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Random Walks in I.I.D. Random Environment on Non-Abelian Free Groups

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Abstract

We consider the random walk on an *independent and identically distributed (i.i.d.)* random environment on a Cayley graph of a finitely generated non-abelian free group. Such a Cayley graph is readily seen to be a regular tree with even degree. Under a non-degeneracy assumption we show that the walk is always transient.

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1 Introduction

In this short note we show that the random walk in random environment model on a regular tree with even degree where the environment at each vertex is independent and are also "identically" distributed is transient. We make this notion of i.i.d.-ness of the environment rigorous by defining the model on a finitely generated non-abelian free group and then transfer it back to an appropriate even degree regular tree which is essentially same as a Cayley graph associated with the free group.

1.1 Basic Setup

Cayley Graph: Let G be a finitely generated non-abelian free group and let $S = \{s_1, s_2, \ldots s_d\} \subseteq G$ be a *minimal* generating set. Note that it is then necessary that $s \in S \Rightarrow s^{-1} \in S$, that is, S is a *symmetric* set. Moreover we must also have $d \geq 4$ is an even integer. We further assume the convention that the elements of S are ordered in such a way that the first $\frac{d}{2}$ elements are inverses (respectively) of the second $\frac{d}{2}$ elements.

We now define a graph $\bar{G}$ with vertex set as G and edge set $E := \left\{ \{x, y\} \,\middle|\, xy^{-1} \in S \right\}$. Such a graph $\bar{G}$, is called a *(left)-Cayley Graph* of G with respect to the generating set S. Since G is a

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free group so it is easy to see that $\bar{G}$ is a graph with no loops and is regular with degree d, thus is isomorphic to $\mathbb{T}_d$. We will abuse the terminology a bit and will write $\mathbb{T}_d$ is the Cayley graph of G. We will assume that $\mathbb{T}_d$ is rooted and we will take the identity element e of G as the root. We will write $N(x)$ for the set of all neighbors of a vertex $x \in G$. Notationally, $N(x) = \left\{ y \in G \,\middle|\, xy^{-1} \in S \right\}$. Observe that from definition $N(e) = S$. For $x \in G$ define $\theta_x : G \to G$ by $\theta_x(y) = yx$, then θ_x is an automorphism of $\mathbb{T}_d$. We will call θ_x the *translation by* x.

Random Environment: Let $\mathcal{S} := \mathcal{S}_e$ be a collection of probability measures on the d elements of $N(e) = S$. To simplify the presentation and avoid various mesurability issues, we assume that $\mathcal{S}$ is a Polish space (including the possibilities that $\mathcal{S}$ is finite or countably infinite). For each $x \in \mathbb{T}_d$, $\mathcal{S}_x$ denotes a copy of $\mathcal{S}$, with all elements of $\mathcal{S}$ translated by θ_x, so as to have support on $N(x)$. Formally, an element $\omega(x, \cdot)$ of $\mathcal{S}_x$, is a probability measure satisfying

$$\omega(x, y) \geq 0 \;\; \forall \;\, x, y \in \mathbb{T}_d \quad \text{and} \quad \sum_{y \in N(x)} \omega(x, y) = 1$$

Let $\mathcal{B}_{\mathcal{S}_x}$ denote the Borel σ-algebra on $\mathcal{S}_x$. The *environment space* is defined as the measurable space $(\Omega, \mathcal{F})$ where

$$\Omega := \prod_{x \in \mathbb{T}_d} \mathcal{S}_x, \quad \mathcal{F} := \bigotimes_{x \in \mathbb{T}_d} \mathcal{B}_{\mathcal{S}_x}, \tag{1.1}$$

An element $\omega \in \Omega$ will be written as $\left\{ \omega(x, \cdot) \,\middle|\, x \in \mathbb{T}^d \right\}$. An environment distribution is a probability P on this measurable space.

Random Walk: We now turn to define a random walk $(X_n)_{n \geq 0}$. Given an environment $\omega \in \Omega$, $(X_n)_{n \geq 0}$ is a time homogeneous Markov chain taking values in $\mathbb{T}_d$ with transition probabilities

$$\mathbf{P}_\omega \left(X_{n+1} = y \,\middle|\, X_n = x \right) = \omega(x, y).$$

Let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For each $\omega \in \Omega$, we denote by $\mathbf{P}_\omega^x$ the law induced by $(X_n)_{n \geq 0}$ on $\left((\mathbb{T}_d)^{\mathbb{N}_0}, \mathcal{G} \right)$, where $\mathcal{G}$ is the σ-algebra generated by the cylinder sets, such that

$$\mathbf{P}_\omega^x (X_0 = \mathbf{x}) = 1. \tag{1.2}$$

$\mathbf{P}_\omega^x$ is called the *quenched law* of the random walk $\{X_n\}_{n \geq 0}$, starting at x. We will use the notation $\mathbf{E}_\omega^x$ for the expectation under the quenched measure $\mathbf{P}_\omega^x$.

We note that for every $G \in \mathcal{G}$, the function

$$\omega \mapsto \mathbf{P}_\omega^{\mathbf{x}}(G)$$

is $\mathcal{F}$-measurable. Hence, we may define the measure $\mathbb{P}^x$ on $\left(\Omega \times (\mathbb{T}_d)^{\mathbb{N}_0}, \mathcal{F} \otimes \mathcal{G} \right)$ from the relation

$$\mathbb{P}^x (F \times G) = \int_F \mathbf{P}_\omega^x (G) \, P(d\omega), \quad \forall \;\, F \in \mathcal{F}, G \in \mathcal{G}.$$

With a slight abuse of notation, we also denote the marginal of $\mathbb{P}^x$ on $(\mathbb{T}_d)^{\mathbb{N}_0}$ by $\mathbb{P}^x$, whenever no confusion occurs. This probability distribution is called the *annealed law* of the random walk $(X_n)_{n \geq 0}$, starting at x. Note that under $\mathbb{P}^x$, the random walk $(X_n)_{n \geq 0}$ is not, in general, a Markov chain. We will use the notation $\mathbb{E}^x$ for the expectation under the quenched measure $\mathbb{P}^x$

1.2 Main Results

Throughout this paper we will assume the following hold,

(A1) P is a product measure on $(\Omega, \mathcal{F})$ with "*identical*" marginals, that is, under P the random probability laws $\left\{\omega(x,\cdot) \,\middle|\, x \in \mathbb{T}^d\right\}$ are independent and "identically" distributed in the sense that

$$P \circ \theta_x^{-1} = P, \tag{1.3}$$

for all $x \in G$.

(A2) For all $1 \le i \le d$,

$$\mathbf{E}\left[\left|\log \omega(e, s_i)\right|\right] < \infty. \tag{1.4}$$

Following is our main result.

Theorem 1 *Under assumptions (A1) and (A2) the random walk $(X_n)_{n\ge 0}$ is transient $\mathbb{P}^e$ almost surely.*

An immediate question that arises is whether the above walk has a speed which may be zero. Following result provides a partial answer to this question with (A2) replaced by the usual *uniform ellipticity* condition.

(A3) There exists $\epsilon > 0$ such that

$$P\left(\omega(e, s_i) > \epsilon \;\forall\; 1 \le i \le d\right) = 1. \tag{1.5}$$

Theorem 2 *Under assumptions (A1) and (A3) with $\epsilon \ge \frac{1}{2(d-1)} + \delta$ where $\delta > 0$ we have $\mathbb{P}^e$ almost surely*

$$\liminf_{n\to\infty} \frac{|X_n|}{n} > 0, \tag{1.6}$$

where $|X_n|$ denotes the length of the unique path in $\mathbb{T}_d$ from the root e to X_n.

1.3 Remarks

Random walk in Random Environment (RWRE) models on the one dimensional integer lattice $\mathbb{Z}$ was first introduced by Solomon in [11] where he gave explicit criteria for the recurrence and transience of the walk for *independent and identically distributed (i.i.d.)* environment distribution. Since then a large variety of results have been discovered for RWRE in $\mathbb{Z}^d$, yet there are many challenging problems which are still left open (see [13] and [12]).

Perhaps the earliest known results for RWRE on trees is by Pemantle and Lyons [7]. In that paper they consider a model on rooted tress, which later got to known as *random conductance model*. In that model, the random conductances along each path from vertices to the root are assumed to be independent and identically distributed. The random walk is then shown to be recurrent or transient depending on how large is the value of the average conductance. Later [5] considered the same model under additional assumption of the jump probabilities are also i.i.d. and have studied the speed of the walk in the recurrent regime.

In our set up, the assumption (A1) essentially says that that the random transition laws $\left\{\omega(x,\cdot) \,\middle|\, x \in \mathbb{T}^d\right\}$ are *independent and identically distributed (i.i.d.)*. On $\mathbb{T}_d$ we introduced the group structure to define *identically distributed* and we made the probability law P invariants under the *translations* by the group elements. Hence the RWRE model in this article is different then the studied *random conductance model* discussed above. It is interesting to note that the only example where the two models agree is the deterministic environment of the *simple symmetric walk* on $\mathbb{T}_d$.

There have also been several other contributions on random trees, particularly on random walk on Galton-Watson trees [6, 8, 9, 2, 10]. It is worth to point out here that a random walk on a Galton-Watson tree [6] satisfies the assumption (A1) and so does a random walk on a multi-type Galton-Watson tree [3].

Our last result (Theorem 2) is certainly far from being satisfactory. We strongly believe that under the assumptions (A1) and (A3) the sequence of random variables $\left(\frac{|X_n|}{n}\right)_{n\geq 0}$ has a $\mathbb{P}^e$-almost sure limit which is non-random and strictly positive. A similar conclusion has been derived for the special case of random walk on Galton-Watson trees [8]. This and the central limit theorem for such walks will be studied in future work.

In the entire article we have worked with a regular tree with even degree and considered it as a Cayley graph of a non-abelian free group. However the same argument presented in the main theorem of the article will also work if the tree is of odd degree. We did not present the result as the proof will be largely repetitive once the i.i.d. structure is set up via a torsion element in the asymmetric set of generators.

2 Proofs of the Main Results

2.1 Proof of Theorem 1

Given an environment $\omega \in \Omega$, we can define the conductance at a vertex σ_n where $|\sigma_n| = n$, to be

$$\Phi_n(\sigma_n) = \omega(e, x_1) \prod_{k=2}^{n-1} \frac{\omega(x_k, x_{k+1})}{\omega(x_k, x_{k-1})}, \tag{2.1}$$

where x_i's are the vertices on the unique path from e to σ_n with $x_0 = e$ and $x_n = \sigma_n$. Suppose $\sigma_n = \prod_{i=1}^{n} \alpha_{n-i+1}$, with $\alpha_i \in S$ and $\alpha_i \neq \alpha_{i+1}^{-1}$. We can re-write $\Phi_n(\sigma_n)$ as

$$\Phi_n(\sigma_n) = \omega_1(e, \alpha_1) \prod_{k=1}^{n-2} \frac{\omega_k(e, \alpha_k)}{\omega_{k+1}(e, \alpha_k^{-1})}, \tag{2.2}$$

with $\omega_k\,(e, s) = \omega(x_k, sx_k)$ for any $s \in S$.

Let $\mathcal{B}_{\mathbb{N}_0}$ denote the product σ-algebra on $S^{\mathbb{N}_0}$, and μ be a Probability measure on $\left(S^{\mathbb{N}_0}, \mathcal{B}_{\mathbb{N}_0}\right)$ such that the coordinate variables, say, $(Y_n)_{n\geq 1}$ is a Markov chain on S with

$$\mu\left(Y_n = s \,\middle|\, Y_{n-1} = t\right) = \frac{1}{d-1}, \; s, t \in S \text{ with } s \neq t^{-1}. \tag{2.3}$$

It is easy to see that the chain (Y_n) is an aperiodic, irreducible and finite state Markov chain and its stationary distribution is the uniform distribution on S. We shall assume that Y_0 is uniformly distributed on S.

Let $\eta_n = \prod_{i=1}^{n} Y_i$. From equation (2.3) η_n is a uniformly distributed on the set of vertices $\mathbb{T}_d^n := \left\{ x \in \mathbb{T}_d \,\middle|\, |x| = n \right\}$. Now

$$\begin{aligned} \frac{1}{n}\log\Phi_n(\eta_n) &= \frac{1}{n}\log\omega_1(e,Y_1) + \frac{1}{n}\sum_{k=1}^{n-2}\log\omega_k(e,Y_k) - \log\omega_{k+1}(e,Y_k^{-1}) \\ &= \frac{1}{n}\log\omega_1(e,Y_1) + \frac{1}{n}\sum_{i=1}^{d}[\sum_{j=1}^{N_n^{Y,s_i}}(\log\omega_{k_j}(e,s_i) - \log\omega_{k_j+1}(e,s_i^{-1}))], \end{aligned} \tag{2.4}$$

where

$$N_n^{Y,s_i} = \sum_{k=1}^{n}\mathbf{1}\left(Y_k = s_i\right). \tag{2.5}$$

Now consider the product space $\left(\Omega\times S^{\mathbb{N}_0}, \mathcal{B}_\Omega\otimes\mathcal{B}_W, P\otimes\mu\right)$. By Erdös-Feller-Pollard Theorem [4] we have $P\otimes\mu$-almost surely

$$\lim_{n\to\infty}\frac{N_n^{Y,s_i}}{n} = \frac{1}{d}. \tag{2.6}$$

Further under assumption (A2) and using the Strong Law of Large Numbers for i.i.d. random variables we have P-almost surely

$$\lim_{n\to\infty}\frac{1}{N_n^{Y,s_i}}\sum_{j=1}^{N_n^{Y,s_i}}\left(\log\omega_{k_j}\left(e,s_i\right) - \log\omega_{k_j+1}\left(e,s_i^{-1}\right)\right) = E\left[\log\omega_1\left(e,s_i\right) - \log\omega_1\left(e,s_i^{-1}\right)\right].$$

As S is a symmetric set of generators for G therefore $P\otimes\mu$-almost surely,

$$\lim_{n\to\infty}\frac{1}{n}\log\Phi_n(\eta_n) = \frac{1}{d}\sum_{i=1}^{d}E\left[\log\omega_1\left(e,s_i\right) - \log\omega_1\left(e,s_i^{-1}\right)\right] = 0. \tag{2.7}$$

So by Fubini's Theorem, for P-almost surely all $\omega\in\Omega$, we must have that the equation (2.7) holds μ-almost surely. Fix such an $\omega\in\Omega$. Let $\frac{1}{d-1} < \Delta < 1$ be a fixed positive real number. We can find such a Δ since $d\geq 3$. Since convergence almost surely with respect to μ implies convergence in probability. So given any $0<p<1$ we can find $N_\mu^\omega(p)\in\mathbb{N}$ and $B_n^\omega\subseteq\mathbb{T}_d^n$ such that

$$\Phi_n(\eta_n) < \left(\frac{1}{\sqrt{\Delta}}\right)^n \ \forall\ \eta_n\in B_n^\omega,\ \forall\ n\geq N_\mu^\omega(p)$$

and

$$\lim_{n\to\infty}\frac{\#B_n^\omega}{d(d-1)^n} = p.$$

Let $\beta_n = \Delta^{\frac{n}{2}}$ then for $n\geq N_\mu^\omega(p)$

$$\sum_{\sigma_n\in\mathbb{T}_d^n}\beta_n\left(\Phi_n(\sigma_n)\right)^{-1} \geq \sum_{\eta_n\in B_n^\omega}\beta_n\left(\Phi_n(\sigma_n)\right)^{-1} \geq d(d-1)^n\Delta^n\frac{\#B_n^\omega}{d(d-1)^n}. \tag{2.8}$$

Finally by choice of Δ we have

$$\lim_{n\to\infty} \sum_{\sigma_n \in \mathbb{T}_d^n} \beta_n \left(\Phi_n(\sigma_n)\right)^{-1} = \infty. \tag{2.9}$$

So we conclude that for $d \geq 3$, there is a positive (non-random) sequence $(\beta_n)_{n\geq 1}$ such that $\sum_{n=1}^{\infty} \beta_n < \infty$ but

$$\lim_{n\to\infty} \sum_{\sigma_n \in \mathbb{T}_d^n} \beta_n \left(\Phi_n(\sigma_n)\right)^{-1} = \infty \ \ P\text{-almost surely.} \tag{2.10}$$

By Corollary 4.2 in [6], the random walk has to be transient. $\square$

2.2 Proof of Theorem 2

Let $D_n := |X_n|$ be the distance from the root e of the position of the walker at time n on the tree $\mathbb{T}_d$. Then

$$\begin{aligned} D_n &= \sum_{i=1}^{n} (D_i - D_{i-1}) \\ &= \sum_{i=1}^{n} \left(D_i - D_{i-1} - \mathbb{E}_\omega^e \left[D_i - D_{i-1} \,\middle|\, X_0, \ldots, X_{i-1} \right] \right) \\ &\quad + \sum_{i=1}^{n} \mathbb{E}_\omega^e \left[D_i - D_{i-1} \,\middle|\, X_0, \ldots, X_{i-1} \right] \end{aligned} \tag{2.11}$$

But then $M_n := \sum_{i=1}^{n} \left(D_i - D_{i-1} - \mathbb{E}_\omega^e \left[D_i - D_{i-1} \,\middle|\, X_0, \ldots, X_{i-1} \right] \right)$ is a martingale with bounded increments, so by Azuma's Inequality [1]

$$\frac{M_n}{n} \to 0 \ \ \mathbb{P}^e\text{-almost surely.} \tag{2.12}$$

Further it is easy to see that

$$\sum_{i=1}^{n} \mathbb{E}_\omega^e \left[D_i - D_{i-1} \,\middle|\, X_0, \ldots, X_{i-1} \right] = 1 - 2\,\mathbf{1}\left(X_{i-1} \neq e\right) \omega \left(X_{i-1}, \overleftarrow{X}_{i-1} \right).$$

where a vertex $x \in \mathbb{T}_d$ we define $\overleftarrow{x}$ as the *parent* of x, that is, the first vertex on the unique path from x to e.

Now under our assumption (A3) with $\epsilon \geq \frac{1}{2(d-1)} + \delta$ where $\delta > 0$ we have

$$P\left(\omega\left(x, \overleftarrow{x}\right) < \frac{1}{2} - \delta\,(d-1) \ \ \forall\, x \in \mathbb{T}_d \right) = 1.$$

Thus $\mathbb{P}^e$-almost surely

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \left(1 - 2\,\mathbf{1}\left(X_{i-1} \neq e\right) \omega \left(X_{i-1}, \overleftarrow{X}_{i-1} \right) \right) > 2\delta\,(d-1) > 0. \tag{2.13}$$

Finally, by (2.11) $D_n = M_n + \sum_{i=1}^{n}(1 - 2 1_{X_{i-1} \neq e}\, \omega(X_{i-1}, \overleftarrow{X}_{i-1})$, so using equations (2.12) and (2.13) we conclude that (1.6) holds. $\square$

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