isid/ms/2013/09
September 10, 2013
http://www.isid.ac.in/~statmath/eprints

Maximum eigenvalue of symmetric random matrices with dependent heavy tailed entries

Arijit Chakrabarty, Rajat Subhra Hazra and Parthanil Roy

Indian Statistical Institute, Delhi Centre
7, SJSS Marg, New Delhi–110 016, India

MAXIMUM EIGENVALUE OF SYMMETRIC RANDOM MATRICES WITH DEPENDENT HEAVY TAILED ENTRIES

ARIJIT CHAKRABARTY, RAJAT SUBHRA HAZRA, AND PARTHANIL ROY

Abstract. This paper deals with symmetric random matrices whose upper diagonal entries are obtained from a linear random field with heavy tailed noise. It is shown that the maximum eigenvalue and the spectral radius of such a random matrix with dependent entries converge to the Frechét distribution after appropriate scaling. This extends a seminal result of Soshnikov (2004) when the tail index is strictly less than one.

1. Introduction

In this article, we study the asymptotic behaviour of the maximum eigenvalue of an $n \times n$ symmetric random matrix A_n, whose upper diagonal entries are given by the linear random field

$$Y_{k,l} := \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} c_{ij} X_{i+k,j+l},\ 1 \leq\ k \leq l \leq n, \tag{1.1}$$

where $\{c_{i,j}\}_{i,j\geq 0}$ is a sequence of real numbers satisfying

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} |c_{i,j}|^{\delta} < \infty \tag{1.2}$$

for some $\delta \in (0, \alpha)$, and $\{X_{i,j}, i, j \in \mathbb{N}\}$ is a family of i.i.d. positive random variables with distribution function F satisfying

$$1 - F(x) = L(x)x^{-\alpha},\ x > 0 \tag{1.3}$$

for some slowly varying function L and for some $0 < \alpha < 1$. It is easy to check, following the arguments of Cline (1983) (see also Davis and Resnick (1985)), that (1.2) ensures the almost sure convergence of the series in (1.1).

Random matrices with heavy tailed entries have generated considerable interest in the recent years; see Soshnikov (2004), Ben Arous and Guionnet (2008), Belinschi et al. (2009), Auffinger et al. (2009), Davis et al. (2011), Bordenave et al. (2011). Soshnikov (2004) investigated the edge behavior of Wigner matrices with

2010 *Mathematics Subject Classification.* Primary 60B20; Secondary 60B10, 60G70.

Key words and phrases. Random matrix, heavy tails, regular variation, linear random field, maximum eigenvalue, spectral radius.

The first author's research was partially supported by the INSPIRE grant at the Indian Statistical Institute, Delhi Centre.

The second author acknowledges the support from Swiss National Science Foundation, grant 132909.

The third author's research was supported in part by the FP7 grant PIRSES-GA-2012-318984 at Indian Statistical Institute Kolkata.

i.i.d. heavy tailed upper diagonal entries whose distribution function F satisfies (1.3) for some $0 < \alpha < 2$. It was established that in this case, the largest eigenvalue converges to Frechét distribution after scaling by

$$b_n := \inf\left\{x : 1 - F(x) \le \frac{2}{n(n+1)}\right\}. \tag{1.4}$$

This result was later extended by Auffinger et al. (2009) to the case $2 \le \alpha < 4$ with centered entries. One of the important features of the proof of the above results is that eigenvalues behave similar to the largest entries of the matrix in absolute value, and as a consequence, the point process of the normalized positive eigenvalues converges to a Poisson point process.

This edge behavior is drastically different from the case of $\alpha > 4$ which is supposed to be governed by the Tracy-Widom law; see, for example, Lee and Yin (2013). For a relaxation of identically distributed condition and further results on edge universality see Bourgade et al. (2013). Few similar results are also known for sample covariance matrices; see, for example, Yin et al. (1988) and Auffinger et al. (2009). Davis et al. (2011) studied the edge behavior for sample covariance matrix XX^T, where the rows of X are independent copies of a linear process with heavy tailed noise. Dependence across both rows and columns have also been investigated in the context of bulk asymptotics of sample covariance matrices whose entries have lighter tails; see Hachem et al. (2005) and Pfaffel and Schlemm (2012). For a review of the existing literature on random matrices, we refer the readers to the articles Ben Arous and Guionnet (2011), Erdős and Yau (2012).

Consider the Hilbert space $l^2 := \{(a_n : n \in \mathbb{Z}) \subset \mathbb{R} : \sum_{n\in\mathbb{Z}} a_n^2 < \infty\}$. We shall define an operator T on l^2 as follows. For $i, j \in \mathbb{Z}$, let

$$T(i,j) := \begin{cases} c_{i+1,j-1}, & i \le -1, j \ge 1\,, \\ c_{j+1,i-1}, & j \le -1, i \ge 1\,, \\ 0, & \text{otherwise}\,. \end{cases} \tag{1.5}$$

T acts on l^2 in the natural way by $(Ta)_i := \sum_{j=-\infty}^{\infty} T(i,j)a_j$, $i \in \mathbb{Z}$. By the Cauchy-Schwarz inequality, the operator norm $\|T\|$ of T can be bounded above by $\left(2\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} c_{i,j}^2\right)^{1/2}$, which is finite because of (1.2). Let $\rho(A_n)$ and $\lambda_{\max}(A_n)$ denote the spectral radius (same as the spectral norm in this case) and the maximum eigenvalue of A_n, respectively. With these notations, we can now state the main result of this paper.

Theorem 1.1. *Let A_n be as above and b_n be as in (1.4), then for all $x > 0$,*

$$\lim_{n\to\infty} P\big(\lambda_{\max}(A_n) \le \|T\|\, b_n x\big) = \lim_{n\to\infty} P\big(\rho(A_n) \le \|T\|\, b_n x\big) = e^{-x^{-\alpha}}.$$

Note that we partially recover Theorem 1.1 of Soshnikov (2004) as a consequence of the above result by choosing $c_{ij} = 1$ when $i = j$ and $c_{ij} = 0$ when $i \neq j$. However our methods are limited to $0 < \alpha < 1$ and the case $1 \le \alpha < 4$ is still open. We would also like to point that in general, it is very difficult to calculate $\|T\|$. In the following special case, it can be computed using Lemma 3.2 below.

Corollary 1.2. *Let A_n, b_n be as in Theorem 1.1 with $c_{ij} = \alpha_i\beta_j$, where $\{\alpha_i\}_{i\ge 0}$ and $\{\beta_j\}_{j\ge 0}$ satisfy the summability conditions $\sum_{i=0}^{\infty} |\alpha_i|^\delta < \infty$ and $\sum_{j=0}^{\infty} |\beta_j|^\delta < \infty$,* *respectively, for some $\delta \in (0, \alpha)$. Then for all $x > 0$,*

$$\lim_{n\to\infty} P\big(\lambda_{\max}(A_n) \le \|\alpha\|_2 \|\beta\|_2 \, b_n x\big) = \lim_{n\to\infty} P\big(\rho(A_n) \le \|\alpha\|_2 \|\beta\|_2 \, b_n x\big) = e^{-x^{-\alpha}} \,,$$

where $\|\cdot\|_2$ denotes the ℓ^2 norm.

The setup of Davis et al. (2011) can be thought of as a special case of the above example (except that their random matrix X is not necessarily symmetric) with only nonzero α_i being $\alpha_0 = 1$. They also obtained a similar constant in the limit.

In the rest of the paper, we prove Theorem 1.1. In Section 2, we deal with the finite linear random fields and then using a truncation argument pass on to the infinite case in Section 3. The novel technique used in this paper is the well-known fact from linear algebra that for any matrix norm $|||\cdot|||$,

$$\rho(A) = \lim_{r\to\infty} \left|\left|\left| A^{2r} \right|\right|\right|^{\frac{1}{2r}}. \tag{1.6}$$

In this paper, for any $n \times n$ matrix M, the $(i,j)^{\text{th}}$ entry is denoted by $M(i,j)$ and the matrix norms $\|\cdot\|_\infty$ and $\max(\cdot)$ are defined by $\|M\|_\infty = \max_{1\le i\le n} \sum_{j=1}^n |M(i,j)|$ and $\max(M) = \max_{1\le i,j\le n} |M(i,j)|$, respectively.

2. Finite linear random field

In this section, we first show an upper bound of the spectral radius of A_n using (1.6) and by splitting the matrix A_n^{2r} into two parts, one having entries only with the $2r^{\text{th}}$ powers and the other with lower order terms. The matrix with lower order terms has negligible contribution and the main contribution comes from the $2r^{\text{th}}$ powers. A careful analysis of the coefficient matrices leads to the upper bound. Then we prove a lower bound of maximum eigenvalue using the Rayleigh's characterisation.

For each $N \ge 1$, let $A_n^{(N)}$ be the $n \times n$ symmetric matrix whose upper diagonal elements are given by the finite linear random field

$$Y_{k,l}^{(N)} := \sum_{i=0}^{N} \sum_{j=0}^{N} c_{ij} X_{i+k,j+l}, \ 1 \le k \le l \le n, \tag{2.1}$$

where $\{X_{i,j}, i,j \in \mathbb{N}\}$ and $\{c_{ij}\}_{i,j\ge 0}$ are as in Section 1 and C_N be the matrix

$$C_N := \begin{bmatrix} \mathbf{0} & \widehat{C}_N \\ \widehat{C}_N^T & \mathbf{0} \end{bmatrix}, \tag{2.2}$$

where $\widehat{C}_N$ is the $(N+1)\times(N+1)$ matrix whose $(i,j)^{\text{th}}$ entry is $c_{N+1-i,N+1-j}$.

2.1. Upper bound for spectral radius.

Our first lemma gives an upper bound for the spectral radius of $A_n^{(N)}$ and is the most crucial step towards proving Theorem 1.1.

Lemma 2.1.

$$\lim_{n\to\infty} P\left(\frac{\rho(A_n^{(N)})}{\max_{1\le k,l\le n} |X_{k,l}|} > \rho(C_N) + \varepsilon \right) = 0\,, \tag{2.3}$$

for all $\varepsilon > 0$.

Proof. Fix an integer $r \geq 1$. Write $\left(A_n^{(N)}\right)^{2r} = U_n + V_n$, where U_n contains the terms $X_{k,l}^{2r}$ with their coefficients and V_n consists of the cross terms. For all $n > 2N$, we shall decompose U_n further into the sum of two matrices W_n and Z_n, show that W_n gives the correct upper bound and $\|Z_n\|_\infty$, $\|V_n\|_\infty$ have negligible contributions. These steps are elaborated below.

Step 1: **Decomposition of U_n.**

Fix $n > 2N$. Let k, l be integers such that $N < k < l - N \leq n - N$ and $\widetilde{T}$ be the set of all pairs (k, l) satisfying these inequalities. Define $\tilde{C}_U$ to be the $n \times n$ matrix whose $(N+1) \times (N+1)$ submatrix formed by the $(l-N)^{\text{th}}$, $(l-N+1)^{\text{th}}$, $\ldots$, l^{th} rows and $(k-N)^{\text{th}}$, $(k-N+1)^{\text{th}}$, $\ldots$, k^{th} columns is $\widehat{C}_N$ defined as above and the other entries are all zero. The condition $k < l - N$ ensures that $\tilde{C}_U$ is upper triangular with diagonal entries zero. Set $\tilde{C}_{k,l} := \tilde{C}_U + \tilde{C}_U^T$. Thus, $\tilde{C}_{k,l}$ is a symmetric matrix, and the coefficient of $X_{k,l}^{2r}$ in the $(i,j)^{\text{th}}$ entry of U_n equals $\tilde{C}_{k,l}^{2r}(i,j)$. It is easy to see that $\tilde{C}_{k,l}^{2r}(i,j)$ is zero unless (i,j) belongs to either $[k-N, k] \times [l-n, l]$ or $[l-N, l] \times [k-n, k]$. Write $U_n = W_n + Z_n$, where W_n contains the entry $X_{k,l}^{2r}$ (with its coefficient) if and only if $(k,l) \in \widetilde{T}$ and the remaining entries are in Z_n.

Step 2: **Bound for $\|W_n\|_\infty$.**

Define an $n \times n$ symmetric matrix B_n whose $(u,v)^{\text{th}}$ upper diagonal entry is $X_{u,v}^{2r}$. Fix $1 \leq i, j \leq n$. Clearly, if $(k,l) \notin S_{ij}$, the union of $[i, i+N] \times [j, j+N]$ and $[j, j+N] \times [i, i+N]$, then the coefficient of $X_{k,l}^{2r}$ in $W_n(i,j)$ is zero. Thus,

$$|W_n(i,j)| \leq \sum_{(k,l) \in S_{ij} \cap \widetilde{T}} \tilde{C}_{k,l}^{2r}(i,j) X_{k,l}^{2r} \leq \max(C_N^{2r}) \sum_{(k,l) \in S_{ij} \cap \widetilde{T}} X_{k,l}^{2r}\,. \tag{2.4}$$

Using the above inequality, we have that for each i,

$$\sum_{j=i}^{n} |W_n(i,j)| \leq \max(C_N^{2r}) \sum_{(k,l) \in \widetilde{T}} \#\{j \in [i,n] : (k,l) \in S_{ij}\} B_n(k,l).$$

Since $(k,l) \in S_{ij} \cap \widetilde{T}$ and $i \leq j$ yield $k - N \leq i \leq k < l - N \leq j \leq l \leq n$, the above upper bound can be further bounded by

$$\leq \max(C_N^{2r}) \sum_{k=i}^{i+N} \sum_{l=1}^{n} \#\{j \in [i,n] : (k,l) \in S_{ij}\} B_n(k,l) \leq (N+1)^2 \max(C_N^{2r}) \|B_n\|_\infty\,.$$

Similarly, using the symmetry of B_n, $\sum_{j=1}^{i-1} |W_n(i,j)|$ can also be bounded by the same quantity. Thus, we have $\|W_n\|_\infty \leq 2(N+1)^2 \max(C_N^{2r}) \|B_n\|_\infty$, from which using equation (34) in Soshnikov (2004) it follows that

$$\lim_{n\to\infty} P\left(\frac{\|W_n\|_\infty}{\max_{1 \leq k \leq l \leq n} X_{k,l}^{2r}} > 2(N+1)^2 \max(C^{2r}) + \varepsilon \right) = 0$$

for all $\varepsilon > 0$.

Step 3: **Negligibility of $\|Z_n\|_\infty$.**

Let us now try to upper bound $\|Z_n\|_\infty$. For all $0 \le k \le N$, let $\bar{C}_k$ be a symmetric $(k+N+1)\times(k+N+1)$ matrix whose $(i,j)^{\text{th}}$ upper triangular entry is $c_{N+1-i,N+k+1-j}$ whenever $1 \le i \le N+1$ and $k+1 \le j \le N+k+1$ (with $i \le j$) and other upper triangular entries are all zero. By a reasoning similar to the proof of (2.4), it follows that

$$|Z_n(i,j)| \le \max_{0\le k\le N} \max(\bar{C}_k) \sum_{(u,v)\in\widetilde{T}^c} X_{u,v}^{2r},$$

for all $1 \le i,j \le n$, where $\widetilde{T}^c := \{(u,v) : 1 \le u \le v \le n+N\} \setminus \widetilde{T}$. It is easy to see that the cardinality of $\widetilde{T}^c$ and the number of non-zero entries of Z_n are both $O(n)$. Therefore, there exists a constant K independent of n such that

$$\|Z_n\|_\infty \le \sum_{i=1}^n \sum_{j=1}^n |Z_n(i,j)| \le Kn \sum_{(u,v)\in\widetilde{T}^c} X_{u,v}^{2r},$$

which implies $\|Z_n\|_\infty = o_p\big(n^{4r/\alpha}\big)$.

Step 4: **Negligibility of $\|V_n\|_\infty$.**

We shall show that

$$P[\|V_n\|_\infty > b_n^{2r}\epsilon] \to 0 \text{ as } n \to \infty. \tag{2.5}$$

To this end, note that a typical entry of V_n has the following form:

$$V(k,l) = {\sum_{i_1,i_2,\cdots,i_{2r-1}=1}^{n}}' \; {\sum_{\substack{m_1,\cdots m_{2r}=0\\ n_1,\cdots,n_{2r}=0}}^{N}}' \prod_{i=1}^{2r} c_{m_i,n_i} X_{m_1+k,n_1+i_1}\cdots X_{m_{2r}+i_{2r-1},n_{2r}+l},$$

where $\sum'$ denotes that the sum is over all those indices which do not give $2r$-th power. To simplify the notations we denote by $\mathcal{I}$ the set of indices $1 \le l \le n$, $\{i_k, 1 \le k \le 2r\}$ and $\{m_i,n_i\}_{1\le i\le 2r}$ such that $i_k \in \{1,\cdots,n\}$ and $m_i,n_i \in \{0,\cdots,N\}$ with the constraint that $(m_1+k,n_1+i_1) \ne \cdots \ne (m_{2r}+i_{2r-1},n_{2r}+l)$. Also for any index $\mathbf{j} \in \mathcal{I}$ we denote the product of random variables $X_{m_1+k,n_1+i_1}\cdots X_{m_{2r}+i_{2r-1},n_{2r}+l}$ by $Z_{\mathbf{j}}$ and product of coefficients $\prod_{i=1}^{2r} c_{m_i,n_i}$ by $c_{\mathbf{j}}$.

Now note that for any $\mathbf{u} \in \mathcal{I}$, if $\{Y_i, 1 \le i \le 2r\}$ are i.i.d. random variables with regularly varying tail of index $-\alpha$ then, $Z_{\mathbf{u}} \overset{d}{=} Y_1^{l_1}\cdots Y_{2r}^{l_{2r}}$ for some $0 \le l_i \le 2r-1$ satisfying $l_1+l_2+\cdots+l_{2r} = 2r$. It is easy to see that $Z_{\mathbf{u}}$ is regularly varying with index $-\alpha/k$ for some $1 \le k \le 2r-1$. Now partition $\mathcal{I}$ into equivalence classes $\mathbf{C}_k$ such that for any $\mathbf{u} \in \mathbf{C}_k$, $Z_{\mathbf{u}}$ is regularly varying $-\alpha/k$. Also note that there exists constants L_k such that, $|\,\mathbf{C}_k\,| \le L_k N^{4r} n^{2r}$. As $\sum_{l=1}^n |V(k,l)| \le \sum_{\mathbf{u}\in\mathcal{I}} |c_{\mathbf{u}}| Z_{\mathbf{u}}$, it is enough to show that for some $\delta > 0$, we have $P\left[\sum_{\mathbf{u}\in\mathcal{I}} |c_{\mathbf{u}}| Z_{\mathbf{u}} > b_n^{2r}\eta\right] \le Cn^{-(1+\delta)}$, where C does not depend on k.

For $k \geq 2$ and $r \geq 2$ denote

$$\beta_{\alpha,r}(k) := \frac{k(4r-(1+2r)\alpha)}{\alpha(k-\alpha)} > 0.$$

Let $\delta_1 = 4r/\alpha$ if $r = 1$ and also for $k = 1$. For $k \geq 2$ choose $\delta_k \in (0, \beta_{\alpha,r}(k))$.

$$P\left[\sum_{\mathbf{u}\in\mathbf{C}_k} |c_{\mathbf{u}}|Z_{\mathbf{u}} > b_n^{2r}\eta\right] \leq P\left[\sum_{\mathbf{u}\in\mathbf{C}_k} |c_{\mathbf{u}}||Z_{\mathbf{u}}|\mathbb{I}(|Z_{\mathbf{u}} \leq n^{\delta_k}) > b_n^{2r}\epsilon\right]$$
$$+ P\left[\max_{\mathbf{u}\in\mathbf{C}_k} Z_{\mathbf{u}} > n^{\delta_k}\right] =: P_1 + P_2.$$

First we bound P_2. Choose κ such that $\kappa > (\frac{2r+1}{\delta_k} \wedge \frac{\alpha}{k})$.

$$P\left[\max_{\mathbf{u}\in\mathbf{C}_k} |Z_{\mathbf{u}}| > n^{\delta_k}\right] \leq \sum_{\mathbf{u}\in\mathbf{C}_k} P\left[|Z_{\mathbf{u}}| > n^{\delta_k}\right] \leq n^{-\delta_k\kappa} \sum_{\mathbf{u}\in\mathbf{C}_k} E[|Z_{\mathbf{u}}|^{\kappa}]$$
$$\leq l_k n^{2r-\delta_k\kappa} = l_k n^{-(1+\epsilon_1)}.$$

Where we have used that $|\,\mathbf{C}_k\,| \leq L_k N^{4r} n^{2r}$ and $l_k = L_k N^{4r}$. Also note that $\epsilon_1 = \delta_k\kappa - 2r - 1 > 0$.

Now to bound P_1 choose $\gamma > (1 \wedge \frac{\alpha}{k})$. Using Hölder's inequality we get,

$$P\left[\sum_{\mathbf{u}\in\mathbf{C}_k} |c_{\mathbf{u}}||Z_{\mathbf{u}}|\mathbb{I}(|Z_{\mathbf{u}}| \leq n^{\delta_k}) > b_n^{2r}\epsilon\right]$$
$$\leq P\left[\sum_{\mathbf{u}\in\mathbf{C}_k} |c_{\mathbf{u}}|^{\gamma}|Z_{\mathbf{u}}|^{\gamma}\mathbb{I}(|Z_{\mathbf{u}}| \leq n^{\delta_k}) > K_1 b_n^{2\gamma r} n^{2r(\gamma-1)}\right]$$
$$\leq K_2 b_n^{-2\gamma r} n^{-2r\gamma} E[|Z_{\mathbf{u}}|^{\gamma}\mathbb{I}(|Z_{\mathbf{u}}| \leq n^{\delta_k})].$$

By Karamata's theorem (see Theorem 2.1 in Resnick (2007)), we get,

$$E[|Z_{\mathbf{u}}|^{\gamma}\mathbb{I}(|Z_{\mathbf{u}}| \leq n^{\delta_k})] \sim K_3 n^{\gamma\delta_k} n^{-\frac{\delta_k\alpha}{k}}.$$

As $b_n^{2\gamma r} \sim n^{\frac{4\gamma r}{\alpha}} L_1(n)$ for some slowly varying function L_1 we have

$$b_n^{-2\gamma r} n^{-2r\gamma} E[|Z_i|^{\gamma}\mathbb{I}(|Z_i| \leq n^{\delta_k})] \sim n^{-\frac{4\gamma r}{\alpha} - 2r\gamma + \gamma\delta_k - \frac{\delta_k\alpha}{k}} L_l(n).$$

Now choosing appropriately γ and using the fact that $\alpha < 1$ it easily follows that, $P_1 \leq l'_j n^{-(1+\epsilon_2)}$ for some $\epsilon_2 > 0$. Note that the constants only depend on ϵ, N, r. So there exists C independent of k such that, $P\left[\sum_{\mathbf{u}\in\mathcal{I}} |c_{\mathbf{u}}|Z_{\mathbf{u}} > b_n^{2r}\eta\right] \leq Cn^{-(1+\delta)}$ This establishes (2.5).

Step 5: **Combine Steps 1 - 4.**

Combining the above steps, we get that there is $K > 0$ such that for all $r \geq 1$ and $\varepsilon > 0$,

$$\lim_{n\to\infty} P\left(\frac{\rho(A_n)}{\max_{1\leq k\leq l\leq n} |X_{k,l}|} > K^{1/r} \max(C^{2r})^{1/2r} + \varepsilon\right) = 0,$$

from which (2.3) follows using (1.6). □

2.2. **Lower bound for maximum eigenvalue.** Let us now proceed to lower bound $\lambda_{\max}(A_n^{(N)})$.

Lemma 2.2.

$$\lim_{n\to\infty} P\left(\frac{\lambda_{\max}(A_n^{(N)})}{|X_{i^*,j^*}|} \le \lambda_{\max}(C_N) - \varepsilon\right) = 0\,.$$

Proof. Let $(i^*, j^*) := \arg\max_{1\le i\le j\le n} |X_{i,j}|$ and $E_n := \{(i^*, j^*) \in \widetilde{T}\}$, where $\widetilde{T}$ is as defined in Step 1 of the proof of Lemma 2.1. It's easy to see that

$$\lim_{n\to\infty} P(E_n) = 1\,. \tag{2.6}$$

Let $v \in \mathbb{R}^n$ be the unit vector whose $(i^* - N)^{\text{th}}, (i^* - N + 1)^{\text{th}}, \ldots, i^{*\text{th}}$ and $(j^* - N)^{\text{th}}, (j^* - N + 1)^{\text{th}}, \ldots, j^{*\text{th}}$ coordinates are $1/\sqrt{2(N+1)}$ and the other coordinates are zero. Now it is easy to see that on E_n we have,

$$\left|v^T A_n^{(N)} v - |X_{i^*,j^*}| v^T C_N v\right| \le \sum_{(k,l)\in([i^*-N,i^*]\times[j^*-N,j^*])\setminus\{(i^*,j^*)\}} |u_{kl}||X_{k,l}|\,,$$

where $u_{k,l}$ are fixed numbers independent of n. Thus by Rayleigh's characterization, we have that for fixed $\varepsilon > 0$,

$$\begin{aligned}
&P\left(\frac{\lambda_{\max}(A_n^{(N)})}{|X_{i^*,j^*}|} \le \lambda_{\max}(C_N) - \varepsilon\right) \\
&\le P(E_n^c) + P\left(\sum_{(k,l)\in([i^*-N,i^*]\times[j^*-N,j^*])\setminus\{(i^*,j^*)\}} |u_{kl}||X_{k,l}| > \varepsilon |X_{i^*,j^*}|\right),
\end{aligned}$$

from which Lemma 2.2 follows using (2.6) since for each (k,l) in $[i^* - N, i^*] \times [j^* - N, j^*])$, the ratio $X_{k,l}/X_{i^*,j^*}$ converges to zero in probability. □

3. Infinite linear random field

Let $\{X_{i,j}\}$ be as before, that is, regularly varying $-\alpha$ with $\alpha \in (0,1)$ and it is easy to see that b_n satisfies

$$n^2 P(X_{i,j} > b_n x) \to x^{-\alpha}.$$

Note that $b_n = n^{\frac{2}{\alpha}} \tilde{L}(n)$ for slowly varying function $\tilde{L}(n)$. The following lemma asserts that if the truncation level of the linear random field goes to infinity then one can approximate the original matrix A_n with the matrix with truncated entries.

Lemma 3.1. *Let $A_n^{(N)}$ and A_n are $n \times n$ symmetric matrices with entries given by (2.1) and (1.1), respectively. Then the following holds,*

$$\lim_{N\to\infty} \limsup_{n\to\infty} P\left(|\rho(A_n) - \rho(A_n^{(N)})| > b_n \eta\right) = 0,$$

and,

$$\lim_{N\to\infty} \limsup_{n\to\infty} P\left(|\lambda_{\max}(A_n) - \lambda_{\max}(A_n^{(N)})| > b_n \eta\right) = 0.$$

Proof. It is well known that

$$\left|\lambda_{\max}(A_n) - \lambda_{\max}(A_n^{(N)})\right| \le \left|\rho(A_n) - \rho(A_n^{(N)})\right| \le 2 \max_{1\le k\le n} \sum_{l=k}^{n} |Y_{k,l} - Y_{k,l}^{(N)}|.$$

Therefore, it is enough to show the following two limits

$$\lim_{N\to\infty} \limsup_{n\to\infty} \sum_{k=1}^{n} P\left(\sum_{l=k}^{n}\sum_{i=0}^{N}\sum_{j=N+1}^{\infty} |c_{i,j}| X_{i+k,j+l} > b_n \frac{\eta}{4}\right) = 0, \tag{3.1}$$

and

$$\lim_{N\to\infty} \limsup_{n\to\infty} \sum_{k=1}^{n} P\left(\sum_{l=k}^{n}\sum_{i=N+1}^{\infty}\sum_{j=0}^{\infty} |c_{i,j}| X_{i+k,j+l} > b_n \frac{\eta}{4}\right) = 0. \tag{3.2}$$

To establish (3.1), note that the expression inside the limits can be bounded by

$$\sum_{k=1}^{n}\sum_{l=k}^{n}\sum_{i=0}^{N}\sum_{j=N+1}^{\infty} P\left(|c_{i,j}| X_{i+k,j+l} > b_n\frac{\eta}{4}\right) + \sum_{k=1}^{n}\sum_{l=k}^{n}\sum_{i=0}^{N}\sum_{j=N+1}^{\infty} \frac{4|c_{i,j}|}{b_n\eta} E\left(X_{i+k,j+l}\mathbb{I}(|c_{i,j}|X_{i+k,j+l} \le b_n\frac{\eta}{4})\right).$$

Since $\alpha \in (0,1)$, by Karamata's theorem each $E\left(X_{i+k,j+l}\mathbb{I}(X_{i+k,j+l} \le t)\right) \sim \frac{\alpha}{1-\alpha} t(1-F(t))$ as $t \to \infty$ and hence by applying Potter bound (see Proposition 2.6 in Resnick (2007)) on both the terms above, we can bound their sum by

$$\le \sum_{k=1}^{n}\sum_{l=k}^{n}\sum_{i=0}^{N}\sum_{j=N+1}^{\infty} O(n^{-2})|c_{ij}|^{\delta},$$

from which (3.1) follows using (1.2) as the $O(n^{-2})$ term above does not depend on i, j, k and l. Similarly, we can also establish (3.2). This completes the proof. ☐

The next lemma evaluates the limit of spectral radius and maximum eigenvalue of the coefficient matrix C_N defined in (2.2). This result was applied to establish Corollary 1.2 and will be used in the proof of Theorem 1.1 as well.

Lemma 3.2. *Let C_N be as in (2.2) and the coefficients satisfy (1.2) with $0 < \alpha < 1$. Then we have,*

(a) *As $N \to \infty$, $\rho(C_N) \to \|T\|$ where T is defined in (1.5) and $\|\cdot\|$ denotes the operator norm.*
(b) *$\lambda_{\max}(C_N) = \rho(C_N)$ and hence the convergence of maximum eigenvalue follows from (a).*

Proof. Part (b) is obvious from the structure of the matrix C_N and hence we just briefly sketch a proof of Part (a). To this end, note that by pre and post-multiplying C_N by a permutation matrix, it is easy to see that for all $N \ge 1$, $\rho(C_N) = \rho(T_N)$, where $T_N(i,j) := T(i,j)\mathbf{1}(|i| \vee |j| \le N)$ with T is given by (1.5). Furthermore, T_N

is a self adjoint operator, and hence by Spectral Theorem, its spectral radius and operator norm are equal. As $N \to \infty$ we have,

$$\|T_N - T\|^2 \leq \sum_{i,j} (T_N(i,j) - T(i,j))^2 = \sum_{(i,j)\in\mathbb{Z}^2:|i|\vee|j|>N} T(i,j)^2 \to 0,$$

from which Part (a) follows. □

3.1. **Proof of Theorem 1.1.** We denote by Z_α a Frechét random variable with parameter α. Using Proposition 1.11 of Resnick (1987) and Lemmas 2.1, 2.2 and 3.2 above, we have that for each fixed truncation level N when $n \to \infty$, both $\lambda_{\max}(A_n^{(N)})/b_n$ and $\rho(A_n^{(N)})/b_n$ converge weakly to $\rho(C_N)Z_\alpha$, which in turn converges to $\|T\|Z_\alpha$ when $N \to \infty$. Therefore by Lemma 3.1, Theorem 1.1 follows.

Remark 3.1. The proofs of Lemma 3.1 and Step 4 of Lemma 2.1 rely heavily on the fact that $0 < \alpha < 1$. If these can be established (with additional assumptions if necessary) for higher values of α, then our result can be extended.

Acknowledgment. The authors are thankful to Robert Adler, Debashish Goswami, Debashis Paul and Ofer Zeitouni for their valuable suggestions.

References

A. Auffinger, G. Ben Arous, and S. Péché. Poisson convergence for the largest eigenvalues of heavy tailed random matrices. *Ann. Inst. Henri Poincaré Probab. Stat.*, 45(3):589–610, 2009.

S. Belinschi, A. Dembo, and A. Guionnet. Spectral measure of heavy tailed band and covariance random matrices. *Comm. Math. Phys.*, 289(3):1023–1055, 2009.

G. Ben Arous and A. Guionnet. The spectrum of heavy tailed random matrices. *Comm. Math. Phys.*, 278(3):715–751, 2008.

G. Ben Arous and A. Guionnet. Wigner matrices. In *The Oxford handbook of random matrix theory*, pages 433–451. Oxford Univ. Press, Oxford, 2011.

C. Bordenave, P. Caputo, and D. Chafaï. Spectrum of non-Hermitian heavy tailed random matrices. *Comm. Math. Phys.*, 307(2):513–560, 2011.

P. Bourgade, L. Erdős, and H.-T. Yau. Edge universality for beta ensembles. *arXiv: 1306.5728*, 2013.

D. B. H. Cline. Infinite series of random variables with regularly varying tails. Technical Report 83–24, Insitutute of Applied Mathematics and Statistics, University of British Columbia, Vancouver, B. C., 1983. URL `http://www.stat.tamu.edu/~dcline/Papers/infiniteseries.pdf`.

R. A. Davis and S. I. Resnick. Limit theory for moving averages of random variables with regularly varying tail probabilities. *The Annals of Probability*, 13(1):179–195, 1985.

R. A. Davis, O. Pfaffel, and R. Stelzer. Limit theory for the largest eigenvalues of sample covariance matrices with heavy-tails. *arXiv:1108.5464*, to appear in *Stochastic Processes and their Applications*, 2011.

L. Erdős and H.-T. Yau. Universality of local spectral statistics of random matrices. *Bull. Amer. Math. Soc. (N.S.)*, 49(3):377–414, 2012.

W. Hachem, P. Loubaton, and J. Najim. The empirical eigenvalue distribution of a gram matrix: from independence to stationarity. *Markov Process. Related Fields*, 11(4):629–648, 2005.

J. O. Lee and J. Yin. A necessary and sufficient condition for edge universality of wigner matrices. *arXiv: 1306.5728*, 2013.

O. Pfaffel and E. Schlemm. Limiting spectral distribution of a new random matrix model with dependence across rows and columns. *Linear Algebra Appl.*, 436(9): 2966–2979, 2012.

S. I. Resnick. *Extreme Values, Regular Variation and Point Processes*. Springer-Verlag, New York, 1987.

S. I. Resnick. *Heavy-Tail Phenomena : Probabilistic and Statistical Modeling*. Springer, New York, 2007.

A. Soshnikov. Poisson statistics for the largest eigenvalues of Wigner random matrices with heavy tails. *Electron. Comm. Probab.*, 9:82–91 (electronic), 2004. ISSN 1083-589X. doi: 10.1214/ECP.v9-1112.

Y. Q. Yin, Z. D. Bai, and P. R. Krishnaiah. On the limit of the largest eigenvalue of the large-dimensional sample covariance matrix. *Probab. Theory Related Fields*, 78(4):509–521, 1988.

Theoretical Statistics and Mathematics Unit, Indian Statistical Institute, New Delhi
E-mail address: arijit@isid.ac.in

Institut für Mathematik,Universität Zürich, Zürich
E-mail address: rajat.hazra@math.uzh.ch

Theoretical Statistics and Mathematics Unit, Indian Statistical Institute, Kolkata
E-mail address: parthanil@isical.ac.in