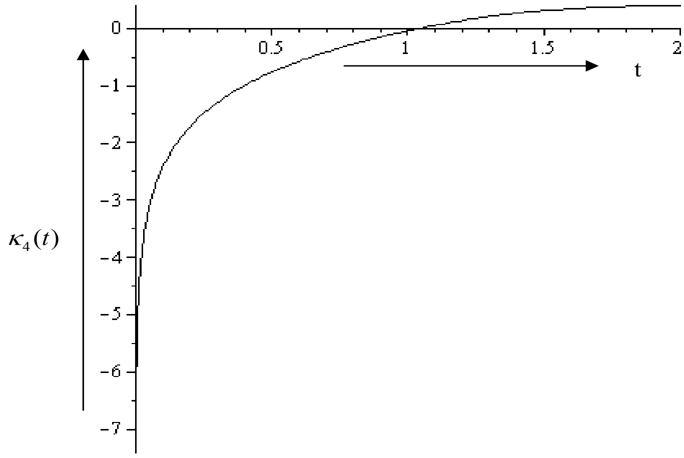

Figure 9: Plot of $\kappa_4(t)$ for and $t \in [0, 2]$ (Counterexample 4.2)

Theorem 4.1 *Let X be a nonnegative absolutely continuous random variable having probability density function $f_X(t)$ and the distribution function $F_X(t)$. If*

(i) $\bar{H}_1^{\beta}(X;t)$ is increasing in t, then $\bar{H}_1^{\omega^\beta}(X;t)$ uniquely determine $F_X(t)$;

(ii) $\bar{H}_2^{\beta}(X;t)$ is increasing in t, then $\bar{H}_2^{\omega^\beta}(X;t)$ uniquely determine $F_X(t)$.

Proof: (i) From (4.4), we have

$$\begin{aligned} 1-(\beta-1)\bar{H}_1^{\omega^\beta}(X;t) &= \frac{1}{[E(X|X<t)]^\beta}\Big[t^\beta\Big(1-(\beta-1)\bar{H}_1^{\beta}(X;t)\Big) \\ &\quad -\beta\int_0^t y^{\beta-1}\left(\frac{F_X(y)}{F_X(t)}\right)^\beta\Big(1-(\beta-1)\bar{H}_1^{\beta}(X;y)\Big)\,dy\Big] \\ &= \frac{J(t)}{[g_1(t)]^\beta}, \end{aligned} \tag{4.6}$$

where $g_1(t) = E(X|X<t)$ and

$$J(t) = t^\beta\Big(1-(\beta-1)\bar{H}_1^{\beta}(X;t)\Big) - \beta\int_0^t y^{\beta-1}\left(\frac{F_X(y)}{F_X(t)}\right)^\beta\Big(1-(\beta-1)\bar{H}_1^{\beta}(X;y)\Big)\,dy.$$

Differentiating $J(t)$ with respect to t, we get

$$J'(t) = -(\beta-1)t^\beta\frac{d}{dt}\bar{H}_1^{\beta}(X;t) + \beta^2\nu_X(t)\int_0^t y^{\beta-1}\left(\frac{F_X(y)}{F_X(t)}\right)^\beta\Big(1-(\beta-1)\bar{H}_1^{\beta}(X;y)\Big)\,dy.$$

Differentiating (4.6) with respect to t, we get

$$-(\beta-1)\frac{d}{dt}\bar{H}_1^{\omega^\beta}(X;t) = -\beta\frac{g_1'(t)}{g_1^{\beta+1}(t)}J(t) + \frac{J'(t)}{g_1^{\beta}(t)}. \tag{4.7}$$

Again, from (4.2), we have

$$1-(\beta-1)\bar{H}_1^{\omega^\beta}(X;t) = \frac{1}{g_1^{\beta}(t)}\int_0^t y^\beta\left(\frac{f_X(y)}{F_X(t)}\right)^\beta dy. \tag{4.8}$$

Differentiating (4.8) with respect to t, we get

$$
\begin{aligned}
-(\beta-1)\frac{d}{dt}\bar{H}_1^{\omega^\beta}(X;t) &= -\beta\frac{g_1'(t)}{g_1^{\beta+1}(t)}\int_0^t y^\beta\left(\frac{f_X(y)}{F_X(t)}\right)^\beta dy \\
&\quad -\frac{\beta}{g_1^\beta(t)}\nu_X(t)\int_0^t y^\beta\left(\frac{f_X(y)}{F_X(t)}\right)^\beta dy+\frac{t^\beta}{g_1^\beta(t)}\nu_X^\beta(t). \qquad (4.9)
\end{aligned}
$$

From (4.7) and (4.9), we have

$$
\begin{aligned}
\beta\frac{g_1'(t)}{g_1^{\beta+1}(t)}J(t)-\frac{J'(t)}{g_1^\beta(t)} &= \beta\frac{g_1'(t)}{g_1^{\beta+1}(t)}\int_0^t y^\beta\left(\frac{f_X(y)}{F_X(t)}\right)^\beta dy+\frac{\beta}{g_1^\beta(t)}\nu_X(t)\int_0^t y^\beta\left(\frac{f_X(y)}{F_X(t)}\right)^\beta dy \\
&\quad -\frac{t^\beta}{g_1^\beta(t)}\nu_X^\beta(t),
\end{aligned}
$$

or, equivalently,

$$
\begin{aligned}
&\beta\frac{g_1'(t)}{g_1^{\beta+1}(t)}\left[J(t)-\int_0^t y^\beta\left(\frac{f_X(y)}{F_X(t)}\right)^\beta dy\right]+\frac{\beta-1}{g_1^\beta}t^\beta\frac{d}{dt}\bar{H}_1^\beta(X;y)-\frac{\beta\nu_X(t)}{g_1^\beta(t)} \\
&\left[\beta\int_0^t y^{\beta-1}\left(\frac{F_X(y)}{F_X(t)}\right)^\beta\left(1-(\beta-1)\bar{H}_1^\beta(X;y)\right)dy+\int_0^t y^\beta\left(\frac{f_X(y)}{F_X(t)}\right)^\beta dy\right]+\frac{t^\beta}{g_1^\beta(t)}\nu_X^\beta(t)=0.
\end{aligned}
$$

For a fixed $t>0$, $\nu_X(t)$ is a solution of $B(x)=0$, where

$$
\begin{aligned}
B(x) &= \beta\frac{g_1'(t)}{g_1^{\beta+1}(t)}\left[J(t)-\int_0^t y^\beta\left(\frac{f_X(y)}{F_X(t)}\right)^\beta dy\right]+\frac{\beta-1}{g_1^\beta}t^\beta\frac{d}{dt}\bar{H}_1^\beta(X;y)-\frac{\beta}{g_1^\beta(t)} \\
&\quad \left[\beta\int_0^t y^{\beta-1}\left(\frac{F_X(y)}{F_X(t)}\right)^\beta\left(1-(\beta-1)\bar{H}_1^\beta(X;y)\right)dy+\int_0^t y^\beta\left(\frac{f_X(y)}{F_X(t)}\right)^\beta dy\right]x+\frac{t^\beta}{g_1^\beta(t)}x^\beta.
\end{aligned}
$$

Differentiating $B(t)$ with respect to x, we get

$$
\begin{aligned}
B'(x) &= -\frac{\beta}{g_1^\beta(t)}\left[\beta\int_0^t y^{\beta-1}\left(\frac{F_X(y)}{F_X(t)}\right)^\beta\left(1-(\beta-1)\bar{H}_1^\beta(X;y)\right)dy+\int_0^t y^\beta\left(\frac{f_X(y)}{F_X(t)}\right)^\beta dy\right] \\
&\quad +\frac{\beta t^\beta}{g_1^\beta(t)}x^{\beta-1}.
\end{aligned}
$$

Now, $B'(x)=0$ gives

$$
\begin{aligned}
x &= \left[\frac{1}{t^\beta}\left(\int_0^t y^\beta\left(\frac{f_X(y)}{F_X(t)}\right)^\beta dy+\beta\int_0^t y^{\beta-1}\left(\frac{F_X(y)}{F_X(t)}\right)^\beta\left(1-(\beta-1)\bar{H}_1^\beta(X;y)\right)dy\right)\right]^{\frac{1}{\beta-1}} \\
&= t_2,\ \textit{say}.
\end{aligned}
$$

Again, we see that

$$
B(0) = \frac{\beta-1}{g_1^\beta(t)}t^\beta\frac{d}{dt}\bar{H}_1^\beta(X;t).
$$

Case I : For $\beta>1$, $B(0)>0$, since $\bar{H}_1^\beta(X;t)$ is increasing in t and $B(x)$ is convex function with minimum occurring at $x=t_2$. So $B(x)=0$ has unique solution when $B(t_2)=0$.

Case II : For $\beta < 1$, $B(0) < 0$, since $\bar{H}_1^{\beta}(X;t)$ is increasing in t and $B(x)$ is concave function with maximum occurring at $x = t_2$. So $B(x) = 0$ has unique solution when $B(t_2) = 0$.

Therefore, from the two cases it can be conclude that if $\bar{H}_1^{\beta}(X;t)$ is increasing in $t > 0$ and $B(t_2) = 0$ then $B(x) = 0$ has unique solution. Since $\nu_X(t)$ is the solution of $B(x) = 0$ then $\bar{H}_1^{\omega^{\beta}}(X;t)$ determines $\nu_X(t)$ uniquely. Again, $\nu_X(t)$ uniquely determine $F_X(t)$. Hence, the result follows.

To proof (ii), we get, from (4.3)

$$\begin{aligned}\exp\left[(1-\beta)\bar{H}_2^{\omega^{\beta}}(X;t)\right] &= \frac{1}{[E(X|X<t)]^{\beta}}\int_0^t y^{\beta}\left(\frac{f_X(y)}{F_X(t)}\right)^{\beta}dy\\ &= \frac{1}{g_1^{\beta}(t)}\int_0^t y^{\beta}\left(\frac{f_X(y)}{F_X(t)}\right)^{\beta}dy. \qquad (4.10)\end{aligned}$$

Differentiating (4.10) with respect to t, we get

$$\begin{aligned}\frac{d}{dt}\left(\exp\left[(1-\beta)\bar{H}_2^{\omega^{\beta}}\right]\right) &= -\beta\frac{g_1'(t)}{g_1^{\beta+1}(t)}\int_0^t y^{\beta}\left(\frac{f_X(y)}{F_X(t)}\right)^{\beta}dy - \frac{\beta}{g_1^{\beta}(t)}\nu_X(t)\int_0^t y^{\beta}\left(\frac{f_X(y)}{F_X(t)}\right)^{\beta}dy\\ &\quad+\frac{t^{\beta}}{g_1^{\beta}(t)}\nu_X^{\beta}(t). \qquad (4.11)\end{aligned}$$

Again, from (4.5), we have

$$\begin{aligned}\exp\left[(1-\beta)\bar{H}_2^{\omega^{\beta}}(X;t)\right] &= \frac{1}{g_1^{\beta}(t)}\Big[t^{\beta}\exp\left((1-\beta)\bar{H}_2^{\beta}(X;t)\right)\\ &\quad-\beta\int_0^t y^{\beta-1}\left(\frac{F_X(y)}{F_X(t)}\right)^{\beta}\exp\left((1-\beta)\bar{H}_2^{\beta}(X;y)\right)dy\Big]\\ &= \frac{J_1(t)}{g_1^{\beta}(t)}, \qquad (4.12)\end{aligned}$$

where $J_1(t) = t^{\beta}\exp\left((1-\beta)\bar{H}_2^{\beta}(X;t)\right) - \beta\int_0^t y^{\beta-1}\left(\frac{F_X(y)}{F_X(t)}\right)^{\beta}\exp\left((1-\beta)\bar{H}_2^{\beta}(X;y)\right)dy$. Differentiating $J_1(t)$ with respect to t, we get

$$\begin{aligned}J_1'(t) &= t^{\beta}(1-\beta)\exp\left((1-\beta)\bar{H}_2^{\beta}(X;t)\right)\frac{d}{dt}\bar{H}_2^{\beta}(X;t) + \beta^2\nu_X(t)\\ &\quad\int_0^t y^{\beta-1}\left(\frac{F_X(y)}{F_X(t)}\right)^{\beta}\exp\left((1-\beta)\bar{H}_2^{\beta}(X;y)\right)dy.\end{aligned}$$

Differentiating (4.12) with respect to t, we get

$$\begin{aligned}\frac{d}{dt}\left(\exp\left[(1-\beta)\bar{H}_2^{\omega^{\beta}}(X;t)\right]\right) &= \frac{1}{g_1^{\beta}(t)}\Big(t^{\beta}(1-\beta)\exp\left((1-\beta)\bar{H}_2^{\beta}(X;t)\right)\frac{d}{dt}\bar{H}_2^{\beta}(X;t)\\ &\quad+\beta^2\nu_X(t)\int_0^t y^{\beta-1}\left(\frac{F_X(y)}{F_X(t)}\right)^{\beta}\exp\left((1-\beta)\bar{H}_2^{\beta}(X;y)\right)dy\Big)\\ &\quad-\beta\frac{g_1'(t)}{g_1^{\beta+1}(t)}J_1(t). \qquad (4.13)\end{aligned}$$

From (4.11) and (4.13), we have

$$\beta\frac{g_1'(t)}{g_1^{\beta+1}(t)}\left[J_1(t)-\int_0^t y^\beta\left(\frac{f_X(y)}{F_X(t)}\right)^\beta dy\right]-\frac{1-\beta}{g_1^\beta(t)}\left[t^\beta\exp\Big((1-\beta)\bar{H}_2^\beta(X;t)\Big)\frac{d}{dt}\bar{H}_2^\beta(X;t)\right]$$
$$-\frac{\beta}{g_1^\beta(t)}\nu_X(t)\left[\beta\int_0^t y^{\beta-1}\left(\frac{F_X(y)}{F_X(t)}\right)^\beta\exp\Big((1-\beta)\bar{H}_2^\beta(X;y)\Big)dy+\int_0^t y^\beta\left(\frac{f_X(y)}{F_X(t)}\right)^\beta dy\right]$$
$$+\frac{t^\beta}{g_1^\beta(t)}\nu X^\beta(t)=0.$$

For a fixed $t>0$, $\nu_X(t)$ is a solution of $B_1(x)=0$, where

$$\begin{aligned}B_1(x) &= \beta\frac{g_1'(t)}{g_1^{\beta+1}(t)}\left[J_1(t)-\int_0^t y^\beta\left(\frac{f_X(y)}{F_X(t)}\right)^\beta dy\right]-\frac{1-\beta}{g_1^\beta(t)}\left[t^\beta\exp\Big((1-\beta)\bar{H}_2^\beta(X;t)\Big)\frac{d}{dt}\bar{H}_2^\beta(X;t)\right]\\ &\quad-\frac{\beta}{g_1^\beta(t)}\left[\beta\int_0^t y^{\beta-1}\left(\frac{F_X(y)}{F_X(t)}\right)^\beta\exp\Big((1-\beta)\bar{H}_2^\beta(X;y)\Big)dy+\int_0^t y^\beta\left(\frac{f_X(y)}{F_X(t)}\right)^\beta dy\right]x\\ &\quad+\frac{t^\beta}{g_1^\beta(t)}x^\beta.\end{aligned}$$

Now, $B_1'(x)=0$ gives

$$\frac{\beta t^\beta}{g_1^\beta(t)}x^{\beta-1}-\frac{\beta}{g_1^\beta(t)}\left[\beta\int_0^t y^{\beta-1}\left(\frac{F_X(y)}{F_X(t)}\right)^\beta\exp\Big((1-\beta)\bar{H}_2^\beta(X;y)\Big)dy+\int_0^t y^\beta\left(\frac{f_X(y)}{F_X(t)}\right)^\beta dy\right] = 0,$$

which is equivalent to

$$\begin{aligned}x &= \left[\frac{1}{t^\beta}\left(\int_0^t y^\beta\left(\frac{f_X(y)}{F_X(t)}\right)^\beta dy+\beta\int_0^t y^{\beta-1}\left(\frac{F_X(y)}{F_X(t)}\right)^\beta\exp\Big((1-\beta)\bar{H}_2^\beta(X;y)\Big)dy\right)\right]^{\frac{1}{\beta-1}}\\ &= t_3,\ \textit{say}.\end{aligned}$$

Again, $B_1(0)=-\frac{1-\beta}{g_1^\beta(t)}t^\beta\exp\Big((1-\beta)\bar{H}_2^\beta(X;t)\Big)\frac{d}{dt}\bar{H}_2^\beta(X;t)$.

Case I : For $\beta>1$, $B_1(0)>0$, since $\bar{H}_2^\beta(X;t)$ is increasing in t and $B_1(x)$ is convex function with minimum occurring at $x=t_3$. So $B_1(x)=0$ has unique solution when $B_1(t_3)=0$.

Case II : For $\beta<1$, $B_1(0)<0$, since $\bar{H}_2^\beta(X;t)$ is increasing in t and $B_1(x)$ is concave function with maximum occurring at $x=t_3$. So $B_1(x)=0$ has unique solution when $B_1(t_3)=0$.

Therefore, from the two cases it can be conclude that if $\bar{H}_2^\beta(X;t)$ is increasing in $t>0$ and $B_1(t_3)=0$ then $B_1(x)=0$ has unique solution. Since $\nu_X(t)$ is the solution of $B_1(x)=0$ then $\bar{H}_2^{\omega^\beta}(X;t)$ determines $\nu_X(t)$ uniquely. Again, $\nu_X(t)$ uniquely determine $F_X(t)$. Hence, the result follows. □

The following theorem gives the upper bound to the reversed failure rate function $\nu_X(t)$, in terms of $\bar{H}_1^{\omega^\beta}(X;t)$ and $\bar{H}_2^{\omega^\beta}(X;t)$. The proof is omitted.

Theorem 4.2 *(i) If X is IWUPLF(β), then*

$$\nu_X(t)\leqslant\frac{E(X|X<t)}{t}\left[\beta\Big(1-(\beta-1)\bar{H}_1^{\omega^\beta}(X;t)\Big)\right]^{\frac{1}{\beta-1}};$$

(ii) If X is IWUPLS(β), then

$$\nu_X(t)\leqslant\frac{E(X|X<t)}{t}\left[\beta\exp\Big((1-\beta)\bar{H}_2^{\omega^\beta}(X;t)\Big)\right]^{\frac{1}{\beta-1}},$$

for all $t>0$. □

By taking $\beta \to 1$, we have the following corollary. The proof is omitted.

Corollary 4.1 *If X is IWUPL, then*

$$\nu_X(t) \leqslant \frac{E\left(X|X<t\right)}{t} \exp\left[1-\bar{H}^{\omega}\left(X;t\right)\right],$$

for all $t > 0$. □

5 Concluding Remarks

In literature, generalized entropy is a very well known concept which can always gives a nonnegative measure of uncertainty. But in many practical situations for modeling statistical data, sometime a certain amount of information may be lost. With this in mind, here the concept of weighted generalized entropy has been introduced. In this paper, several results on weighted generalized residual and past entropies also have been discussed. Here it has been shown that generalized entropy uniquely determines the distribution of the random variables. Some nonparametric classes of distribution based on the monotonicity properties of weighted generalized entropy have been defined. A partial order based on weighted generalized entropy has been given.

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