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Limiting spectral distribution for Wigner matrices with dependent entries

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LIMITING SPECTRAL DISTRIBUTION FOR WIGNER MATRICES WITH DEPENDENT ENTRIES

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Abstract. In this article we show the existence of limiting spectral distribution of a symmetric random matrix whose entries come from a stationary Gaussian process with covariances satisfying a summability condition. We provide an explicit description of the moments of the limiting measure. We also show that in some special cases the Gaussian assumption can be relaxed. The description of the limiting measure can also be made via its Stieltjes transform which is characterized as the solution of a functional equation. In two special cases, we get a description of the limiting measure - one as a free product convolution of two distributions, and the other one as a dilation of the Wigner semicircular law.

1. Introduction

In his seminal paper, Wigner (1958) showed that for a symmetric random matrix with independent on and off diagonal entries satisfying some moment conditions, the empirical spectral distribution (henceforth ESD) converges to the Wigner semicircle law (defined in (7.2), henceforth WSL). Subsequent work has tried to obtain a better understanding of the spectrum of such matrices, which plays an important role in physics as well as other branches of mathematics such as operator algebras. Recently, there has been interest in how far the independence assumption and the moment conditions can be relaxed. The reader may refer to the recent review article by Ben Arous and Guionnet (2011) and the references therein for an overview of currently available results.

Relaxation of the independence assumption has been investigated by Chatterjee (2006), Götze and Tikhomirov (2005), Hofmann-Credner and Stolz (2008), Rashidi Far et al. (2008). Adamczak (2011), Pfaffel and Schlemm (2012), and Hachem et al. (2005) have studied the sample covariance matrix imposing some dependence on the rows and columns. However, the limiting spectral distributions (henceforth LSD) obtained by considering symmetric matrices with the independence assumption weakened have stayed within the WSL regime for the most part. One exception is Anderson and Zeitouni

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(2008), who considered the LSD of Wigner matrices where on and off diagonal elements form a finite-range dependent random field; in particular, the entries are assumed to be independent beyond a finite range, and within the finite range the correlation structure is given by a kernel function.

Motivation. We begin with a few examples to motivate the problem studied in this article. In each of the following examples, a random field $\{Z_{i,j} : i, j \geq 1\}$ is developed. For $n \geq 1$, let A_n be the $n \times n$ matrix whose (i, j)-th entry is $Z_{i\wedge j, i\vee j}$. The question is whether the ESD of $A_n/\sqrt{n}$ converges as $n \to \infty$, and if so, where.

Example 1. Let $\{Z_{i,j} : i, j \geq 1\}$ be a mean zero Gaussian process such that $E\left[Z_{i,j} Z_{i+k,j+l}\right] = \rho^{|k|+|l|}$ for integers i, j, k, l such that $i, j, i + k, j + l \geq 1$, where $|\rho| < 1$ is fixed. This process can be thought of as a "two dimensional AR(1) process", because $\{Z_{i,j} : j \geq 1\}$ is an AR(1) process for fixed i, as is $\{Z_{i,j} : i \geq 1\}$ for fixed j.

Example 2. Assume that $\{G_{i,j} : i, j \geq 1\}$ are i.i.d. standard Gaussian random variables, and N is a fixed positive integer. Define

$$Z_{i,j} := \sum_{k=0}^{N} \sum_{l=0}^{N} G_{i+k,j+l},\ i, j \geq 1\,.$$

Example 3. Suppose that $(G_n : n \in \mathbb{Z})$ is a mean zero variance one stationary Gaussian process. Let $(G_n^i : n \in \mathbb{Z})$ be i.i.d. copies of $(G_n : n \in \mathbb{Z})$ for $i = \ldots, -2, -1, 0, 1, 2, \ldots$. Set $Z_{i,j} := G_i^{i-j},\ i, j \in \mathbb{Z}$.

Example 4. Let $\{c_{k,l}\}$ be real numbers such that

$$\begin{aligned}
\sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} c_{k,l}^2 &< \infty\,,\\
c_{k,l} &= c_{l,k} \text{ for all } k, l \in \mathbb{Z}\,,\\
\sum_{l=-\infty}^{\infty} c_{k,l} c_{k',l} &= 0 \text{ for all } k \neq k'\,.
\end{aligned}$$

As in Example 2, let $\{G_{i,j} : i, j \geq 1\}$ be i.i.d. standard Gaussian random variables. Define

$$Z_{i,j} := \sum_{k,l\in\mathbb{Z}} c_{k,l} G_{i-k,j-l},\ i, j \in \mathbb{Z}\,.$$

It is shown later in Section 7 that for Examples 1 and 2, the LSD of $A_n/\sqrt{n}$ is the free product convolution of the WSL with a distribution supported on a compact subset of $[0, \infty)$, and for Examples 3 (under the additional assumption that $\sum_{n=1}^{\infty} |E(G_0 G_n)| < \infty$) and 4, the LSD is a dilation of the WSL. To the best of our understanding, Example 2 is the only one of the above examples where the result follows from the work of Anderson and Zeitouni (2008), because that is the only example where two entries are independent if their distance is above a threshold.

Motivated by these examples, this article considers a random matrix model where on and off diagonal entries form a stationary Gaussian field, with the covariance of the entries satisfying a summability condition. It is shown using the method of moments that the ESD converges to a non-degenerate measure. The combinatorial approach we have adopted for calculating the traces of powers of the matrices avoids the use of independence in any stage. Unlike Anderson and Zeitouni (2008) where the use of independence facilitates the negligibility of certain partitions, sharper estimates are needed on a class of partitions. These sharper estimates on the set of partitions and Wick's formula are used to derive the limiting moments. The assumption of Gaussianity, although important in the proof, is relaxed to allow for a fairly general class of input sequences using the Lindeberg type argument developed in Chatterjee (2005). An interpretation of the limiting moments in terms of functions of non-crossing pair partitions is used to derive the Stieltjes transform of the measure. The form of the Stieltjes transform indicates a relationship with operator-valued semicircular variables studied in Speicher (1998) (for the application of free probability to random matrices, see the recent review by Speicher (2011)).

Outline of our contribution. Let $(Z_{i,j} : i, j \in \mathbb{Z})$ be a stationary, mean zero, variance one Gaussian process. Stationarity here means that for $k, l \in \mathbb{Z}$,

$$(Z_{i+k,j+l} : i, j \in \mathbb{Z}) \stackrel{d}{=} (Z_{i,j} : i, j \in \mathbb{Z}) \,.$$

For $i, j \geq 1$, set

$$X_{i,j} := Z_{i\wedge j, i\vee j}\,,$$

and let

$$A_n := ((X_{i,j}))_{n\times n},\ n \geq 1\,. \tag{1.1}$$

Let $\lambda_1 \leq \ldots \leq \lambda_n$ be the eigenvalues of A_n, which are real because A_n is symmetric, and denote

$$\mu_n := \frac{1}{n}\sum_{i=1}^{n} \delta_{\{\lambda_i/\sqrt{n}\}}\,. \tag{1.2}$$

The main result of this article is Theorem 2.1, stated in Section 2 along with an outline of the proof, which gives a set of conditions on the covariance of $\{X_{i,j}\}$ under which the ESD μ_n converges weakly in probability. In Section 3, some combinatorial results are proven, which are used in Section 4 for the proof of Theorem 2.1. In Section 5, we show that by specializing on an infinite order moving average process with independent inputs satisfying the Pastur condition, Theorem 2.1 and an invariance principle can be used to establish the convergence of the ESD. In Section 6, an explicit description of the Stieltjes transform is provided using the moment formula and some properties of the Kreweras complement. In Section 7, two explicit examples are described where we get better descriptions of the limit: Theorem 7.1 gives conditions under which the LSD is the free multiplicative convolution of the WSL and another distribution. Theorem 7.2 gives conditions under which the LSD is the WSL. Finally, in Section 8, Theorem 7.1 is extended to entail the case where the correlations are not necessarily summable. That assumption is replaced there by the weaker assumption of absolute continuity of the spectral measure.

2. The main result

In this section, we state the main result, and give an outline of the proof. Let the $n \times n$ random symmetric matrix A_n be as in (1.1), and set μ_n to be ESD of $A_n/\sqrt{n}$, as defined in (1.2). Before stating the main result, we need a few more notations and assumptions. Define

$$R(u,v) := E\left[Z_{0,0} Z_{-u,v}\right],\ u,v \in \mathbb{Z}\,. \tag{2.1}$$

The assumptions are the following.

Assumption 1: $R(\cdot,\cdot)$ is symmetric, that is,

$$R(u,v) = R(v,u) \text{ for all } u,v \in \mathbb{Z}\,. \tag{2.2}$$

Assumption 2: $R(\cdot,\cdot)$ is absolutely summable, that is,

$$\bar{R} := \sum_{u,v\in\mathbb{Z}} |R(u,v)| < \infty\,. \tag{2.3}$$

An immediate consequence of Assumption 1 and stationarity is that

$$R(u,v) = R(-v,-u),\ u,v \in \mathbb{Z}\,. \tag{2.4}$$

A consequence of Assumption 2 is the following. Fix $\sigma \in NC_2(2m)$, the set of non-crossing pair partitions of $\{1,\ldots,2m\}$. Let $(V_1,\ldots,V_{m+1})$ denote the Kreweras complement of σ, which is the maximal partition $\overline{\sigma}$ of $\{\overline{1},\ldots,\overline{2m}\}$ such that $\sigma\cup\overline{\sigma}$ is a non-crossing partition of $\{1,\overline{1},\ldots,2m,\overline{2m}\}$. For $1 \le i \le m+1$, denote

$$V_i := \{v_1^i,\ldots,v_{l_i}^i\}\,. \tag{2.5}$$

Define

$$S(\sigma) := \left\{(k_1,\ldots,k_{2m}) \in \mathbb{Z}^{2m} : \sum_{j=1}^{l_s} k_{v_j^s} = 0,\ s = 1,\ldots,m+1\right\}. \tag{2.6}$$

If $\sigma = \{(u_1,u_{m+1}),\ldots,(u_m,u_{2m})\}$, then notice that

$$\sum_{(k_1,\ldots,k_{2m})\in S(\sigma)} \prod_{(u,v)\in\sigma} |R(k_u,k_v)|$$

$$
\begin{aligned}
&= \sum_{i\in\mathbb{Z}^{2m}} \Bigg[\#\Big\{ k \in S(\sigma) :\ \text{there exists a permutation } \pi \text{ of } \{1,\ldots,m\} \\
&\qquad \text{such that } k_{u_j} = i_{\pi(j)} \text{ and } k_{u_{m+j}} = i_{m+\pi(j)},\ j=1,\ldots,m \Big\} \\
&\qquad \prod_{j=1}^{m} |R(i_j, i_{m+j})| \Bigg] \\
&\le \sum_{i\in\mathbb{Z}^{2m}} \left[m! \prod_{j=1}^{m} |R(i_j, i_{m+j})| \right] \\
&= m!\bar{R}^m < \infty\,. \qquad (2.7)
\end{aligned}
$$

In view of the above calculation, it makes sense to define

$$\beta_{2m} := \sum_{\sigma\in NC_2(2m)} \sum_{k\in S(\sigma)} \prod_{(u,v)\in\sigma} R(k_u, k_v),\ m \ge 1\,. \tag{2.8}$$

The main result of this article is the following.

Theorem 2.1. *Under Assumption 1 and Assumption 2, μ_n converges weakly in probability to a distribution μ. The k-th moment of μ is zero if k is odd, and β_k if k is even. Furthermore, μ is uniquely determined by its moments, that is, if a distribution has the same moments as that of μ, then the distribution equals μ.*

Remark 1. *As is common in the literature, the phrase "μ_n converges weakly in probability to a distribution μ" means that*

$$L(\mu_n, \mu) \xrightarrow{P} 0\,,$$

as $n \to \infty$, where L, the Lévy distance, is defined by

$$L(\nu_1,\nu_2) := \inf\Big\{ \varepsilon > 0 : \nu_1((-\infty, x-\varepsilon]) - \varepsilon \le \nu_2((-\infty, x]) \le \nu_1((-\infty, x+\varepsilon]) + \varepsilon \text{ for all } x \in \mathbb{R} \Big\}\,, \tag{2.9}$$

for probability measures ν_1, ν_2 on $\mathbb{R}$.

We end this section with a brief outline of the proof of the above result. As is standard in a proof by the method of moments, what needs to be shown is that for fixed $m \ge 1$,

$$\lim_{n\to\infty} n^{-(m+1)} \sum_{i_1,\ldots,i_{2m}=1}^{n} E\left[X_{i_1,i_2} \ldots X_{i_{2m-1},i_{2m}} X_{i_{2m},i_1} \right] = \beta_{2m}\,. \tag{2.10}$$

As in the proof of the classical Wigner's result, the first step is to get rid of the "non-pair matched" tuples $i = (i_1, \ldots i_{2m})$ in the above sum. Fix $N \ge 1$, and say that a tuple $i \in \{1,\ldots,n\}^{2m}$ is N-pair matched if there exists a pair partition π of $\{1,\ldots,2m\}$ such that for all $(u,v) \in \pi$,

$$|i_{u-1} \wedge i_u - i_{v-1} \wedge i_v| \vee |i_{u-1} \vee i_u - i_{v-1} \vee i_v| \le N\,,$$

with the convention $i_0 := i_{2m}$. It needs to be shown that if $C_{N,n}$ denotes the set of tuples in $\{1,\ldots,n\}^{2m}$ which are not N-pair matched, then

$$\lim_{N\to\infty}\limsup_{n\to\infty} n^{-(m+1)}\left|\sum_{i\in C_{N,n}}^{n} E\left[X_{i_1,i_2}\ldots X_{i_{2m-1},i_{2m}}X_{i_{2m},i_1}\right]\right| = 0\,. \tag{2.11}$$

Unlike in the classical Wigner's result, this is a non-trivial step in our situation because not only does the above sum not vanish for N large, even showing that the expectation in modulus is less than some ε is not enough because $\#C_{N,n} \sim n^{2m}$ as $n\to\infty$, and the sum is scaled only by $n^{(m+1)}$. This is precisely the step where Assumption 2 plays an important role.

Once (2.11) is established, what remains to be shown for (2.10) is that for fixed N,

$$\begin{aligned}\lim_{n\to\infty} n^{-(m+1)}\sum_{i\in C^c_{N,n}}^{n} E\left[X_{i_1,i_2}\ldots X_{i_{2m-1},i_{2m}}X_{i_{2m},i_1}\right]\\ = \sum_{\sigma\in NC_2(2m)}\ \sum_{k\in S(\sigma):\max_j|k_j|\le N}\ \prod_{(u,v)\in\sigma} R(k_u,k_v)\,.\end{aligned} \tag{2.12}$$

By standard combinatorial arguments, the sum over $C^c_{N,n}$ can be shown to be asymptotically equivalent to the sum over all tuples that are Catalan with respect to some $\sigma\in NC_2(2m)$, that is, whenever $(j,k)\in\sigma$,

$$|i_{j-1}-i_k|\vee|i_j-i_{k-1}|\le N\,.$$

The final step is to show that for fixed $\sigma\in NC_2(2m)$, if D_σ denotes the set of tuples in $\{1,\ldots,n\}^{2m}$ which are Catalan with respect to σ, then

$$\begin{aligned}\lim_{m\to\infty}\sum_{i\in D_\sigma} E\left[X_{i_1,i_2}\ldots X_{i_{2m-1},i_{2m}}X_{i_{2m},i_1}\right]\\ = \sum_{k\in S(\sigma):\max_j|k_j|\le N}\ \prod_{(u,v)\in\sigma} R(k_u,k_v)\,.\end{aligned}$$

This follows by computing the expectation via Wick's formula, and observing that in that formula, the contribution of all the pair partitions excluding σ is asymptotically negligible. This final step establishes (2.12).

3. Some combinatorics

In this section, we recall some elementary combinatorial notions, and prove a few results related to them. The results of this section are not of independent interest, but will be used in Section 4.

There is an infinite totally ordered set called the "alphabet" whose elements are called "letters". The order with which the alphabet is endowed is the "alphabetical ordering". A "word" is an ordered finite collection of not necessarily distinct letters. While the actual description of the alphabet is irrelevant, to fix ideas, we shall consider the set of natural numbers with the natural ordering to be the alphabet. Two words are "distinct" if one cannot be obtained from the other by relabeling letters. For example, the words 1112 and 2224 are not distinct, and the words 4612 and 2181 are distinct. If the lengths of two words are different, then they are necessarily distinct.

Let $P(2m)$ and $NC_2(2m)$ denote the set of pair partitions and non-crossing pair partitions of $\{1,\ldots,2m\}$ respectively. Clearly, $NC_2(2m)$ is a proper subset of $P(2m)$ for $m \geq 2$. For example, $\{(1,3),(2,4),(5,6)\} \in P(6) \setminus NC_2(6)$ and $\{(1,4),(2,3),(5,6)\} \in NC_2(6)$. Recall that for $\sigma \in NC_2(2m)$, the Kreweras complement $K(\sigma)$ is the maximal partition $\overline{\sigma}$ of $\{\overline{1},\ldots,\overline{2m}\}$ such that $\sigma \cup \overline{\sigma}$ is a non-crossing partition of $\{1,\overline{1},\ldots,2m,\overline{2m}\}$. For example,

$$K\left(\{(1,4),(2,3),(5,6)\}\right) = \{(\overline{1},\overline{3}),(\overline{2}),(\overline{4},\overline{6}),(\overline{5})\}\,.$$

For $m \geq 1$, π is a "pairing" of $\{1,\ldots,2m\}$ if it is a permutation of that set satisfying

$$\pi(j) \neq j = \pi(\pi(j)),\ 1 \leq j \leq 2m\,.$$

Call π to be an "almost pairing" of $\{1,\ldots,2m\}$ if it is a permutation satisfying

$$\#\{1 \leq j \leq 2m : \pi(j) = j\} = 2\,,$$

and

$$\pi(\pi(j)) = j,\ 1 \leq j \leq 2m\,.$$

There is a clear bijection between the set of pairings and $P(2m)$, namely for any pairing π, $\{(j,\pi(j)) : 1 \leq j \leq 2m\} \in P(2m)$. Keeping this bijection in mind, we shall use the words pairing and pair partition interchangeably.

Recall that a word $A := a_1 \ldots a_{2m}$ is "pair matched" if there exists a pairing π of $\{1,\ldots,2m\}$ such that

$$a_j = a_{\pi(j)},\ 1 \leq j \leq 2m\,.$$

If there exists an almost pairing π satisfying the above, then A is "almost pair matched". Examples of pair matched words are 1212, 122221 etc., and that of almost pair matched words are 1231, 2111, 1221 etc. An example of a non-pair matched word is 111222. Notice that while the set of pairings and almost pairings are disjoint, a pair matched word is necessarily almost pair matched. Another useful observation is that a word is almost pair matched if and only if it be can made pair matched by changing at most one letter.

Recall that a pair matched word is a "Catalan word" if successive deletions of double letters lead to the empty word. Examples of Catalan words are 1221, 123321, 122122 etc., while 1212 is an example of a pair-matched word which is not a Catalan word.

The conventions that we now discuss will be assumed throughout the article. Any tuple $i \in \mathbb{Z}^k$ is taken to be of the form $i := (i_1,\ldots,i_k)$, and furthermore, implicitly defines $i_0 := i_k$. The same convention also applies to words, that is, for a word $A = a_1 \ldots a_k$, $a_0 := a_k$. The next convention is that, for alphabets or integers a,b,c,d, we say

$$(a,b) \approx (c,d) \tag{3.1}$$

if $a \wedge b = c \wedge d$ and $a \vee b = c \vee d$. For integers "$\wedge$" and "$\vee$" have the usual interpretation of minimum and maximum respectively, whereas for letters u and v, $u \wedge v$ and $u \vee v$ mean the one that comes first in the alphabetical ordering, and the one that comes later, respectively.

Given words $A := a_1 \ldots a_m$ and $B := b_1 \ldots b_m$ of the same length, say that B is an "offspring" of A if the following is true. Whenever $a_j = a_k$ for some $1 \leq j, k \leq m$, it holds that

$$(b_{j-1}, b_j) \approx (b_{k-1}, b_k) \ .$$

For example, 1213 is an offspring of both 1221 and 5789, and both 7565 and 1111 are offsprings of 2211.

The next notion we need is that of a "compound offspring word". A word $B = b_1 \ldots b_{2m}$ is a compound offspring of $A = a_1 \ldots a_{2m}$ if $b_1 \ldots b_m$ and $b_{m+1} \ldots b_{2m}$ are offsprings of $a_1 \ldots a_m$ and $a_{m+1} \ldots a_{2m}$ respectively, and furthermore, whenever $a_j = a_k$, it holds that

$$\left(b_j, b_{\gamma(j)}\right) \approx \left(b_k, b_{\gamma(k)}\right) \ ,$$

where

$$\gamma(j) := \begin{cases} j-1, & j \in \{1, \ldots, 2m\} \setminus \{1, m+1\}\,, \\ m, & j = 1\,, \\ 2m, & j = m+1\,. \end{cases}$$

For a word A, we denote by $\#A$ the number of distinct letters in A (and **not the length** of A).

The following result is well known in the literature, but in different settings. One can look at, for example, equation (34) in the proof of Theorem 4 in Bose and Sen (2008) where the same claim has been restated in a slightly different language. Hence we omit the proof.

Lemma 3.1. *(a) Let A be a pair matched word of length $2m$ for some $m \geq 1$. Then A has an offspring word B with*

$$\#B = m + 1 \tag{3.2}$$

if and only if A is a Catalan word with

$$\#A = m\,.$$

In this case, the offspring word is unique upto relabeling of letters.
(b) Assume that A_1 and A_2 are distinct Catalan words of length $2m$, that is, one cannot be obtained from the other by relabeling letters. If B_1 and B_2 are offsprings of A_1 and A_2 respectively such that

$$\#B_1 = \#B_2 = m + 1\,,$$

then B_1 and B_2 are distinct.
(c) Furthermore, if $B = b_1 \ldots b_{2m}$ is an offspring of $A = a_1 \ldots a_{2m}$ satisfying

(3.2), *then it is necessary that whenever* $a_j = a_k$ *for some* $j < k$, *it holds that*

$$b_{j-1} = b_k\,, \text{ and} \tag{3.3}$$

$$b_j = b_{k-1}\,. \tag{3.4}$$

As mentioned at the beginning of the section, the remaining results will be used for later results in this section or the ones in Section 4.

Lemma 3.2. *Let* $B = b_1 \ldots b_{2k}$ *be a compound offspring word of* $A = a_1 \ldots a_{2k}$ *for some* $k \geq 2$. *Define*

$$n_j := \begin{cases} \#(b_k b_1 \ldots b_j) - \#(a_1 \ldots a_j), & 1 \leq j \leq k-1\,, \\ \#(b_{2k} b_1 \ldots b_j) - \#(a_1 \ldots a_j), & k+1 \leq j \leq 2k-1\,. \end{cases}$$

Then,

$$1 \geq n_1 \geq \ldots \geq n_{k-1} \geq n_{k+1} - 1 \geq \ldots \geq n_{2k-1} - 1\,. \tag{3.5}$$

Consequently, if D *is an offspring word of* $C := c_1 \ldots c_k$ *for some* $k \geq 2$, *then*

$$\#D \leq 1 + \#(c_1 \ldots c_{k-1})\,. \tag{3.6}$$

Proof. It is easy to see that the inequality $n_{k-1} \leq 1$ in (3.5) implies (3.6). So the former claim is the one that needs a proof. The leftmost inequality in (3.5) is trivial. For the subsequent inequalities, fix $1 \leq j \leq k-2$, and we shall show that

$$n_{j+1} \leq n_j\,. \tag{3.7}$$

The proof will be separate for the two cases:

$$\textbf{(Case 1)}\ a_{j+1} \neq a_i \text{ for all } 1 \leq i \leq j\,,$$

and

$$\textbf{(Case 2)}\ a_{j+1} = a_i \text{ for some } 1 \leq i \leq j\,.$$

Observe that in Case 1,

$$\begin{aligned} n_{j+1} &= \#(b_k b_1 \ldots b_{j+1}) - \#(a_1 \ldots a_{j+1}) \\ &\leq 1 + \#(b_k b_1 \ldots b_j) - \#(a_1 \ldots a_{j+1}) \\ &= \#(b_k b_1 \ldots b_j) - \#(a_1 \ldots a_j) \\ &= n_j\,. \end{aligned}$$

In Case 2, if $i \geq 2$, then b_{j+1} equals b_i or b_{i-1}, and if $i = 1$, then b_{j+1} equals b_i or b_k. Hence

$$\begin{aligned} \#(b_k b_1 \ldots b_{j+1}) &= \#(b_k b_1 \ldots b_j)\,, \\ \text{and } \#(a_1 \ldots a_{j+1}) &= \#(a_1 \ldots a_j)\,, \end{aligned}$$

which shows that

$$n_{j+1} = n_j\,.$$

This establishes (3.7) for $1 \le j \le k-2$. Similar arguments establish the same claim for $k+1 \le j \le 2k-2$, and that

$$n_{k+1} \le 1 + n_{k-1}\,.$$

This completes the proof of (3.5), and thereby establishes the lemma. □

Lemma 3.3. *Suppose that A is an almost pair matched word of length $2m$ with $m \ge 2$, and B is an offspring of A. Then,*

$$\#B \le m+1\,.$$

Proof. Since A is almost pair matched, clearly $\#A \le m+1$. If $\#A \le m$, then by (3.6), it follows that

$$\#B \le 1 + \#A \le 1 + m\,.$$

So, without loss of generality, let us assume that $\#A = m+1$. We start with the observation that if $b_1 \ldots b_{2m}$ is an offspring of $a_1 \ldots a_{2m}$, then $b_2 \ldots b_{2m} b_1$ is an offspring of $a_2 \ldots a_{2m} a_1$. Therefore, once again without loss of generality, we can and do assume that

$$A = W_1 c W_2 d\,,$$

where W_1 and W_2 are possibly empty words such that $W_1 W_2$ is pair matched, and c, d are distinct letters which do not occur in $W_1 W_2$. Therefore, by (3.6),

$$\begin{aligned} \#B &\le 1 + \#(W_1 c W_2) \\ &= 1 + m\,. \end{aligned}$$

This completes the proof. □

Lemma 3.4. *Suppose that A is an almost pair matched word of length $2m$ with $m \ge 2$, and B is a compound offspring of A. Then,*

$$\#B \le m+2\,.$$

Proof. Denote $A = a_1 \ldots a_{2m}$. By the fact that the rightmost quantity in (3.5) is at most 1, it follows that

$$\#B \le 2 + \#(a_1 \ldots a_{2m-1})\,. \tag{3.8}$$

Thus, the claim of the lemma follows if $\#A \le m$. So assume without loss of generality that

$$\#A = m+1\,.$$

It is easy to see that since $b_1 \ldots b_{2m}$ is a compound offspring of $a_1 \ldots a_{2m}$, so are $b_{m+1} \ldots b_{2m} b_1 \ldots b_m$ and $b_1 \ldots b_m b_{m+2} \ldots b_{2m} b_{m+1}$ of $a_{m+1} \ldots a_{2m} a_1 \ldots a_m$ and $a_1 \ldots a_m a_{m+2} \ldots a_{2m} a_{m+1}$ respectively. Therefore, as in the proof of Lemma 3.3, we assume without loss of generality that

$$a_{2m} \ne a_j,\ j = 1, \ldots, 2m-1\,. \tag{3.9}$$

Clearly, (3.8) and the observation that under this assumption

$$\#(a_1 \ldots a_{2m-1}) = m\,,$$

completes the proof. ☐

Lemma 3.5. *Suppose that A is a pair matched word of length $4m$ for some $m \geq 1$. Then, A has a compound offspring word B with*

$$\#B = 2m + 2 \tag{3.10}$$

if and only if

$$A = A_1 A_2 \tag{3.11}$$

where A_1 and A_2 are Catalan words of length $2m$ with no common letters, and

$$\#A_1 = \#A_2 = m \,. \tag{3.12}$$

In this case,

$$B = B_1 B_2 \,,$$

where B_1 and B_2 are offspring words of A_1 and A_2 respectively, do not have a common letter, and satisfy

$$\#B_1 = \#B_2 = m + 1 \,. \tag{3.13}$$

Furthermore, the compound offspring word B satisfying (3.10) is unique up to relabeling, and if B and B' are compound offspring words of distinct pair matched words of length $4m$ satisfying

$$\#B = \#B' = 2m + 2 \,,$$

then B and B' are distinct.

Proof. Assume for a moment that the "if and only if" claim has been shown. Let B be a compound offspring of A such that (3.10) holds. By definition of a compound offspring word, we can write

$$B = B_1 B_2$$

where B_1 and B_2 are offspring words of A_1 and A_2 respectively. Now notice that by (3.12) and Lemma 3.4,

$$\#B_i \leq m + 1, \; i = 1, 2 \,.$$

Thus,

$$\begin{aligned} 2m + 2 &= \#B \\ &\leq \#B_1 + \#B_2 \\ &\leq 2m + 2 \,. \end{aligned}$$

Therefore, B_1 and B_2 cannot have a common letter, because otherwise, the inequality in the second line becomes strict. Also, the inequality in the last line must be an equality, proving (3.13). The final claim follows from Lemma 3.1 (b).

So the "if and only if" claim is the only part that needs a proof. Once again, the "if" part follows trivially from Lemma 3.1 (a). Let us proceed towards the "only if" part. So assume that A has a compound offspring word B such that (3.10) holds. Let $A := a_1 \dots a_{4m}$ and $B = b_1 \dots b_{4m}$. Set

$$\begin{aligned} A_1 &:= a_1 \dots a_{2m}\,, \\ A_2 &:= a_{2m+1} \dots a_{4m}\,. \end{aligned}$$

Thus, (3.11) trivially holds. We start with showing that A_1 and A_2 do not have a common letter. Assume for the sake of contradiction that they have a common letter. The arguments that justify the assumption (3.9) in the proof of Lemma 3.4 show that in this case, a_{2m+1} can be chosen to be that letter without loss of generality, that is,

$$a_{2m+1} = a_j \text{ for some } 1 \le j \le 2m\,. \tag{3.14}$$

Therefore,

$$(b_{4m}, b_{2m+1}) \approx (b_j, b_{\gamma(j)})\,, \tag{3.15}$$

where "$\approx$" and $\gamma(\cdot)$ are as in (3.1) and the definition of a compound offspring word respectively. Let n_j for $j = 1, \dots, 2m-1, 2m+1, \dots, 4m-1$, be as in the statement of Lemma 3.2 with $k = 2m$. By that result, it follows that

$$1 \ge n_1 \ge \dots \ge n_{2m-1}\,,$$

and

$$n_{2m+1} \ge \dots \ge n_{4m-1}\,.$$

Thus,

$$\begin{aligned} n_{4m-1} &\le n_{2m+1} \\ &= \#(b_{4m} b_1 \dots b_{2m+1}) - \#(a_1 \dots a_{2m+1}) \\ &= \#(b_1 \dots b_{2m}) - \#(a_1 \dots a_{2m+1}) \\ &\le \#(b_1 \dots b_{2m}) - \#(a_1 \dots a_{2m-1}) \\ &= n_{2m-1} \\ &\le 1\,, \end{aligned}$$

the equality in the third line following by (3.15). However, notice that

$$n_{4m-1} = \#B - \#(a_1 \dots a_{4m-1}) = \#B - \#A \ge 2\,, \tag{3.16}$$

the second equality following from the fact that A is pair matched, and the inequality following from (3.10). This clearly is a contradiction, thus showing that A_1 and A_2 have no common letters. An immediate consequence is that A_1 and A_2 are pair matched words.

Next, we proceed towards showing (3.12). Lemma 3.2 implies that

$$1 \ge n_{2m-1} \ge n_{2m+1} - 1 \ge n_{4m-1} - 1 \ge 1\,, \tag{3.17}$$

the rightmost one following from (3.16) which is clearly valid regardless of the assumption (3.14). Thus,

$$\#A = \#B - n_{4m-1} = 2m\,.$$

Hence each letter in A comes exactly twice, and so (3.12) holds. Another consequence of (3.17) is that

$$n_{2m-1} = 1\,,$$

a restatement of which in view of the fact that A_1 is pair matched is

$$\#(b_1 \dots b_{2m}) = 1 + \#A_1 = m+1\,.$$

Since $b_1 \dots b_{2m}$ is an offspring word of A_1, by Lemma 3.1, it follows that A_1 is a Catalan word. A similar argument holds for A_2, and completes the proof. □

Lemma 3.6. *Let $A := A_1 \dots A_{2m}$ be an almost pair matched word of length $2m$ where $m \geq 2$. Assume that A has $m+1$ distinct letters $a_1, \dots, a_{m+1}$ with each of $a_1, \dots, a_{m-1}$ occurring twice. Fix*

$$u := (u_1, \dots, u_{m-1}), v := (v_1, \dots, v_{m-1}) \in \mathbb{Z}^{m-1}\,.$$

Given a $2m$-tuple $(i_1, \dots, i_{2m})$ in $\mathbb{N}^{2m}$, say that it is uv-matched if the following is true:

$$\textit{whenever } A_j = A_k = a_l \textit{ for some } 1 \leq j < k \leq 2m \textit{ and } 1 \leq l \leq m-1\,,$$

it holds that

$$(i_{j-1} \wedge i_j) - (i_{k-1} \wedge i_k) = u_l\,,$$

and

$$(i_{j-1} \vee i_j) - (i_{k-1} \vee i_k) = v_l\,,$$

where $i_0 := i_{2m}$, as usual. Let U_n denote the set of uv-matched tuples in $\{1, \dots, n\}^{2m}$ for $n \geq 1$. Then,

$$\#U_n \leq 4^m n^{m+1} \textit{ for all } n \geq 1\,.$$

Proof. Let Π denote the set of all functions from $\{1, 2, \dots, 2m\}$ to $\{0, 1\}$. Clearly, if $(i_1, \dots, i_{2m}) \in U_n$, then the following is true. There exists $\pi \in \Pi$ such that whenever $A_j = A_k = a_l$ for some $1 \leq j < k \leq 2m$ and $1 \leq l \leq m-1$, it holds that

$$i_{j-\pi(j)} - i_{k-\pi(k)} = u_l\,, \tag{3.18}$$

and

$$i_{j-(1-\pi(j))} - i_{k-(1-\pi(k))} = v_l\,. \tag{3.19}$$

It is easy to see that the number of $(i_1, \dots, i_{2m})$ in $\{1, \dots, n\}^{2m}$ satisfying (3.18) and (3.19) is at most the number of those satisfying the same equations with u_l and v_l replaced by 0 for all l.

Fix $\pi \in \Pi$. We shall now show that the number of $i := (i_1, \dots, i_{2m}) \in \{1, \dots, n\}^{2m}$ satisfying (3.18) and (3.19) with u_l and v_l replaced by 0 for all l is at most n^{m+1}. Clearly, for any such i, the word $B = i_1 i_2 \dots i_{2m}$ is an offspring word of A. Since B can have at most $m+1$ distinct letters by Lemma 3.3, it follows that the number of such i's is at most n^{m+1}. Thus,

$$\#U_n \leq n^{m+1} \#\Pi = 4^m n^{m+1}\,,$$

which completes the proof. □

Define a binary operation $\star$ on $\mathbb{Z}^2$, that is, a function from $\mathbb{Z}^2 \times \mathbb{Z}^2$ to $\mathbb{Z}^2$ as follows:

$$(i,j) \star (k,l) := (i \wedge j - k \wedge l, k \vee l - i \vee j),\ i,j,k,l \in \mathbb{Z}\,. \tag{3.20}$$

Fix

$$u := (u_1, \ldots, u_{m-1}), v := (v_1, \ldots, v_{m-1}) \in \mathbb{Z}^{m-1}\,.$$

For $n \geq 1$, let $V_n(u,v)$ denote the set of all tuples i in $\{1,\ldots,n\}^{2m}$ for which there exists an almost pairing π of $\{1,\ldots,2m\}$ and an onto function ϕ from $W := \{1 \leq j \leq 2m : \pi(j) \neq j\}$ to $\{1,\ldots,m-1\}$ such that for all $j \in W$,

$$\phi(j) = \phi(\pi(j))\,, \tag{3.21}$$

and

$$(i_{j-1}, i_j) \star (i_{\pi(j)-1}, i_{\pi(j)}) = (u_{\phi(j)}, v_{\phi(j)})\,. \tag{3.22}$$

Lemma 3.7. *There exists a finite constant $C(m)$ depending only on m such that*

$$\# V_n(u,v) \leq C(m) n^{m+1},\ n \geq 1\,. \tag{3.23}$$

Proof. Since there are only finitely many almost pair matched words of length $2m$ and each of them has finitely many offspring words, there are only finitely many almost pairings π, and given any π the number of functions ϕ satisfying (3.21) and (3.22) is also finite, the proof follows from the conclusion of Lemma 3.6. □

4. Proof of the main result

For the proof of Theorem 2.1, we shall need a few notations. Fix $N \in \mathbb{N}$. Call a $2m$-tuple $i := (i_1, \ldots, i_{2m}) \in \mathbb{N}^{2m}$ "N-Catalan corresponding to σ" if there exists $\sigma \in NC_2(2m)$ such that whenever $(j,k) \in \sigma$,

$$|i_{j-1} - i_k| \vee |i_j - i_{k-1}| \leq N\,. \tag{4.1}$$

For $i,j,k,l \geq 1$, say that

$$(i,j) \sim (k,l) \tag{4.2}$$

if

$$|(i \wedge j) - (k \wedge l)| \vee |(i \vee j) - (k \vee l)| \leq N\,.$$

Say that a $(2m)$-tuple $(i_1, \ldots, i_{2m})$ is "N-pair matched" if there exists a pairing π of $\{1,\ldots,2m\}$ such that

$$(i_{j-1}, i_j) \sim (i_{\pi(j-1)}, i_{\pi(j-1)+1})\,,\ \text{for } j = 1, \ldots, 2m\,,$$

where $\pi(0) := \pi(2m)$. We shall suppress the "N" in N-Catalan and N-pair matched if the N of interest is clear from the context.

Lemma 4.1. *Fix $N, m, n \geq 1$ and $\sigma \in NC_2(2m)$. Let $V_1, \ldots, V_{m+1}$ denote the blocks of the Kreweras complement of σ. Write*

$$V_u = \{v_1^u, \ldots, v_{l_u}^u\},\ u = 1, \ldots, m+1\,,$$

where

$$v_1^u \leq \ldots \leq v_{l_u}^u\,. \tag{4.3}$$

Then $i := (i_1, \ldots, i_{2m}) \in \{1, \ldots, n\}^{2m}$ is a N-Catalan tuple corresponding to σ if and only if there exists $k := (k_1, \ldots, k_{2m}) \in S(\sigma, N)$ where

$$S(\sigma, N) := \left\{ k \in \{-N, \ldots, N\}^{2m} : \sum_{j=1}^{l_s} k_{v_j^s} = 0,\ s = 1, \ldots, k+1 \right\}, \tag{4.4}$$

and a 0-Catalan tuple $j := (j_1, \ldots, j_{2m}) \in \{1, \ldots, n\}^{2m}$ such that

$$i_{v_x^u} = j_{v_x^u} + \sum_{w=1}^{x} k_{v_w^u},\ x = 1, \ldots, l_u,\ u = 1, \ldots, m+1\,. \tag{4.5}$$

Furthermore, the j and k satisfying (4.5) are unique.

Proof. It is clear that $i := (i_1, \ldots, i_{2m}) \in \mathbb{N}^{2m}$ is a 0-Catalan tuple corresponding to σ if and only if

$$i_{v_1^u} = \ldots = i_{v_{l_u}^u},\ u = 1, \ldots, m+1\,.$$

In view of the ordering (4.3), similar reasoning as that leading to the above equivalence will yield that i is a N-Catalan tuple corresponding to σ if and only if

$$\left|i_{v_1^u} - i_{v_2^u}\right| \vee \left|i_{v_2^u} - i_{v_3^u}\right| \vee \ldots \vee \left|i_{v_{l_u}^u} - i_{v_1^u}\right| \leq N,\ u = 1, \ldots, m+1\,. \tag{4.6}$$

Now, suppose that i is a N-Catalan tuple corresponding to σ. Define

$$k_{v_w^u} := i_{v_{w+1}^u} - i_{v_w^u},\ w = 1, \ldots, l_u,\ u = 1, \ldots, m+1\,,$$

where $v_{l_u+1}^u := v_1^u$ for $u = 1, \ldots, m+1$. It is easy to see because of (4.6) that $k := (k_1, \ldots, k_{2m})$ thus defined, belongs to $S(\sigma, N)$. Define

$$j_{v_w^u} := i_{v_{l_u}^u},\ w = 1, \ldots, l_u,\ u = 1, \ldots, m+1\,.$$

Then, clearly (4.5) holds, and from the equivalence mentioned at the beginning of this proof, it is easy to see that $j := (j_1, \ldots, j_{2m})$ is a 0-Catalan word corresponding to σ. This completes the proof of the "only if" part.

For the "if" part, let j and k be given, and define i by (4.5). Then, (4.6) is immediate, and by the equivalence mentioned just above that equation, it follows that i is a N-Catalan tuple.

Finally, for the uniqueness, assume that i is given. If j and k satisfy (4.5), then from the fact that

$$\sum_{w=1}^{l_u} k_{v_w^u} = 0\,,$$

it follows that

$$j_{v^u_{l_u}} = i_{v^u_{l_u}}\,,$$

This, along with the fact that j is 0-Catalan, specifies j. Then, (4.5) determines k. This completes the proof. $\square$

Lemma 4.2. *Let $\sigma \in NC_2(2m)$, j be a 0-Catalan tuple corresponding to σ, $k \in S(\sigma, N)$ and i be given by (4.5). Assume furthermore that*

$$\#\{\textit{distinct numbers in } (j_1, \ldots, j_{2m})\} = m + 1\,, \tag{4.7}$$

and that

$$\min\{|j_u - j_v| : 1 \le u, v \le 2m,\ j_u \ne j_v\} > 4mN\,. \tag{4.8}$$

Then, for all $(u, v) \in \sigma$,

$$E\left(X_{i_{u-1},i_u} X_{i_{v-1},i_v}\right) = R(k_u, k_v)\,.$$

Proof. We start by showing that

$$i_u - i_{v-1} \;=\; k_u\,, \tag{4.9}$$

$$\text{and } i_v - i_{u-1} \;=\; k_v\,. \tag{4.10}$$

Assume without loss of generality that $u < v$. Therefore, $\overline{u}$ and $\overline{v-1}$ belong to the same block in $K(\sigma)$, and furthermore, the block containing them is a subset of $\{\overline{u}, \overline{u+1}, \ldots, \overline{v-1}\}$. Thus, (4.9) follows from (4.5). We show (4.10) separately for the cases $u \ge 2$ and $u = 1$. If $u \ge 2$, then $\overline{u-1}$ and $\overline{v}$ are in the same block of $K(\sigma)$, and furthermore that block does not intersect with $\{\overline{u}, \overline{u+1}, \ldots, \overline{v-1}\}$. This shows (4.10), once again with the help of (4.5). If $u = 1$, then $\overline{2m}$ and $\overline{v}$ are in the same block of $K(\sigma)$. Obviously, $\overline{2m}$ has to be the last member of its block. Since $(1, v) \in \sigma$, it follows that $\overline{v}$ is the first member of the block containing itself and $\overline{2m}$, showing that

$$i_v = i_{2m} + k_v = i_0 + k_v = i_{u-1} + k_v\,.$$

This complete the proof of (4.10)

Our next aim is to show that either

$$i_u \vee i_{v-1} < i_{u-1} \wedge i_v\,, \tag{4.11}$$

or

$$i_u \wedge i_{v-1} > i_{u-1} \vee i_v\,, \tag{4.12}$$

holds. Since $\overline{v-1}$ and $\overline{v}$ belong to distinct blocks of $K(\sigma)$, (4.7) implies that

$$j_v \ne j_{v-1}\,.$$

This, in conjunction with (4.8) establishes that

$$|j_v - j_{v-1}| > 4mN\,.$$

In view of (4.5), it follows that

$$|i_v - j_v| \vee |i_{u-1} - j_v| \vee |i_{v-1} - j_{v-1}| \vee |i_u - j_{v-1}| \le 2mN\,.$$

If $j_v > j_{v-1}$, then in view of the above two inequalities, it is easy to see that

$$i_{u-1} \wedge i_v \geq j_v - 2mN > j_{v-1} + 2mN \geq i_{v-1} \vee i_u \,,$$

showing that (4.11) holds. Similarly, if $j_v < j_{v-1}$, then (4.12) holds.

Finally to see the claim of the lemma, assume that (4.12) holds. Then, by stationarity, (4.9) and (4.10), it follows that

$$\begin{aligned} E\left(X_{i_{u-1},i_u} X_{i_{v-1},i_v}\right) &= R(-k_v, -k_u) \\ &= R(k_u, k_v)\,, \end{aligned}$$

the second equality following from (2.4). It is easy to see that when (4.11) holds, then the claim also holds. This completes the proof. ☐

Lemma 4.3. *Fix $N \geq 0$, $m \geq 1$. In what follows, "pair matched" and "Catalan" mean "N-pair matched" and "N-Catalan" respectively.*
(a) If $\pi \in P(2m) \setminus NC_2(2m)$, then

$$\lim_{n\to\infty} n^{-(m+1)} \#\{\textit{pair matched tuples in } \{1,\ldots,n\}^{2m} \textit{ corresponding to } \pi\}$$
$$= 0\,.$$

(b) If $\sigma \in NC_2(2m)$, then

$$\lim_{n\to\infty} n^{-(m+1)} \#\Big[\{\textit{pair matched tuples in } \{1,\ldots,n\}^{2m} \textit{ corresponding to } \sigma\}$$

$$\setminus \{\textit{Catalan tuples in } \{1,\ldots,n\}^{2m} \textit{ corresponding to } \sigma\}\Big] = 0\,. \tag{4.13}$$

(c) If $\sigma \in NC_2(2m)$ and $\pi \in P(2m) \setminus \{\sigma\}$, then

$$\lim_{n\to\infty} n^{-(m+1)} \#\Big[\{\textit{Catalan tuples in } \{1,\ldots,n\}^{2m} \textit{ corresponding to } \sigma\}$$

$$\cap \{\textit{pair matched tuples in } \{1,\ldots,n\}^{2m} \textit{ corresponding to } \pi\}\Big] = 0\,.$$

Proof. We shall prove the claims only for $N = 0$. The generalization to the case when $N \neq 0$ is trivial, and follows, for example, by arguments similar to the ones that allow replacement of u_l and v_l by zero in (3.18) and (3.19) respectively.

Proof of (a). Fix $\pi \in P(2m) \setminus NC_2(2m)$. Let $A = a_1 \ldots a_{2m}$ be a word of length $2m$ such that for $j < k$, $a_j = a_k$ if and only if $(j, k) \in \pi$. That is, A is a pair matched word. It is easy to see that any 0-pair matched tuple is actually an offspring word (considering the entries to be letters) of A. Since A is not a Catalan word, by Lemma 3.1 (a), it follows that for all offspring word B of A,

$$\#B \leq m\,. \tag{4.14}$$

This completes the proof of (a). ☐

Proof of (b). Let $A = a_1 \dots a_{2m}$ be a word of length $2m$ such that for $j < k$, $a_j = a_k$ if and only if $(j,k) \in \sigma$. It is easy to see that a 0-pair matched tuple which is not a 0-Catalan tuple generates an offspring word $B = b_1 \dots b_{2m}$ such that at least one of (3.3) or (3.4) is violated for some $(j,k) \in \sigma$. Therefore, by Lemma 3.1 (c), (4.14) follows, thus proving (b). ☐

Proof of (c). Once again, we prove this for $N = 0$. If $\pi \notin NC_2(2m)$, then the claim follows by part (a) which has already been proved. So assume that $\pi \in NC_2(2m)$. Let A_1 and A_2 be Catalan words of length $2m$ corresponding to σ and π respectively, as in the proof of (a). Since $\pi \neq \sigma$, A_1 and A_2 are distinct, that is neither of them can be obtained from the other by relabeling letters. Lemma 3.1 (b) implies that if B is an offspring of both A_1 and A_2, then (4.14) holds, thereby establishing (c). ☐

Since all the claims have been established, this completes the proof of the lemma. ☐

Remark 2. *The number on the left hand side of* (4.13) *is not necessarily zero. For example,* $(1,3,2,1,3,2)$ *is* 1*-pair matched but not* 1*-Catalan corresponding to* $\{(1,6),(2,5),(3,4)\}$.

Proof of Theorem 2.1. Our first task is to show the existence of a probability measure μ whose odd moments are zero and the $2m$-th moment is β_{2m} for all $m \geq 1$. To that end, define the expected ESD $\hat{\mu}_n$ of $A_n/\sqrt{n}$ as

$$\hat{\mu}_n(B) := \frac{1}{n}\sum_{j=1}^{n} P(\lambda_j/\sqrt{n} \in B),\ n \geq 1\,,$$

for all Borel sets B. Clearly,

$$\int x^m \hat{\mu}_n(dx) = n^{-(m/2+1)} E\left[\mathrm{Tr}(A_n^m)\right],\ m,n \geq 1\,,$$

which is zero if m is odd. If it can be shown that for $m \geq 1$,

$$\lim_{n\to\infty} n^{-(m+1)} E\left[\mathrm{Tr}(A_n^{2m})\right] = \beta_{2m}\,, \tag{4.15}$$

then existence of μ will follow. In addition, the above is also a significant step in proving that μ_n converges in probability to μ. We shall come to that a moment later. Before that let us quickly dispose off the issue of uniqueness of μ. It will be shown in Section 6 that β_{2m} are the moments of a compactly supported probability measure which automatically ensures uniqueness; see Remark 3. However, for the sake of completeness, we provide a quick proof of uniqueness via Carleman's condition. In view of Carleman's condition, it suffices to show that

$$\sum_{m=1}^{\infty} \beta_{2m}^{-1/2m} = \infty\,. \tag{4.16}$$

By (2.7), it follows that

$$\begin{aligned}\sum_{m=1}^{\infty}\beta_{2m}^{-1/2m} &\geq \bar{R}^{-1/2}\sum_{m=1}^{\infty}\left(m!\ \#NC_2(2m)\right)^{-1/2m}\\ &\geq \frac{1}{2}\bar{R}^{-1/2}\sum_{m=1}^{\infty}(m!)^{-1/2m}\\ &\geq \frac{1}{2}\bar{R}^{-1/2}\sum_{m=1}^{\infty}m^{-1}\,,\end{aligned}$$

the inequality in the last line following from the fact that $m! \leq m^{2m}$ for all $m \geq 1$. This establishes (4.16). Consequently, there is at most one measure μ whose odd moments vanish and the $2m$-th moment is β_{2m}. Thus, to complete the proof, we need to show (4.15) and that

$$\lim_{n\to\infty} n^{-2(m+1)}\mathrm{Var}\left[\mathrm{Tr}(A_n^{2m})\right] = 0\,. \tag{4.17}$$

We now proceed towards showing (4.15). Recall that

$$\mathrm{Tr}(A_n^{2m}) = \sum_{i_1,\ldots,i_{2m}=1}^{n} X_{i_1,i_2}\ldots X_{i_{2m-1},i_{2m}}X_{i_{2m},i_1}\,,$$

and therefore

$$E\left[\mathrm{Tr}(A_n^{2m})\right] = \sum_{i\in\{1,\ldots,n\}^{2m}} E_i\,,$$

where

$$E_i := E\left[\prod_{j=1}^{2m} X_{i_{j-1},i_j}\right],\ i\in\mathbb{Z}^{2m}\,,$$

the convention being that for $i = (i_1,\ldots,i_{2m}) \in \mathbb{Z}^{2m}$, $i_0 := i_{2m}$. Recall the definition of $S(\sigma, N)$ from (4.4) for $\sigma \in NC_2(2m)$. Set

$$\beta_{2m}^{(N)} := \sum_{\sigma\in NC_2(2m)}\sum_{k\in S(\sigma,N)}\prod_{(u,v)\in\sigma} R(k_u,k_v),\ m,N\geq 1\,.$$

In view of (2.7), it follows that

$$\lim_{N\to\infty}\beta_{2m}^{(N)} = \beta_{2m}\,.$$

Fix $\varepsilon > 0$. Let N be such that

$$|R(u,v)| \leq \varepsilon$$

whenever $u \wedge v \geq N$, and

$$\left|\beta_{2m}^{(N)} - \beta_{2m}\right| \leq \varepsilon\,. \tag{4.18}$$

Recall the definitions of "$\sim$", N-pair matched and N-Catalan words from (4.2) and the text following it. Let PM denote set of the N-pair matched tuples in $\{1,\ldots,n\}^{2m}$, and let NPM denote the corresponding set for the ones that are not N-pair matched. Write

$$\begin{aligned} E[\mathrm{Tr}(A_n^{2m})] &= \sum_{i\in PM} E_i + \sum_{i\in NPM} E_i \\ &=: T_1 + T_2 \,. \end{aligned}$$

We shall apply Wick's formula for estimating E_i. Fix $i \in NPM$. Recalling that $P(2m)$ is the set of all pair partitions of $\{1,\ldots,2m\}$, given $\pi \in P(2m)$ there exists $(u,v)\in\pi$ such that

$$(i_{u-1}, i_u) \not\sim (i_{v-1}, i_v)\,.$$

Therefore, every $\pi\in P(2m)$ can be written as

$$\pi = \{(u_1^\pi, v_1^\pi),\ldots,(u_m^\pi, v_m^\pi)\}\,, \tag{4.19}$$

where

$$(i_{u_1^\pi -1}, i_{u_1^\pi}) \not\sim (i_{v_1^\pi -1}, i_{v_1^\pi})\,. \tag{4.20}$$

Notice that

$$E[X_{i,j}X_{k,l}] = R((i,j)\star(k,l))\,, \tag{4.21}$$

where $\star$ is as defined in (3.20). By Wick's formula, it follows that

$$\begin{aligned} |E_i| &\le \varepsilon \sum_{\pi\in P(2m)} \prod_{j=2}^m \left| E\left[X_{i_{u_j^\pi -1}, i_{u_j^\pi}} X_{i_{v_j^\pi -1}, i_{v_j^\pi}} \right]\right| \\ &= \varepsilon \sum_{\pi\in P(2m)} \prod_{j=2}^m \left| R((i_{u_j^\pi -1}, i_{u_j^\pi}) \star (i_{v_j^\pi -1}, i_{v_j^\pi}))\right|\,, \end{aligned}$$

the equality following by (4.21). Therefore,

$$\begin{aligned} |T_2| &\le \sum_{u,v\in\mathbb{Z}^{m-1}} \varepsilon \# V_n(u,v) \prod_{j=1}^{m-1} |R(u_j,v_j)|\,, \\ &\le \varepsilon C(m) n^{m+1} \bar{R}^{m-1}\,, \end{aligned} \tag{4.22}$$

where V_n and $C(m)$ are as in (3.23) and $\bar{R}$ is as in (2.3), the last inequality following by Lemma 3.7. Thus,

$$\limsup_{n\to\infty} n^{-(m+1)} |T_2| \le \varepsilon C(m) \bar{R}^{m-1}\,. \tag{4.23}$$

We now work with T_1. For $\sigma\in NC_2(2m)$, let

$$CT(\sigma) := \{\text{Catalan tuples in } \{1,\ldots,n\}^{2m} \text{ corresponding to } \sigma\}\,,$$

$$\begin{aligned} CT'(\sigma) &:= CT(\sigma) \setminus \Bigg(\bigcup_{\pi\in P(2m)\setminus\{\sigma\}} \{\text{Pair matched tuples in } \{1,\ldots,n\}^{2m} \\ &\qquad\qquad \text{corresponding to } \pi\} \Bigg)\,, \end{aligned}$$

and

$$NCT := PM \setminus \left(\bigcup_{\sigma \in NC_2(2m)} CT'(\sigma) \right).$$

Clearly,

$$\begin{aligned} & PM \setminus \left(\bigcup_{\sigma \in NC_2(2m)} CT(\sigma) \right) \\ \subset & \left(\bigcup_{\pi \in P(2m) \setminus NC_2(2m)} \{\text{Pair matched tuples corresponding to } \pi\} \right) \\ & \cup \Bigg(\bigcup_{\sigma \in NC_2(2m)} \Big[\{\text{Pair matched tuples corresponding to } \sigma\} \\ & \qquad \setminus \{\text{Catalan tuples corresponding to } \sigma\} \Big] \Bigg). \end{aligned}$$

By Lemma 4.3 (a) and (b) respectively, the cardinality of the first and the second set on the right hand side is $o(n^{m+1})$. Also,

$$\begin{aligned} & \left(\bigcup_{\sigma \in NC_2(2m)} CT(\sigma) \right) \setminus \left(\bigcup_{\sigma \in NC_2(2m)} CT'(\sigma) \right) \\ \subset & \bigcup_{\sigma \in NC_2(2m)} \left(CT(\sigma) \setminus CT'(\sigma) \right). \end{aligned}$$

By Lemma 4.3 (c), it follows that the cardinality of the set on the right hand side is $o(n^{m+1})$. All the above put together imply that

$$\lim_{n \to \infty} n^{-(m+1)} \# NCT = 0\,. \tag{4.24}$$

Clearly, by definition, if $\sigma, \sigma' \in NC_2(2m)$ and $\sigma \neq \sigma'$, then

$$CT'(\sigma) \cap CT'(\sigma') = \phi\,.$$

Therefore,

$$\begin{aligned} T_1 &= \sum_{\sigma \in NC_2(2m)} \sum_{i \in CT'(\sigma)} E_i + \sum_{i \in NCT} E_i \\ &=: T_{11} + T_{12}\,. \end{aligned}$$

By Wick's formula, it follows that

$$|E_i| \leq (2m)! \text{ for all } i \in \mathbb{N}^{2m}\,,$$

which in view of (4.24), shows that

$$\lim_{n \to \infty} n^{-(m+1)} T_{12} = 0\,. \tag{4.25}$$

Fix $\sigma \in NC_2(2m)$ and $i \in CT'(\sigma)$. By definition of $CT'(\sigma)$, it is easy to see that every $\pi \in P(2m) \setminus \{\sigma\}$ can be written as (4.19) such that (4.20) holds. Therefore,

$$\left| E_i - \prod_{(u,v)\in\sigma} E\left[X_{i_{u-1},i_u} X_{i_v-1,i_v}\right] \right| \le \varepsilon \sum_{\pi\in P(2m)\setminus\{\sigma\}} \prod_{j=2}^{m} \left| R((i_{u_j^\pi-1}, i_{u_j^\pi}) \star (i_{v_j^\pi-1}, i_{v_j^\pi})) \right| . \tag{4.26}$$

By Lemma 4.1, there exist a 0-Catalan tuple j and a $k \in S(\sigma, N)$ satisfying (4.5). Fix $k \in S(\sigma, N)$, $n > 4mN$ and define the sets

$$\begin{aligned}
B_1(k) &= \Big\{ i \in \mathbb{Z}^{2m} : (4.5) \text{ holds for some 0-Catalan tuple} \\
&\qquad j \in \{1, \ldots, n\}^{2m} \text{ corresponding to } \sigma \Big\}, \\
B_2(k) &= \Big\{ i \in \{1, \ldots, n\}^{2m} : (4.5) \text{ holds for some 0-Catalan tuple} \\
&\qquad j \in \{1, \ldots, n\}^{2m} \text{ corresponding to } \sigma \Big\}, \\
B_3(k) &= \Big\{ i \in \mathbb{Z}^{2m} : (4.5) \text{ holds for some 0-Catalan tuple} \\
&\qquad j \in \{4mN+1, \ldots, n-4mN\}^{2m} \text{ corresponding to } \sigma \Big\}, \\
B_4(k) &= \Big\{ i \in CT'(\sigma) : (4.5) \text{ holds for some 0-Catalan tuple} \\
&\qquad j \in \{1, \ldots, n\}^{2m} \text{ corresponding to } \sigma \Big\}, \\
B_5(k) &= \Big\{ i \in CT'(\sigma) : (4.5),\ (4.7) \text{ and } (4.8) \text{ hold for some 0-Catalan} \\
&\qquad \text{tuple } j \in \{1, \ldots, n\}^{2m} \text{ corresponding to } \sigma \Big\}.
\end{aligned}$$

Close inspection reveals that

$$B_3(k) \subset B_2(k) \subset B_1(k) \, ,$$

and

$$\begin{aligned}
\#B_1(k) &= n^{m+1} \, , \\
\#B_3(k) &= (n - 8mN)^{m+1} \, .
\end{aligned}$$

Therefore,

$$\lim_{n\to\infty} n^{-(m+1)} \#B_2(k) = 1 \, . \tag{4.27}$$

Lemma 4.1 asserts that

$$\begin{aligned} CT(\sigma) &= \bigcup_{k\in S(\sigma,N)} B_2(k)\,, \\ CT'(\sigma) &= \bigcup_{k\in S(\sigma,N)} B_4(k)\,. \end{aligned} \tag{4.28}$$

An outcome of the above is that

$$B_2(k)\setminus B_4(k)\subset CT(\sigma)\setminus CT'(\sigma)\,. \tag{4.29}$$

By Lemma 4.3 (c), it follows that

$$\lim_{n\to\infty} n^{-(m+1)}\#\left[CT(\sigma)\setminus CT'(\sigma)\right]=0\,,$$

which along with (4.27) and (4.29) show that

$$\lim_{n\to\infty} n^{-(m+1)}\#B_4(k)=1\,.$$

Elementary combinatorics shows that

$$\#[B_4(k)\setminus B_5(k)]=o(n^{m+1})\,. \tag{4.30}$$

Therefore,

$$\lim_{n\to\infty} n^{-(m+1)}\#B_5(k)=1\,. \tag{4.31}$$

By Lemma 4.2, it follows that

$$\prod_{(u,v)\in\sigma} E\left[X_{i_u-1,i_u}X_{i_v-1,i_v}\right]=\prod_{(u,v)\in\sigma} R(k_u,k_v),\ i\in B_5(k)\,.$$

Similar arguments as those leading to (4.22), in view of (4.26) and the fact that the family of sets $(B_5(k):k\in S(\sigma,N))$ are disjoint, now establish

$$\left|\sum_{k\in S(\sigma,N)}\sum_{i\in B_5(k)} E_i-\sum_{k\in S(\sigma,N)}\#B_5(k)\prod_{(u,v)\in\sigma}R(k_u,k_v)\right|\le \varepsilon C(m)n^{m+1}\bar R^{m-1}\,,$$

where $C(m)$ and $\bar R$ are as in (3.23) and (2.3) respectively. From here, the fact that $S(\sigma,N)$ is a finite set, and that (4.31) holds for all $k\in S(\sigma,N)$, imply that

$$\limsup_{n\to\infty}\left|n^{-(m+1)}\sum_{k\in S(\sigma,N)}\sum_{i\in B_5(k)}E_i-\sum_{k\in S(\sigma,N)}\prod_{(u,v)\in\sigma}R(k_u,k_v)\right|\le \varepsilon C(m)\bar R^{m-1}\,.$$

Equation (4.30) allows us to replace $B_5(k)$ by $B_4(k)$ in the above equation which along with (4.28) and the observation that the sets on the right hand side are pairwise disjoint, lead us to

$$\limsup_{n\to\infty}\left|n^{-(m+1)}\sum_{i\in CT'(\sigma)}E_i-\sum_{k\in S(\sigma,N)}\prod_{(u,v)\in\sigma}R(k_u,k_v)\right| \tag{4.32}$$

$$\leq \varepsilon C(m)\bar{R}^{m-1}\,.$$

Adding over $\sigma \in NC_2(2m)$ yields that

$$\limsup_{n\to\infty}\left|n^{-(m+1)}T_{11} - \beta_{2m}^{(N)}\right| \leq \varepsilon C(m)\bar{R}^{m-1}\#NC_2(2m)\,.$$

Since ε is arbitrary, the above in view of (4.18), (4.23) and (4.25) complete the proof of (4.15).

To complete the proof, (4.17) needs to be shown, or equivalently, in view of (4.15),

$$\lim_{n\to\infty} n^{-2(m+1)}E\left[\left\{\mathrm{Tr}(A_n^{2m})\right\}^2\right] = \beta_{2m}^2\,. \tag{4.33}$$

Notice that

$$E\left[\left\{\mathrm{Tr}(A_n^{2m})\right\}^2\right] \quad = \sum_{i\in\{1,\ldots,n\}^{4m}} E\left[\prod_{j=1}^{4m} X_{i_{\gamma(j)},i_j}\right]\,,$$

where

$$\gamma(j) := \begin{cases} j-1, & j\in\{1,\ldots,4m\}\setminus\{1,2m+1\}\,,\\ 2m, & j=1\,,\\ 4m, & j=2m+1\,.\end{cases}$$

Denote for $i\in\{1,\ldots,n\}^{4m}$,

$$E_i := E\left[\prod_{j=1}^{4m} X_{i_{\gamma(j)},i_j}\right]\,.$$

Once again, the above expectation can be computed via Wick's formula. Therefore, a similar combinatorial analysis as that for the expected trace goes through, except that now offspring words are replaced by compound offspring words. A sketch of the proof is given below.

Fix $N\geq 1$, and say that $i\in\{1,\ldots,n\}^{4m}$ is N-pair matched if there exists a pairing π of $\{1,\ldots,4m\}$ such that

$$\left|i_j\wedge i_{\gamma(j)} - i_{\pi(j)}\wedge i_{\gamma(\pi(j))}\right| \vee \left|i_j\vee i_{\gamma(j)} - i_{\pi(j)}\vee i_{\gamma(\pi(j))}\right| \leq N,\ j=1,\ldots,4m\,.$$

Let PM and NPM denote the sets of N-pair matched tuples and non-N-pair matched tuples respectively, in $\{1,\ldots,n\}^{4m}$ with this new definition of "pair matched". By arguments similar to those leading to (4.23), Lemma 3.4 now playing the role of Lemma 3.3, it follows that

$$\lim_{N\to\infty}\limsup_{n\to\infty} n^{-(m+1)}\left|\sum_{i\in NPM} E_i\right| = 0\,. \tag{4.34}$$

Say that $i \in \{1,\ldots,n\}^{4m}$ is Catalan(σ_1,σ_2) for some $\sigma_1,\sigma_2 \in NC_2(2m)$ if $(i_1,\ldots,i_{2m})$ is N-Catalan with respect to σ_1, and $(i_{2m+1},\ldots,i_{4m})$ is N-Catalan with respect to σ_2. By Lemma 3.5, it follows that

$$\lim_{n\to\infty} n^{-(m+1)} \# \left(PM \,\triangle \left(\bigcup_{\sigma_1,\sigma_2\in NC_2(2m)} CT(\sigma_1,\sigma_2) \right) \right) = 0\,,$$

where

$$CT(\sigma_1,\sigma_2) := \left\{ i \in \{1,\ldots,n\}^{4m} : i \text{ is Catalan}(\sigma_1,\sigma_2) \right\},\ \sigma_1,\sigma_2 \in NC_2(2m)\,.$$

By standard combinatorial arguments, it follows from the above equation that

$$\lim_{n\to\infty} n^{-(m+1)} \left| \sum_{i\in PM} E_i - \sum_{\sigma_1,\sigma_2\in NC_2(2m)} \sum_{i\in CT(\sigma_1,\sigma_2)} E_i \right| = 0\,. \tag{4.35}$$

The arguments leading to (4.32), once again with the aid of Lemma 4.2, imply that for $\sigma_1,\sigma_2 \in NC_2(2m)$,

$$\lim_{N\to\infty} \limsup_{n\to\infty} \Bigg| n^{-(m+1)} \sum_{i\in CT(\sigma_1,\sigma_2)} E_i \tag{4.36}$$
$$- \sum_{k^i\in S(\sigma_i,N)} \left(\prod_{(u,v)\in\sigma_1} R(k_u^1,k_v^1) \right) \left(\prod_{(u,v)\in\sigma_2} R(k_u^2,k_v^2) \right) \Bigg| = 0\,.$$

Clearly,

$$\begin{aligned} & \sum_{\sigma_1,\sigma_2\in NC_2(2m)} \sum_{k^i\in S(\sigma_i,N)} \left(\prod_{(u,v)\in\sigma_1} R(k_u^1,k_v^1) \right) \left(\prod_{(u,v)\in\sigma_2} R(k_u^2,k_v^2) \right) \\ = & \prod_{i=1}^{2} \left(\sum_{\sigma_i\in NC_2(2m)} \sum_{k^i\in S(\sigma_i,N)} \prod_{(u,v)\in\sigma_i} R(k_u^i,k_v^i) \right) \\ = & \beta_{2m}^2\,. \end{aligned}$$

This, in view of (4.34) to (4.36), establishes (4.33), and thus completes the proof. $\square$

5. The linear process

In this section, we study the ESD of a random matrix whose entries are generated from a linear process with independent random variables as the input sequence. In particular, let $\{\epsilon_{i,j} : i,j \in \mathbb{Z}\}$ be independent, mean zero, variance one random variables which satisfy the Pastur condition

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{i,j=1}^{n} E[\epsilon_{i,j}^2 \mathbf{1}(|\epsilon_{i,j}| > \varepsilon\sqrt{n})] = 0 \ \text{ for all } \varepsilon > 0\,. \tag{5.1}$$

Let $\{c_{k,l} : k, l \in \mathbb{Z}\}$ be a collection of deterministic real numbers such that

$$0 < \sum_{k,l\in\mathbb{Z}} |c_{k,l}| < \infty\,, \tag{5.2}$$

and

$$c_{k,l} = c_{l,k},\ k, l \in \mathbb{Z}\,. \tag{5.3}$$

Define

$$Z_{i,j} := \sum_{k,l\in\mathbb{Z}} c_{k,l}\epsilon_{i-k,j-l},\ i, j \in \mathbb{Z}\,, \tag{5.4}$$

where the sum on the right hand side converges in L^2 because $c_{k,l}$ are square summable, which is a consequence of (5.2). While the family of random variables $\{Z_{i,j} : i, j \in \mathbb{Z}\}$ need not be stationary because the distributions of $\epsilon_{i,j}$ are not necessarily identical, it is easy to see that

$$\begin{aligned} E(Z_{i,j}) &= 0,\ i, j \in \mathbb{Z}\,, \\ E(Z_{i,j}Z_{i-u,j+v}) &= \sum_{k,l\in\mathbb{Z}} c_{k,l}c_{k-u,l+v} =: R(u, v),\ i, j, u, v \in \mathbb{Z}\,. \end{aligned} \tag{5.5}$$

Define the $n \times n$ symmetric random matrix A_n and μ_n, the ESD of $A_n/\sqrt{n}$ by (1.1) and (1.2) respectively. The assumption (5.2) ensures that

$$\sum_{u\in\mathbb{Z}}\sum_{v\in\mathbb{Z}} |R(u, v)| \le \sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}} \left[|c_{k,l}| \sum_{u\in\mathbb{Z}}\sum_{v\in\mathbb{Z}} |c_{k-u,l+v}|\right] = \left[\sum_{k,l\in\mathbb{Z}} |c_{k,l}|\right]^2 < \infty\,.$$

Therefore, we can and do define β_{2m} by (2.8). Let μ be the unique probability measure whose odd moments are all zero, and for $m \ge 1$, the $2m$-th moment equals β_{2m}.

The content of this section is the following result.

Theorem 5.1. *Under assumptions* (5.1) *to* (5.3), μ_n *converges weakly in probability to* μ.

Proof. We split the proof into two parts. For a finite linear process, we show that the Stieltjes transform of the ESD of a matrix made up of Gaussian random variables and another with general entries satisfying (5.1) are close to each other using Lindeberg type argument developed in Chatterjee (2005). For the second part, we show that the Lévy distance between the truncated linear process and the original process goes to zero as the truncation level goes to infinity.

Fix $m \ge 1$ and let

$$Z_{i,j}^{(m)} = \sum_{k,l=-m}^{m} c_{k,l}\epsilon_{i-k,j-l} \text{ for } i, j \ge 1\,.$$

Define

$$A_n^{(m)} := ((Z_{i,j}^{(m)}))_{n\times n},\ n \ge 1\,.$$

We next define a similar random matrix model, but with Gaussian entries. Let $(G_{i,j} : i,j \in \mathbb{Z})$ be i.i.d. standard Gaussian, and set

$$Y_{i,j}^{(m)} = \sum_{k,l=-m}^{m} c_{k,l} G_{i-k,j-l} \text{ for } i,j \geq 1\,.$$

Denote

$$B_n^{(m)} := ((Y_{i,j}^{(m)}))_{n\times n},\ n \geq 1\,.$$

Assumption (5.3) ensures that the matrices $A_n^{(m)}$ and $B_n^{(m)}$ are symmetric. By the Lindeberg type argument of "replacing $\epsilon_{i,j}$ by $G_{i,j}$ one at a time", it can be shown using the Pastur condition (5.1) that

$$\frac{1}{n}\left[\mathrm{Tr}\left(\left(zI_n - \frac{A_n^{(m)}}{\sqrt{n}}\right)^{-1}\right) - \mathrm{Tr}\left(\left(zI_n - \frac{B_n^{(m)}}{\sqrt{n}}\right)^{-1}\right)\right] \xrightarrow{P} 0\,, \tag{5.6}$$

as $n \to \infty$, for all z in the complex plane with non-zero imaginary part. The arguments for above are very similar to those in Subsections 2.3 and 2.4 of Chatterjee (2005) and hence are omitted.

It is easy to see that

$$R^{(m)}(u,v) := E(Y_{i,j}^{(m)} Y_{i-u,j+v}^{(m)}) = \sum_{k,l=-m}^{m} c_{k,l} c_{k-u,l+v},\ u,v \in \mathbb{Z}\,,$$

and as a trivial consequence of (5.3), it follows that

$$R^{(m)}(u,v) = R^{(m)}(v,u)\,.$$

Clearly, $R^{(m)}(u,v)$ is non-zero for only finitely many u,v, and hence Theorem 2.1 applies, with a scaling because $R^{(m)}(0,0)$ need not be one. By that result, it follows that for fixed m, as $n \to \infty$, the ESD of $B_n^{(m)}/\sqrt{n}$ converges weakly in probability to the probability measure $\mu^{(m)}$ whose odd moments are all zero and for $l \geq 1$, the $2l$-th moment is $\beta_{2l}^{(m)}$ defined by

$$\beta_{2l}^{(m)} := \sum_{\sigma \in NC_2(2l)} \sum_{k \in S(\sigma)} \prod_{(u,v)\in\sigma} R^{(m)}(k_u,k_v),\ l \geq 1\,,$$

with $S(\sigma)$ being as in (2.6). This, along with (5.6), implies that as $n \to \infty$,

$$\frac{1}{n}\mathrm{Tr}\left(\left(zI_n - \frac{A_n^{(m)}}{\sqrt{n}}\right)^{-1}\right) \xrightarrow{P} \int \frac{1}{z-x}\mu^{(m)}(dx),\ z \in \mathbb{C}\setminus\mathbb{R}\,.$$

Recalling from (2.9) the definition of L, a restatement of the above is that

$$L\left(\mu_n^{(m)}, \mu^{(m)}\right) \xrightarrow{P} 0\,, \tag{5.7}$$

as $n \to \infty$, where $\mu_n^{(m)}$ denotes the ESD of $A_n^{(m)}/\sqrt{n}$. Notice that

$$\lim_{m\to\infty} R^{(m)}(u,v) = R(u,v),\ u,v \in \mathbb{Z}\,.$$

By using (5.2) to interchange limit and sum, it follows that

$$\lim_{m\to\infty} \beta_{2l}^{(m)} = \beta_{2l},\, l \geq 1\,.$$

Therefore,

$$\lim_{m\to\infty} L\left(\mu^{(m)}, \mu\right) = 0\,. \tag{5.8}$$

In view of (5.7) and (5.8), to complete the proof of the result, it suffices to show that

$$\lim_{m\to\infty} \limsup_{n\to\infty} E\left[L^3\left(\mu_n^{(m)}, \mu_n\right)\right] = 0\,, \tag{5.9}$$

recalling that μ_n is the ESD of $A_n/\sqrt{n}$.

To that end, we shall use the fact that for $n\times n$ (deterministic) symmetric matrices C and D with ESD ν_C and ν_D respectively,

$$L^3(\nu_C, \nu_D) \leq \frac{1}{n}\operatorname{Tr}\left((C-D)^2\right)\,,$$

which is a consequence of the Hoffman-Wielandt inequality; see Corollary A.41, page 502 in Bai and Silverstein (2010). Using this inequality, it is immediate that

$$\begin{aligned} E\left[L^3\left(\mu_n^{(m)}, \mu_n\right)\right] &\leq \frac{1}{n} E[\operatorname{Tr}[(A_n/\sqrt{n} - A_n^{(m)}/\sqrt{n})^2] \\ &= \sum_{k,l\in\mathbb{Z}:|k|\vee|l|>m} c_{k,l}^2\,. \end{aligned}$$

The assumption (5.2) of course ensures that $\{c_{k,l}\}$ is square summable, and thus establishes (5.9). Combining this with (5.7) and (5.8) completes the proof. □

6. Stieltjes transform

In this section a characterization of the Stieltjes transform of μ, the LSD in Theorems 2.1 and 5.1, is given via a functional equation. As the reader may have already noticed, in both the above results, μ is defined via the correlations $R(u,v)$ which is as in (2.1) or (5.5). For this section, let $R(\cdot,\cdot)$ be the correlations of a weakly stationary mean zero variance one process $(Y_{ij} : i,j\in\mathbb{Z})$, that is,

$$\begin{aligned} E(Y_{i,j}) &= 0,\, i,j\in\mathbb{Z}\,, \\ E(Y_{i,j}^2) &= 1,\, i,j\in\mathbb{Z}\,, \\ E(Y_{i,j}Y_{i-u,j+v}) &=: R(u,v),\, i,j,u,v\in\mathbb{Z}\,. \end{aligned}$$

As in Section 4, we assume (2.2) and (2.3). As before, let μ be the unique even probability measure whose $2m$-th moment equals β_{2m} which is as defined in (2.8). Recall that the Stieltjes transform of the probability measure

μ on $\mathbb{R}$ is denoted by,

$$\mathcal{G}(z) = \int_{\mathbb{R}} \frac{1}{z-x}\mu(dx), z \in \mathbb{C}.$$

The main result of this section is Theorem 6.1 below.

Let the Fourier transform of covariance function $\{R(k,l)\}_{k,l\in\mathbb{Z}}$ be given by

$$f(x,y) = \sum_{k,l\in\mathbb{Z}} R(k,l)\exp(2\pi i(kx+ly)) \text{ for } (x,y) \in [0,1]\times[0,1].$$

Note that by (2.2), it follows that $f(x,y)$ is a real, symmetric function. For stating the main result, we need the following proposition.

Proposition 6.1. *Suppose that $\mathcal{H}_1$ and $\mathcal{H}_2$ are functions from $\mathbb{C}\times[0,1]$ to $\mathbb{C}$ satisfying the following for $i=1,2$:*

(1) *for all $x \in [0,1]$ and $z \in \mathbb{C}$,*

$$z\mathcal{H}_i(z,x) = 1 + \mathcal{H}_i(z,x)\int_0^1 \mathcal{H}_i(z,y)f(x,y)dy\,, \tag{6.1}$$

(2) *there exists a neighborhood N_i (independent of x) of infinity such that for all $x \in [0,1]$, $\mathcal{H}_i(\cdot,x)$ is analytic on N_i,*

(3) *for all $x\in[0,1]$,*

$$\lim_{z\to\infty} z\mathcal{H}_i(z,x) = 1\,, \tag{6.2}$$

(4) *and*

$$\mathcal{H}(-z,x) = -\mathcal{H}(z,x),\ z\in\mathbb{C},\ x\in[0,1]\,. \tag{6.3}$$

Then

$$\mathcal{H}_1 \equiv \mathcal{H}_2 \text{ on } N_1\cap N_2\,.$$

Proof. By the assumption of analyticity on N_1 and (6.3), for all $x\in[0,1]$ and $k\geq 1$, there exist $H_{2k}(x)\in\mathbb{C}$ such that

$$\mathcal{H}_1(z,x) = \sum_{k=0}^{\infty} H_{2k}(x) z^{-(2k+1)},\ z\in N_1,\ x\in[0,1]\,.$$

The condition (6.2) implies that

$$H_0(x) = 1\,. \tag{6.4}$$

By comparing the power series expansion of the LHS and the RHS of (6.1), one will arrive at the recursion

$$H_{2m}(x) = \sum_{k=1}^{m} H_{2(m-k)}(x)\int_0^1 f(x,y)H_{2(k-1)}(y)dy\,. \tag{6.5}$$

Clearly, a power series expansion of $\mathcal{H}_2$ will also satisfy (6.4) and (6.5), and therefore they have to match with that of $\mathcal{H}_1$. This completes the proof. $\square$

The following is the main result.

Theorem 6.1. *There exists a function $\mathcal{H}$ satisfying the assumptions of the Proposition 6.1. The Stieltjes transform $\mathcal{G}$ of the LSD μ is given by*

$$\mathcal{G}(z) := \left[\int_0^1 \mathcal{H}(z,x)dx\right], \, z \in \mathbb{C}.$$

Before proceeding to the proof, we introduce some notations which will be used for the same. Fix $\sigma \in NC_2(2m)$, and denote its Kreweras complement by $(V_1, \ldots, V_{m+1})$. Although the Kreweras complement is a partition of $\{\overline{1}, \ldots, \overline{2m}\}$, for the ease of notation, $V_1, \ldots, V_{m+1}$ will be thought of as subsets of $\{1, \ldots, 2m\}$, that is, the overline will be suppressed. In order to ensure uniqueness in the notation, we impose the requirement that the blocks $V_1, \ldots, V_{m+1}$ are ordered in the following way. If $1 \le i < j \le m+1$, then the **maximal** element of V_i is strictly less than that of V_j. Let $\mathcal{T}_\sigma$ be the unique function from $\{1, \ldots, 2m\}$ to $\{1, \ldots, m+1\}$ satisfying

$$i \in V_{\mathcal{T}_\sigma(i)}, \, 1 \le i \le 2m.$$

For example, if

$$\sigma := \{(1,4),(2,3),(5,6)\},$$

then $\mathcal{T}_\sigma(1) = 2, \mathcal{T}_\sigma(2) = 1, \mathcal{T}_\sigma(3) = 2, \mathcal{T}_\sigma(4) = 4, \mathcal{T}_\sigma(5) = 3, \mathcal{T}_\sigma(6) = 4$. Define the function L_σ from $\mathbb{R}^{m+1}$ to $\mathbb{R}$ by

$$L_\sigma(x) := \prod_{(u,v)\in\sigma} f\left(x_{\mathcal{T}_\sigma(u)}, x_{\mathcal{T}_\sigma(v)}\right), \, x \in \mathbb{R}^{m+1}.$$

Observe that

$$\mathcal{T}_\sigma(2m) = m+1. \tag{6.6}$$

Finally, set

$$h_\sigma(y) := \int_0^1 \ldots \int_0^1 L_\sigma(x_1, \ldots, x_m, y) dx_m \ldots dx_1, \, y \in \mathbb{R}.$$

The following lemma shows how the moments of the LSD μ are related to this function.

Lemma 6.1. *Let β_{2m} be as in (2.8). Then*

$$\beta_{2m} = \sum_{\sigma \in NC_2(2m)} \int_0^1 h_\sigma(y)dy, \, m \ge 1.$$

Proof. Fix $m \ge 1$ and $\sigma \in NC_2(2m)$. All that needs to be shown is

$$\sum_{k\in S(\sigma)} \prod_{(u,v)\in\sigma} R(k_u, k_v) = \int_{[0,1]^{m+1}} L_\sigma(x)\, dx, \tag{6.7}$$

where $S(\sigma)$ is as in (2.6). Observe that

$$\sum_{(u,v)\in\sigma} \left[k_u x_{\mathcal{T}_\sigma(u)} + k_v x_{\mathcal{T}_\sigma(v)}\right] = \sum_{u=1}^{2m} k_u x_{\mathcal{T}_\sigma(u)} = \sum_{l=1}^{m+1} x_l \sum_{j \in V_l} k_j,$$

and hence

$$\begin{aligned}&\int_{[0,1]^{m+1}} L_\sigma(x)\,dx\\ =\;& \int_{[0,1]^{m+1}}\left[\sum_{k\in\mathbb{Z}^{2m}}\exp\left(2\pi i\sum_{l=1}^{m+1}x_l\sum_{j\in V_l}k_j\right)\prod_{(u,v)\in\sigma}R(k_u,k_v)\right]dx\\ =\;& \sum_{k\in\mathbb{Z}^{2m}}\left(\prod_{(u,v)\in\sigma}R(k_u,k_v)\right)\int_{[0,1]^{m+1}}\exp\left(2\pi i\sum_{l=1}^{m+1}x_l\sum_{j\in V_l}k_j\right)dx\\ =\;& \sum_{k\in S(\sigma)}\prod_{(u,v)\in\sigma}R(k_u,k_v)\,,\end{aligned}$$

the interchange of sum and integral in the second last line being justified by assumption (2.3). This completes the proof. □

The next two lemmas give a recursive relation for the function h_σ.

Lemma 6.2. *Assume that $\sigma\in NC_2(2m)$ can be written as*

$$\sigma=\sigma_1\cup\sigma_2\,, \tag{6.8}$$

where $\sigma_1\in NC_2(2k)$ for some $1\le k\le m-1$, and σ_2 is a non-crossing pair partition of $\{2k+1,\ldots,2m\}$. Viewing σ_2 as an element of $NC_2(2m-2k)$ by the obvious relabeling of $2k+1,\ldots,2m$ to $1,\ldots,2m-2k$ respectively, it is true that

$$h_\sigma(y)=h_{\sigma_1}(y)h_{\sigma_2}(y),\ y\in\mathbb{R}\,.$$

Proof. It is easy to see from (6.6) and (6.8) that

$$\mathcal{T}_\sigma(j)\in\{1,\ldots,k,m+1\},\ \text{for } 1\le j\le 2k\,, \tag{6.9}$$

and

$$\mathcal{T}_\sigma(j)\in\{k+1,\ldots,m+1\},\ \text{for } 2k+1\le j\le 2m\,. \tag{6.10}$$

Write

$$L_\sigma(x)=\\ \left(\prod_{(u,v)\in\sigma:u,v\le 2k}f\left(x_{\mathcal{T}_\sigma(u)},x_{\mathcal{T}_\sigma(v)}\right)\right)\left(\prod_{(u,v)\in\sigma:u,v>2k}f\left(x_{\mathcal{T}_\sigma(u)},x_{\mathcal{T}_\sigma(v)}\right)\right)\,.$$

By (6.9) and (6.10), it follows that

$$\begin{aligned}&h_\sigma(x_{m+1})\\ =\;&\left(\int_0^1\cdots\int_0^1\prod_{(u,v)\in\sigma:u,v\le 2k}f\left(x_{\mathcal{T}_\sigma(u)},x_{\mathcal{T}_\sigma(v)}\right)dx_k\ldots dx_1\right)\\ &\left(\int_0^1\cdots\int_0^1\prod_{(u,v)\in\sigma:u,v>2k}f\left(x_{\mathcal{T}_\sigma(u)},x_{\mathcal{T}_\sigma(v)}\right)dx_m\ldots dx_{k+1}\right)\end{aligned}$$

$$= \quad h_{\sigma_1}(x_{m+1})h_{\sigma_2}(x_{m+1})\,.$$

This completes the proof. □

Lemma 6.3. *If*

$$\sigma = \{(1,2m)\} \cup \sigma_1\,,$$

for some non-crossing pair partition σ_1 *of* $\{2,\ldots,2m-1\}$*, then*

$$h_\sigma(z) = \int_0^1 h_{\sigma_1}(y) f(y,z) dy,\ z \in \mathbb{R}\,,$$

where, once again, σ_1 *is viewed as an element of* $NC_2(2m-2)$*.*

Proof. Throughout the proof, σ_1 is to be thought of as an element of $NC_2(2m-2)$. Clearly,

$$\begin{aligned} \mathcal{T}_\sigma(1) &= m\,, \\ \mathcal{T}_\sigma(2m) &= m+1\,, \\ \mathcal{T}_\sigma(j) &= \mathcal{T}_{\sigma_1}(j-1),\ 2 \le j \le 2m-1\,. \end{aligned}$$

The above equations imply that

$$L_\sigma(x) = f(x_m, x_{m+1}) L_{\sigma_1}(x_1,\ldots,x_m),\ x \in \mathbb{R}^{m+1}\,.$$

Thus,

$$\begin{aligned} h_\sigma(z) &= \int_0^1 \cdots \int_0^1 f(x_m,z) L_{\sigma_1}(x_1,\ldots,x_m) dx_m \ldots dx_1 \\ &= \int_0^1 f(x_m,z) \left\{ \int_0^1 \cdots \int_0^1 L_{\sigma_1}(x_1,\ldots,x_m) dx_{m-1} \ldots dx_1 \right\} dx_m \\ &= \int_0^1 f(x_m,z) h_{\sigma_1}(x_m) dx_m\,, \end{aligned}$$

which completes the proof. □

With the aid of the above lemmas, we proceed to the proof of Theorem 6.1.

Proof of Theorem 6.1. We start with defining the functions

$$H_0(x) = 1, \qquad H_{2m}(x) = \sum_{\sigma \in NC_2(2m)} h_\sigma(x).$$

Lemma 6.1 implies that

$$\beta_{2m} = \int_0^1 H_{2m}(x) dx\,. \tag{6.11}$$

By (2.3), it follows that $|f(x,y)| \le \overline{R}$ uniformly for all x and y in $[0,1]$. Lemmas 6.2 and 6.3 applied inductively imply that

$$\sup_{0 \le x \le 1} |h_\sigma(x)| \le \bar{R}^m,\ \sigma \in NC_2(2m),\ m \ge 1\,.$$

Since $\#NC_2(2m) \leq 4^m$, it holds that

$$\sup_{0\leq x\leq 1} |H_{2m}(x)| \leq \bar{R}^m 2^{2m},\ m \geq 1\,.$$

Consequently,

$$\sup_{0\leq x\leq 1} \limsup_{m\to\infty} |H_{2m}^{1/2m}(x)| \leq 2\bar{R}^{1/2} < \infty\,. \tag{6.12}$$

Therefore, the power series

$$\mathcal{H}(z,x) = \sum_{m=0}^{\infty} \frac{H_{2m}(x)}{z^{2m+1}}$$

converges on $\{z \in \mathbb{C} : |z| > 2\bar{R}^{1/2}\}$ for every fixed $x \in [0,1]$. Note that this neighborhood around infinity is independent of $x \in [0,1]$. It is easy to see that $z\mathcal{H}(z,x)$ has a power series expansion with the leading term as 1 and hence $z\mathcal{H}(z,x) \to 1$ as $|z| \to \infty$. It follows from the definition of $\mathcal{H}(z,x)$ that $\mathcal{H}(-z,x) = -\mathcal{H}(z,x)$. Recall that the Stieltjes transform $\mathcal{G}$ of μ satisfies

$$\mathcal{G}(z) = \sum_{m=0}^{\infty} \beta_{2m} z^{-(2m+1)}\,,$$

(with the obvious convention that $\beta_0 := 1$) which yields,

$$\mathcal{G}(z) = \int_0^1 \mathcal{H}(z,x)dx.$$

Equation (6.1) with $\mathcal{H}_i$ replaced by $\mathcal{H}$ is all that remains to be checked.

To that end, we derive a recursion for $\mathcal{H}(z,x)$ using the properties of $h_\sigma(x)$. Recall that there is a natural one-one correspondence between $NC_2(2m)$ and the set of Catalan words of length $2m$ with the understanding that two words will be considered identical if one can be obtained from the other by a relabeling of letters. Keeping this correspondence in mind, by an abuse of notation, we shall now consider $h_w(x)$ for Catalan words w, and denote by $NC_2(2m)$ the set of Catalan words of length $2m$. Note that any Catalan word w of length $2m$ can be written as $w = aw_1aw_2$, for some $w_1 \in NC_2(2k-2)$ and $w_2 \in NC_2(2m-2k)$. So if

$$H_{2m,k}(x) := \sum_{w_1\in NC_2(2k-2)} \sum_{w_2\in NC_2(2m-2k)} h_{aw_1aw_2}(x)\,,$$

then

$$H_{2m}(x) = \sum_{k=1}^{m} H_{2m,k}(x).$$

Notice that

$$H_{2m,k}(x) = \sum_{w_1\in NC_2(2k-2)} h_{aw_1a}(x) \sum_{w_2\in NC_2(2m-2k)} h_{w_2}(x)$$

$$= \sum_{w_1 \in NC_2(2k-2)} \int_0^1 [f(x,y) h_{w_1}(y)]\, dy \sum_{w_2 \in NC_2(2m-2k)} h_{w_2}(x)$$
$$= \int_0^1 \left[f(x,y) H_{2(k-1)}(y) H_{2(m-k)}(x)\right] dy\,,$$

the equalities in the first two lines following from Lemmas 6.2 and 6.3 respectively. As a consequence,

$$H_{2m}(x) = \sum_{k=1}^{m} H_{2(m-k)}(x) \int_0^1 f(x,y) H_{2(k-1)}(y) dy.$$

Now by an easy computation it follows that,

$$\mathcal{H}(z,x) = \sum_{m=0}^{\infty} H_{2m} z^{-(2m+1)} = \frac{1}{z} + \frac{1}{z}\mathcal{H}(z,x) \int_0^1 f(x,y)\mathcal{H}(z,y) dy.$$

Hence,

$$z\mathcal{H}(z,x) = 1 + \mathcal{H}(z,x) \int_0^1 f(x,y)\mathcal{H}(z,y) dy.$$

This completes the proof. $\square$

Remark 3. *Equations* (6.11) *and* (6.12) *imply that*

$$\lim_{m\to\infty} \beta_{2m}^{1/2m} < \infty\,,$$

which implies that the probability measure μ *is compactly supported.*

7. Special cases and examples

In this section, we attempt to give a better description of the probability measure μ, which appears as the LSD in Theorems 2.1 and 5.1, in some special cases. As in Section 6, $R(\cdot,\cdot)$ are the correlations of a weakly stationary mean zero **variance one** process $(Y_{ij} : i, j \in \mathbb{Z})$. As always, (2.2) and (2.3) are assumed, and μ is the unique even probability measure whose $2m$-th moment equals β_{2m} which is as defined in (2.8). The first main result of this section is the following.

Theorem 7.1. *Assume that*

$$R(u,v) = R(u,0)R(0,v),\ u,v \in \mathbb{Z}\,. \tag{7.1}$$

Then, the function $r(\cdot)$ *defined on* $[-\pi,\pi]$ *by*

$$r(x) := \sum_{k=-\infty}^{\infty} R(k,0) e^{-ikx},\ -\pi \le x \le \pi\,,$$

is a well defined function, that is the sum on the right hand side converges absolutely, and its range is contained in $[0,\infty)$*. Furthermore,*

$$\mu = \mu_r \boxtimes \mu_s\,,$$

where μ_r denotes the law of $r(U)$, U being a Uniform $(-\pi, \pi)$ random variable, μ_s denotes the WSL whose density is given by

$$\mu_s(dx) := \frac{\sqrt{4-x^2}}{2\pi}\mathbf{1}(|x| \le 2)dx\,, \tag{7.2}$$

and "$\boxtimes$" denotes the free product convolution.

Remark 4. *In Bercovici and Voiculescu (1993), the free multiplicative convolution of two probability measures with possibly unbounded support,* **at least one of which is supported on a subset of** $[0,\infty)$*, has been defined. Hence, in the above result, the claim that $r(\cdot)$ is non-negative is needed.*

Proof of Theorem 7.1. In view of assumptions (2.3) and (7.1), it is easy to see that

$$\sum_{k=-\infty}^{\infty} |R(k,0)| = \bar{R}^{1/2} < \infty\,, \tag{7.3}$$

and hence the infinite sum defining $r(x)$ is absolutely convergent. Observing that $(R(k,0) : k \in \mathbb{Z})$ is the autocovariance function of the one-dimensional process $(Y_{-k,0} : k \in \mathbb{Z})$, Corollary 4.3.2, page 120 of Brockwell and Davis (1991) implies that

$$r(x) \in [0,\infty),\ -\pi \le x \le \pi\,.$$

This, in view of (7.3), establishes that the range of $r(\cdot)$ is a compact subset of $[0,\infty)$.

For establishing the other claim, we shall use Theorem 14.4 of Nica and Speicher (2006) which applies to compactly supported probability measures. From that result, it follows that $\mu_r \boxtimes \mu_s$ is an even probability measure, and

$$\int x^{2m}(\mu_r \boxtimes \mu_s)(dx) = \sum_{\sigma \in NC_2(2m)} \prod_{j=1}^{m+1} \int x^{l_j^\sigma}\mu_r(dx),\ m \ge 1\,, \tag{7.4}$$

where for any $\sigma \in NC_2(2m)$, $l_1^\sigma, \ldots, l_{m+1}^\sigma$ denote the block sizes of the Kreweras complement of σ. It is easy to see from the definition of $r(\cdot)$ that

$$\int x^j \mu_r(dx) = \sum_{(k_1,\ldots,k_j)\in\mathbb{Z}^j : k_1+\ldots+k_j=0} \prod_{i=1}^{j} R(k_i,0),\ j \ge 1\,.$$

For $\sigma \in NC_2(2m)$ let the notation for its Kreweras complement be as in (2.5), and recall the definition of $S(\sigma)$ from (2.6). The above two identities put together imply that for $m \ge 1$,

$$\begin{aligned}\int x^{2m}(\mu_r \boxtimes \mu_s)(dx) &= \sum_{\sigma\in NC_2(2m)} \sum_{k\in S(\sigma)} \prod_{j=1}^{2m} R(k_j,0)\\ &= \sum_{\sigma\in NC_2(2m)} \sum_{k\in S(\sigma)} \prod_{(u,v)\in\sigma} R(k_u,0)R(k_v,0)\\ &= \beta_{2m}\end{aligned}$$

$$= \int x^{2m} \mu(dx) \,,$$

the equality in the second last line following from (7.1). This completes the proof. ☐

The next result is the other main result of this section.

Theorem 7.2. *If*

$$R(k, 0) = 0 \text{ for all } k \neq 0 \,, \tag{7.5}$$

then $\mu = \mu_s$, where μ_s is the WSL as defined in (7.2).

Proof. Clearly, it suffices to show that for all $m \geq 1$ and $\sigma \in NC_2(2m)$,

$$\sum_{k \in S(\sigma)} \prod_{(u,v) \in \sigma} R(k_u, k_v) = 1 \,, \tag{7.6}$$

where $S(\sigma)$ is as in (2.6). This is because, if the above is established, then it will follow that

$$\beta_{2m} = \# NC_2(2m) = \int x^{2m} \mu_s(dx) \,.$$

In order to show (7.6), fix $m \geq 1$ and $\sigma \in NC_2(2m)$. What we shall show is that if $k \in S(\sigma)$ is such that

$$\prod_{(u,v) \in \sigma} R(k_u, k_v) \neq 0 \,, \tag{7.7}$$

then,

$$k_1 = \ldots = k_{2m} = 0 \,. \tag{7.8}$$

Recalling that $R(0, 0) = 1$, which is a consequence of the assumption that the process $(Y_{ij} : i, j \in \mathbb{Z})$ mentioned at the beginning of this section, has variance one, the above will imply (7.6).

The claim (7.8) is a tautology when $m = 1$. As the induction hypothesis, we assume that for a fixed $m \geq 1$ and **all** $\sigma \in NC_2(2m)$, (7.7) implies (7.8). To complete the induction step, fix $\sigma \in NC_2(2m+2)$, and let (7.7) hold. By the property of non-crossing pair partition, there exists $j \in \{1, \ldots, 2m+1\}$ such that $(j, j+1) \in \sigma$. Recalling that $K(\sigma)$, the Kreweras complement of σ, is the maximal partition $\overline{\sigma}$ of $\{\overline{1}, \ldots, \overline{2m}\}$ such that $\sigma \cup \overline{\sigma}$ is a non-crossing partition of $\{1, \overline{1}, \ldots, 2m, \overline{2m}\}$, it follows that $(\bar{j}) \in K(\sigma)$. Hence,

$$k_j = 0 \,.$$

Since (7.7) holds, it follows that

$$R(k_j, k_{j+1}) \neq 0 \,,$$

which along with (7.5) implies that

$$k_{j+1} = 0 \,.$$

If $\bar{\sigma}$ denotes the element of $NC_2(2m)$ obtained from σ by deleting $(j, j+1)$ and the obvious relabeling, then it is easy to see that

$$(\bar{k}_1, \ldots, \bar{k}_{2m}) := (k_1, \ldots, k_{j-1}, k_{j+2}, \ldots, k_{2m+2}) \in S(\bar{\sigma})\,,$$

and

$$\prod_{(u,v)\in\bar{\sigma}} R(\bar{k}_u, \bar{k}_v) \neq 0\,.$$

By the induction hypothesis, it follows that

$$\bar{k}_1 = \ldots = \bar{k}_{2m} = 0\,,$$

which establishes the induction step, and thereby completes the proof. $\square$

Now, we shall see the relevance of the two main results proved above in the light of Theorem 5.1.

Corollary of Theorem 7.1. Let $\{\epsilon_{i,j} : i, j \in \mathbb{Z}\}$ be as in Section 5; in particular, the Pastur condition (5.1) holds. Let $\{c_k : k \in \mathbb{Z}\}$ be a sequence of real numbers such that

$$\sum_{k=-\infty}^{\infty} |c_k| < \infty\,, \tag{7.9}$$

and

$$\sum_{k=-\infty}^{\infty} c_k^2 = 1\,. \tag{7.10}$$

Set

$$c_{k,l} := c_k c_l,\ k, l \in \mathbb{Z}\,.$$

Define $Z_{i,j}$ and $R(\cdot,\cdot)$ by (5.4) and (5.5) respectively. Clearly, (5.2) and (5.3) hold, and the process $(Z_{i,j} : i, j \in \mathbb{Z})$ is weakly stationary with mean zero and variance one. Also,

$$\begin{aligned}
R(u,v) &= \left(\sum_k c_k c_{k-u}\right)\left(\sum_l c_l c_{l+v}\right) \\
&= \left(\sum_k \sum_{k'} c_k c_{k-u} c_{k'}^2\right)\left(\sum_l \sum_{l'} c_l c_{l+v} c_{l'}^2\right) \\
&= R(u,0)R(0,v)\,,
\end{aligned}$$

the second equality following from (7.10). Let A_n and μ_n be as in (1.1) and (1.2) respectively, that is, the former is the $n \times n$ matrix whose (i,j)-th entry is $Z_{i\wedge j, i\vee j}$, and the latter is the ESD of $A_n/\sqrt{n}$. Let μ_r and μ_s be as in the statement of Theorem 7.1. Then, as a corollary of the result mentioned above and Theorem 5.1, it follows that, μ_n converges weakly in probability to $\mu_r \boxtimes \mu_s$.

Corollary of Theorem 7.2. Once again, let $\{\epsilon_{i,j} : i,j \in \mathbb{Z}\}$ be as in Section 5 satisfying (5.1). Assume that $\{c_{k,l} : k,l \in \mathbb{Z}\} \subset \mathbb{R}$ is such that (5.2) and (5.3) hold, and furthermore

$$\sum_{l=-\infty}^{\infty} c_{k,l}c_{k',l} = \mathbf{1}(k=k') \text{ for all } k,k' \in \mathbb{Z}\,. \tag{7.11}$$

As before, let $Z_{i,j}$, A_n and μ_n be as in (5.4), (1.1) and (1.2) respectively. It is easy to see that the conditions imposed above ensure that $(Z_{i,j} : i,j \in \mathbb{Z})$ is a mean zero variance one weakly stationary process, and that (7.5) holds. Then by Theorem 5.1 and Theorem 7.2, it follows that μ_n converges weakly in probability to μ_s which is the WSL defined in (7.2).

We end this section by revisiting Examples 1 to 4 mentioned in Section 1.

Example 1. To start with, one needs to argue the existence of a stationary centered Gaussian process $\{Z_{i,j} : i,j \in \mathbb{Z}\}$ satisfying

$$E[Z_{0,0}Z_{u,v}] = \rho^{|u|+|v|},\ u,v \in \mathbb{Z}\,.$$

That, however, is obvious from the observation that

$$\rho^{|u|+|v|} = \int_{(-\pi,\pi]^2} e^{i(ux+vy)}F(dx)F(dy),\ u,v \in \mathbb{Z}\,,$$

where F is the spectral measure of the autocovariance function $(\rho^{|h|} : h \in \mathbb{Z})$; see Herglotz theorem (Theorem 4.3.1 in Brockwell and Davis (1991)). By Theorem 7.1 and results about the AR(1) process, it follows that μ_n converges in probability to $\mu_r \boxtimes \mu_s$, where μ_r is the law of $\frac{1-\rho^2}{1-2\rho\cos U+\rho^2}$, U being an Uniform $(-\pi,\pi)$ random variable.

Example 2. Notice that the (i,j)-th entry of A_n is given by

$$(N+1)\sum_{k,l\in\mathbb{Z}} c_kc_lG_{i-k,j-l} =: (N+1)Y_{i,j}\,,$$

where $c_k := (N+1)^{-1/2}\mathbf{1}(-N \le k \le 0)$. Then (7.9) and (7.10) hold, and therefore, the ESD of $((Y_{i,j}/\sqrt{n}))_{n\times n}$ converges to $\mu_r \boxtimes \mu_s$, where μ_r is the law of

$$1 + 2(N+1)^{-2}\sum_{k=1}^{N}(N-k+1)^2\cos(kU)\,,$$

U being distributed as Uniform $(-\pi,\pi)$. Hence the LSD of $A_n/\sqrt{n}$ is the free product convolution of μ_s with the law of

$$N+1+2(N+1)^{-1}\sum_{k=1}^{N}(N-k+1)^2\cos(kU)\,.$$

Example 3. By Theorem 7.2, it follows that under the additional assumption that $\sum_{n=1}^{\infty}|E(G_0G_n)| < \infty$, the LSD of $A_n/\sqrt{n}$ is μ_s.

Example 4. Setting

$$\sigma := \left(\sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} c_{k,l}^2 \right)^{1/2},$$

it is easy to see from the Corollary of Theorem 7.2 that the LSD of $\sigma^{-1} A_n/\sqrt{n}$ is μ_s. Therefore, the LSD of $A_n/\sqrt{n}$ is $\tilde{\mu}_s$ given by

$$\tilde{\mu}_s(dx) := \frac{\sqrt{4 - x^2/\sigma^2}}{2\pi\sigma} \mathbf{1}(|x| \le 2\sigma) dx \,,$$

which is a dilation of the WSL.

8. An extension of Theorem 7.1

In this section, we generalize Theorem 7.1 to the case when the covariances $R(u, v)$ are not necessarily summable. Suppose that $(Z_{i,j} : i, j \in \mathbb{Z})$ is a stationary mean zero variance one Gaussian process. Define A_n, μ_n and $R(\cdot, \cdot)$ by (1.1), (1.2) and (2.1) respectively. The first assumption is, as before, that

$$R(u, v) = R(u, 0)R(v, 0), \; u, v \in \mathbb{Z} \,. \tag{8.1}$$

The second assumption, the one that replaces the summability of $R(u, v)$, is that the spectral measure of the one dimensional process $(Z_{i,0} : i \in \mathbb{Z})$ is absolutely continuous with respect to the Lebesgue measure. This means that there exists a non-negative integrable function r on $[-\pi, \pi]$ such that

$$(2\pi)^{-1} \int_{-\pi}^{\pi} e^{inx} r(x) dx = R(n, 0), \; n \in \mathbb{Z} \,. \tag{8.2}$$

Strictly speaking, $(2\pi)^{-1} r(\cdot)$ is the density of the spectral measure. Since $R(n, 0) = R(-n, 0)$, it follows that $r(\cdot)$ is symmetric. As in Theorem 7.1, denote by μ_r the law of $r(U)$ where U is an Uniform$(-\pi, \pi)$ random variable. The main result of this section is the following.

Theorem 8.1. *As $n \to \infty$, μ_n converges weakly in probability to $\mu_r \boxtimes \mu_s$, where μ_s is the WSL.*

For the proof of Theorem 8.1, we shall need the following lemma, which is an observation of independent interest.

Lemma 8.1. *Define*

$$c_k := (2\pi)^{-1} \int_{-\pi}^{\pi} e^{ikx} \sqrt{r(x)} \, dx, \; k \in \mathbb{Z} \,, \tag{8.3}$$

the integral being defined because $\sqrt{r(\cdot)} \in L^2([-\pi, \pi])$, and real because $r(\cdot)$ is symmetric. Then,

$$\sum_{k=-\infty}^{\infty} c_k^2 < \infty \,, \tag{8.4}$$

and thus the sum $\sum_{k,l\in\mathbb{Z}} c_k c_l G_{i-k,j-l}$ *converges in* L^2 *for all* i,j*, where* $(G_{i,j} : i,j \geq 1)$ *is a family of i.i.d. standard Gaussian random variables. Furthermore,*

$$(Z_{i,j} : i,j \in \mathbb{Z}) \stackrel{d}{=} \left(\sum_{k,l\in\mathbb{Z}} c_k c_l G_{i-k,j-l} : i,j \in \mathbb{Z} \right) .$$

Proof. The claim (8.4) follows from an application of the Parseval identity. Therefore,

$$\left(\sum_{k,l\in\mathbb{Z}} c_k c_l G_{i-k,j-l} : i,j \in \mathbb{Z} \right)$$

is clearly a mean zero stationary Gaussian process. To check that it has the same finite dimensional distributions as $(Z_{i,j} : i,j \in \mathbb{Z})$, it suffices to verify that the correlations match, that is,

$$R(u,v) = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} c_k c_{k-u} c_l c_{l+v},\ u,v \in \mathbb{Z}\,. \tag{8.5}$$

To that end, we start with the observation that

$$\lim_{N\to\infty} \sum_{k=-N}^{N} c_k e^{-ikx} = \sqrt{r(x)},\ -\pi \leq x \leq \pi\,, \tag{8.6}$$

in $L^2([-\pi,\pi])$, which follows from the fact that the Fourier series of a square integrable function converges in the L^2 norm to that function. Therefore, the square of the left hand side converges in L^1 to $r(x)$ as $N \to \infty$. Consequently, for fixed $n \in \mathbb{Z}$,

$$\begin{aligned}
\int_{-\pi}^{\pi} e^{inx} r(x)dx &= \lim_{N\to\infty} \int_{-\pi}^{\pi} e^{inx} \left(\sum_{k=-N}^{N} c_k e^{-ikx} \right)^2 dx \\
&= \lim_{N\to\infty} 2\pi \sum_{k=-N}^{N} c_k c_{n-k} \\
&= 2\pi \sum_{k=-\infty}^{\infty} c_k c_{n-k} \\
&= 2\pi \sum_{k=-\infty}^{\infty} c_k c_{k-n}\,,
\end{aligned}$$

the second last equality following from (8.4), and the last equality being an outcome of the fact that $\sqrt{r(\cdot)}$ is symmetric. Comparing this with (8.2), it follows that

$$R(n,0) = \sum_{k=-\infty}^{\infty} c_k c_{k-n},\ n \in \mathbb{Z}\,.$$

Using (8.1), it follows that for all $u, v \in \mathbb{Z}$,

$$\begin{aligned} R(u,v) &= \left(\sum_{k=-\infty}^{\infty} c_k c_{k-u}\right)\left(\sum_{l=-\infty}^{\infty} c_l c_{l-v}\right) \\ &= \left(\sum_{k=-\infty}^{\infty} c_k c_{k-u}\right)\left(\sum_{l=-\infty}^{\infty} c_{l+v} c_l\right), \end{aligned}$$

thereby establishing (8.5). This completes the proof. □

Proof of Theorem 8.1. Let $(G_{i,j} : i, j \geq 1)$ be a family of i.i.d. standard Gaussian random variables, and let $\{c_k\}$ be as in (8.3). In view of Lemma 8.1, without loss of generality, we can and do assume that

$$Z_{i,j} = \sum_{k,l \in \mathbb{Z}} c_k c_l G_{i-k,j-l},\ i, j \in \mathbb{Z}.$$

Denote

$$\begin{aligned} Z_{i,j}^{(m)} &:= \sum_{k=-m}^{m} \sum_{l=-m}^{m} c_k c_l G_{i-k,j-l},\ i, j \in \mathbb{Z},\ m \geq 1, \\ R^{(m)}(u,v) &:= E\left[Z_{0,0}^{(m)} Z_{-u,v}^{(m)}\right],\ u, v \in \mathbb{Z},\ m \geq 1, \\ A_n^{(m)} &:= ((Z_{i,j}^{(m)}))_{n\times n},\ n, m \geq 1. \end{aligned}$$

If $\mu_n^{(m)}$ denotes the ESD of $A_n^{(m)}/\sqrt{n}$, then exactly same arguments as those in the proof of Theorem 5.1 show that

$$\lim_{m\to\infty} \limsup_{n\to\infty} E\left[L^3\left(\mu_n^{(m)}, \mu_n\right)\right] = 0, \tag{8.7}$$

where L denotes the Lévy distance. Clearly,

$$\lim_{m\to\infty} R^{(m)}(0,0) = R(0,0) = 1. \tag{8.8}$$

Fix m large enough so that $R^{(m)}(0,0) > 0$. By Theorems 2.1 and 7.1, it follows that

$$L\left(\mu_n^{(m)}, \mu_{r_m} \boxtimes \mu_s\right) \xrightarrow{P} 0, \tag{8.9}$$

as $n \to \infty$, where

$$r_m(x) := \frac{1}{\sqrt{R^{(m)}(0,0)}} \sum_{n=-\infty}^{\infty} R^{(m)}(n,0) e^{-inx},\ -\pi \leq x \leq \pi,$$

and μ_{r_m} denotes the law of $r_m(U)$, U being an Uniform $(-\pi,\pi)$ random variable. Notice that in the sum on the right hand side, only finitely many terms are non-zero. In view of (8.7) and (8.9), to complete the proof it suffices to show that

$$\lim_{m\to\infty} L\left(\mu_s \boxtimes \mu_{r_m}, \mu_s \boxtimes \mu_r\right) = 0. \tag{8.10}$$

To that end, define

$$\tilde{r}_m(x) := \sum_{n=-\infty}^{\infty} R^{(m)}(n,0)e^{-inx} = \left(\sum_{k=-m}^{m} c_k e^{-ikx}\right)^2, \ -\pi \le x \le \pi\,.$$

By (8.6), it follows that

$$\lim_{m\to\infty}\int_{-\pi}^{\pi} |\tilde{r}_m(x) - r(x)|\, dx = 0\,,$$

which along with (8.8) ensures that

$$\mu_{r_m} \xrightarrow{w} \mu_r, \text{ as } m \to \infty\,.$$

Define the map $\sqrt{\cdot}$ from the space of non-negative probability measures to that of symmetric probability measures as follows. If a non-negative probability measures ν is the law of a random variable V, then $\sqrt{\nu}$ is the law of $\epsilon\sqrt{V}$, where ϵ takes values $+1$ and -1 each with probability $1/2$ independently of V. Let $\cdot^2$ denote the inverse of $\sqrt{\cdot}$. By Corollary 6.7 of Bercovici and Voiculescu (1993), it follows that

$$\mu_s^2 \boxtimes \mu_{r_m}^2 \xrightarrow{w} \mu_s^2 \boxtimes \mu_r^2, \text{ as } m \to \infty\,.$$

Lemma 8 of Arizmendi and Pérez-Abreu (2009) tells us that for a symmetric probability measure ν_1 and a non-negative probability measure ν_2 such that $\nu_1(\{0\}) \vee \nu_2(\{0\}) < 1$,

$$\nu_1 \boxtimes \nu_2 = \sqrt{\nu_1^2 \boxtimes \nu_2^2}\,.$$

This shows (8.10) which, along with (8.7) and (8.9), completes the proof. $\square$

Next, let us see two examples where Theorem 8.1 applies.

Example 5. Let

$$R(u,v) := \frac{\sin u}{u}\frac{\sin v}{v}, \ u, v \in \mathbb{Z}\,,$$

where $(\sin 0)/0$ is to be interpreted as 1. Clearly,

$$(2\pi)^{-1}\int_{-\pi}^{\pi} e^{inx}\pi\mathbf{1}(|x| \le 1)dx = R(n,0), \ n \in \mathbb{Z}\,.$$

Therefore, by Theorem 8.1 and the fact that the free product convolution of Bernoulli (p) and WSL is same as the classical product convolution of $\sqrt{p}$ times Bernoulli (p) and WSL, it follows that the LSD in this example is the law of ΠW, where W follows WSL, and Π takes the values $\sqrt{\pi}$ and 0 with probabilities $1/\pi$ and $1 - 1/\pi$ respectively, independently of W **in the classical sense**. Note that for this example, even though

$$\sum_{u,v} |R(u,v)| = \infty\,,$$

the LSD is still compactly supported.

Example 6. Define

$$r(x) := \frac{\sqrt{\pi}}{2}|x|^{-1/2}\mathbf{1}(x \neq 0),\ -\pi \leq x \leq \pi\,,$$

and

$$R(u,v) := \left((2\pi)^{-1}\int_{-\pi}^{\pi} e^{iux} r(x)dx\right)\left((2\pi)^{-1}\int_{-\pi}^{\pi} e^{ivx} r(x)dx\right),\ u,v \in \mathbb{Z}\,.$$

The fact that

$$\int_{-\pi}^{\pi} r(x)dx = 2\pi\,,$$

ensures that the assumptions (8.1) and (8.2) hold. Therefore, the LSD in this case is $\mu_r \boxtimes \mu_s$, where μ_r is the law of $\frac{\sqrt{\pi}}{2}|U|^{-1/2}$ and U follows Uniform$(-\pi,\pi)$. The following proposition shows that the fourth moment of the LSD is infinite, which means, in particular, that the largest eigenvalue of $A_n/\sqrt{n}$ goes to infinity in probability.

Proposition 8.1. *For any non-negative probability measure ν and integer $k \geq 1$,*

$$\int_{\mathbb{R}} x^{2k}\mu_s \boxtimes \nu(dx) = \infty\,, \tag{8.11}$$

if and only if

$$\int_{\mathbb{R}} x^k \nu(dx) = \infty\,. \tag{8.12}$$

Proof. We start with the "if" part, that is, assume (8.12). Let X be a random variable whose law is ν. For $n \geq 1$, let ν_n denote the law of $X \wedge n$. In what follows, all integrals are on the whole of $\mathbb{R}$. By (7.4), considering the element $\{(1,2),\ldots,(2k-1,2k)\}$ of $NC_2(2k)$, it follows that

$$\int x^{2k}\mu_s \boxtimes \nu_n(dx) \geq \left(\int x\nu_n(dx)\right)^k \int x^k \nu_n(dx)\,.$$

Proposition 4.15 of Bercovici and Voiculescu (1993) implies that $\mu_s^2 \boxtimes \nu_n^2$ is dominated by $\mu_s^2 \boxtimes \nu^2$, and hence

$$\begin{aligned}
\int x^{2k}\mu_s \boxtimes \nu(dx) &= \int x^k \mu_s^2 \boxtimes \nu^2(dx) \\
&\geq \int x^k \mu_s^2 \boxtimes \nu_n^2(dx) \\
&\geq \left(\int x\nu_n(dx)\right)^k \int x^k \nu_n(dx) \\
&\to \left(\int x\nu(dx)\right)^k \int x^k \nu(dx) = \infty\,,
\end{aligned}$$

the limit in the last line following from the monotone convergence theorem. This establishes the "if" part.

For the "only if" part, assume that

$$\int_{\mathbb{R}} x^k \nu(dx) < \infty .$$

Define ν_n as before for $n \geq 1$. For $\sigma \in NC_2(2k)$, let $l_1^\sigma, \ldots, l_{k+1}^\sigma$ be as in (7.4). By the Skorohod embedding and Fatou's lemma, it follows that

$$\begin{aligned}
\int x^{2k} \mu_s \boxtimes \nu(dx) &\leq \liminf_{n\to\infty} \int x^{2k} \mu_s \boxtimes \nu_n(dx) \\
&= \liminf_{n\to\infty} \sum_{\sigma\in NC_2(2k)} \prod_{j=1}^{k+1} \int x^{l_j^\sigma} \nu_n(dx) \\
&= \sum_{\sigma\in NC_2(2k)} \prod_{j=1}^{k+1} \int x^{l_j^\sigma} \nu(dx) \\
&< \infty ,
\end{aligned}$$

the inequality in the last line following from the observation that $l_j^\sigma \leq k$ for all $\sigma \in NC_2(2k)$ and $1 \leq j \leq k+1$. This completes the proof of the "only if" part. $\square$

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