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Choice Experiments for Estimating Main Effects and Selected Interaction Effects

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Abstract
Choice experiments are often used to gather information on preferences of consumers for products and services when the choice alternatives can be described in terms of attributes, each such attribute having two or more settings or, levels. We assume that each attribute has two levels and the interest is to find optimal or efficient designs for estimating the main effects of the attributes and possibly some interactions as well. In this paper we provide such efficient designs for estimating all the main effects and a subset of two-factor interactions among the attributes.

Keywords: Choice experiments, Hadamard matrices, Pareto optimal subsets.

1 Introduction and Preliminaries

Choice experiments are now widely used to gather information on preferences for products and services. In a choice experiment, respondents are presented sets of profiles, called choice sets, and asked to select the one they consider best. The design and analysis of choice experiments has been studied quite extensively in recent years and for an excellent discussion on these aspects, one might refer to Street and Burgess (2007), where more references can be found.

This paper deals with finding designs for choice experiments where the choice sets are Pareto optimal (Pareto optimal choice sets are defined later in this section). We assume that all the options in each choice set are described by several attributes (or, factors) and that each attribute has two levels. In valuation studies, some profiles are better than others, or dominating, while some profiles are worse, or dominated by others. Choosing a dominating alternative, or not choosing a dominated one, does not involve an economic choice. If a choice

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set has a dominating profile, the respondent's choice is trivially made. Similarly, if a choice set has a dominated profile, it will never be selected. In view of this, in this paper we restrict our attention to choice sets with no dominating or dominated profiles. Such choice sets are known as Pareto Optimal (PO) subsets. If choice sets have a large number of profiles, then they may be difficult to implement. In view of this, one desires strategies for reducing the number of profiles. In this paper, we attempt to provide such a strategy. We consider a situation where apart from the main effects of the attributes, a subset of two-factor interactions is also of interest. Specifically, we consider two-factor interactions that have one attribute in common.

Consider a choice experiment with m attributes each at two levels, coded as, 0, 1. Each profile consists of a specific level of each of the m attributes. Each choice set is a subset of k profiles, $k \leq 2^m$. Let Ω be the set of all 2^m profiles and let a typical profile be denoted as $\boldsymbol{x} = x_1x_2\cdots x_m, x_j = 0, 1, 1 \leq j \leq m$. Without loss of generality, let us suppose that the profiles are lexicographically ordered. Let $y_{\boldsymbol{x}}$ denote the proportion of respondents choosing the profile $\boldsymbol{x} \in \Omega$. We consider a linear model that includes the general mean μ, all the main effects and the two-attribute interaction effects with one common attribute. The model can be written as

$$y_{x_1x_2\ldots x_m} = \mu + \sum_{i=1}^{m} \beta_{x_i}^{F_i} + \sum_{j=2}^{m} \beta_{x_1x_j}^{F_1F_j} + \epsilon_{x_1\ldots x_m} \tag{1}$$

where $y_{x_1\ldots x_m}$ is the proportion of respondents choosing profile $\boldsymbol{x} = (x_1, \ldots, x_m)$ from a given choice set, $x_i = 0, 1$, μ is a general mean, $\beta_{x_i}^{F_i}$ is the effect of attribute F_i at level x_i, $\beta_{x_1x_j}^{F_1F_j}$ is the two-factor interaction component (without loss of generality) between the attribute F_1 and F_j and $\epsilon_{x_1\ldots x_m}$ is the random error term.

According to Raghavarao and Wiley (1998), a design for a choice experiment is said to be connected if the β parameters as in (1) are all estimable under the design. Chen and Chitturi (2012) proved that the PO choice sets $S_1^* = \{100\cdots 0, 010\cdots 0, 000\cdots 1\}$ and $S_2^* = \{x_1x_2\cdots x_m | \sum_{j=1}^{m} x_j = 2, x_1 + x_n \neq 0\}$ give a PO connected design under the model (1). Note that these designs are saturated in the sense that the number of unknown parameters equals the total number of available responses. They also provided a formula for obtaining Information Per Profile (IPP) and showed that for a 2^3 design, the IPP is 0.4 and for a 2^5 design it is 0.1818. In this paper, we provide alternative PO designs under the model (1) which are highly efficient.

2 Main results

In this section, we deal with the derivation of PO designs based on Hadamard matrices under model (1). Recall that a square matrix H_n of order n with entries ± 1 is called a Hadamard matrix if $H_nH_n' = nI_n = H_n'H_n$, where I_n is the identity matrix of order n and primes denote transposition. Without loss of generality, one can assume that the first row and first column of H_n consists of only $+1$'s and in that case, we say that H_n is in its *normal form.* Also, let J denote a matrix (of appropriate order) whose elements are all 1, $\mathbf{1}_u$ be a $u \times 1$ vector of all ones and $\mathbf{0}_u$, a $u \times 1$ null vector. We consider three possible values of m.

Case 1: $m = 2^r,\ r \geq 2$ an integer.

Suppose our objective is to obtain PO designs with $m = 2^r$, for some integer $r(\geq 2)$ attributes. To that end, we begin with H_4, the following Hadamard matrix of order 4:

$$H_4 \quad = \quad \begin{pmatrix} -1 & 1 & 1 & 1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix}.$$

For obtaining the desired choice sets, we define the following matrices:

$$H_m \quad = \quad H_2 \otimes \cdots \otimes H_2 \otimes H_4, \tag{2}$$

where

$$H_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

appears $r-2$ times in the above expression and $\otimes$ denotes the Kronecker (tensor) product of matrices. Clearly, H_m, where $m = 2^r$, is a Hadamard matrix of order m. Furthermore, let

$$U_m = (1/2)[H_m + J], \;\; \bar{U}_m = (1/2)[-H_m + J], \;\; L_m = \begin{pmatrix} U_m \\ \bar{U}_m \end{pmatrix}. \tag{3}$$

With the above background, one can construct PO choice sets for appropriate number of attributes. Raghavarao and Wiley (1998) noted that the choice sets $S_l = \{\boldsymbol{x} = x_1x_2\cdots x_m | \sum x_j = \ell\}, \ell = 0, 1, \cdots, m$ are PO. Moreover, for any fixed $\ell, \ell = 0, 1, \cdots, m$, it is easy to observe that any subset of S_ℓ is also a PO choice set. The following theorem shows that the procedure stated above can be used to obtain PO choice sets from Hadamard matrices of order m (≥ 4).

Theorem 1. For any $m = 2^r, r \geq 2$, the above stated procedure provides set of profiles U_m such that one can obtain the PO choice sets $S^*_{m/4} \subset S_{m/4}$, $S^*_{m/2} \subset S_{m/2}$ and $S^*_{3m/4} \subset S_{3m/4}$ with cardinality 4, $2(m-4)$, and 4 respectively.

Proof. Take $r = 2$. Then,

$$U'_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \;\; \bar{U}'_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \;\; L'_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix},$$

which in turn give the PO choice sets $S_1^* = \{1000, 0100, 0010, 0001\}$ and $S_3^* = \{1110, 1101, 1011, 0111\}$.

In continuation of (3), define

$$\bar{L}_m = \begin{pmatrix} \bar{U}_m \\ U_m \end{pmatrix}. \tag{4}$$

Now suppose $r = 3$. Then from (2) and (3), we get

$$U_8 = \begin{pmatrix} U_4 & U_4 \\ U_4 & \bar{U}_4 \end{pmatrix}, \quad \bar{U}_8 = \begin{pmatrix} \bar{U}_4 & \bar{U}_4 \\ \bar{U}_4 & U_4 \end{pmatrix}, \quad L_8 = \begin{pmatrix} U_8 \\ \bar{U}_8 \end{pmatrix} = \begin{pmatrix} L_4 & L_4 \\ L_4 & \bar{L}_4 \end{pmatrix}.$$

It is then easy to observe that L_8 provides the subsets $S_2^* \subset S_2$, $S_4^* \subset S_4$ and $S_6^* \subset S_6$ with cardinalities 4, 8, and 4 respectively. Suppose Theorem 1 is true for some arbitrary $k, k > 2$. Let $m_1 = 2^k$ and $m_2 = 2^{k+1} = 2m_1$. Then from (2) and (3), we have

$$U_{m_2} = \begin{pmatrix} U_{m_1} & U_{m_1} \\ U_{m_1} & \bar{U}_{m_1} \end{pmatrix}, \quad \bar{U}_{m_2} = \begin{pmatrix} \bar{U}_{m_1} & \bar{U}_{m_1} \\ \bar{U}_{m_1} & U_{m_1} \end{pmatrix}, \quad L_{m_2} = \begin{pmatrix} U_{m_2} \\ \bar{U}_{m_2} \end{pmatrix} = \begin{pmatrix} L_{m_1} & L_{m_1} \\ L_{m_1} & \bar{L}_{m_1} \end{pmatrix}. \tag{5}$$

Since Theorem 1 holds for $m = m_1$, L_{m_1} gives the PO choice sets $S^*_{m_1/4} \subset S_{m_1/4}, S^*_{m_1/2} \subset S_{m_1/2}$ and $S^*_{3m_1/4} \subset S_{3m_1/4}$ with cardinalities 4, $2(m_1 - 4)$ and 4, respectively. This implies that the set of profiles given by (L_{m_1}, L_{m_1}) provides PO choice sets $S^*_{m_2/4} \subset S_{m_2/4}, S^*_{m_2/2} \subset S_{m_2/2}$ and $S^*_{3m_2/4} \subset S_{3m_2/4}$ with cardinalities 4, $2(m_1 - 4)$ and 4, respectively. Again, the set of profiles given by $(L_{m_1}, \bar{L}_{m_1})$ provides choice set $S^*_{m_2/2} \subset S_{m_2/2}$ with cardinality $2m_1$. Hence the set of profiles given by (5) provides PO choice sets $S^*_{m_2/4} \subset S_{m_2/4}, S^*_{m_2/2} \subset S_{m_2/2}$ and $S^*_{3m_2/4} \subset S_{3m_2/4}$ with cardinalities 4, $2(m_2 - 4)$ and 4, respectively. This completes the proof of Theorem 1.

Case 2: $m = 3(2^{k+1}), k > 1$.

Consider

$$H_{12} = \begin{bmatrix}
- & - & - & - & - & - & + & - & - & + & - & + \\
- & - & - & + & - & - & - & + & - & - & + & - \\
- & - & + & - & + & - & + & + & + & - & - & - \\
- & - & + & - & - & + & - & - & + & - & + & + \\
- & - & + & + & + & + & - & - & - & + & - & - \\
- & + & - & + & + & - & - & - & + & - & - & + \\
- & + & - & - & - & + & - & + & + & + & - & - \\
- & + & - & - & + & + & + & - & - & - & + & - \\
- & + & + & + & - & + & + & + & - & - & - & + \\
- & + & + & + & - & - & + & - & + & + & + & - \\
- & + & + & - & + & - & - & + & - & + & + & + \\
- & - & - & + & + & + & + & + & + & + & + & +
\end{bmatrix}, \tag{6}$$

and let

$$H_m = H_2 \otimes \cdots \otimes H_2 \otimes H_{12}, \tag{7}$$

where $m = 3(2^{k+1})$, $k > 1$, H_2 appears in (7) $(k-1)$ times and $+$ and $-$ in (6) stand for $+1$ and -1, respectively. Also, define the matrices $U_m, \bar{U}_m$ and L_m, as in Case 1, with $m = 3(2^{k+1})$. We then have the following result.

Theorem 2. Starting from a Hadamard matrix of order 12 as given by (6) and following (2) and (3) with H_4 there replaced by H_{12}, one can obtain a series of choice sets with $m = 3(2^{k+1}), k > 1$ attributes. These PO choice sets are $S^*_{3m/12} \subset S_{3m/12}$, $S^*_{5m/12} \subset S_{5m/12}$, $S^*_{6m/12} \subset S_{6m/12}$, $S^*_{7m/12} \subset S_{7m/12}$, and $S^*_{9m/12} \subset S_{9m/12}$ with cardinalities 3, 9, $2(m-12)$, 9 and 3 respectively.

Case 3: $m = m = 5(2^{k+1}), k > 1$.

Consider

$$H_{20} = \begin{bmatrix}
- & - & - & - & - & - & - & + & - & - & + & + & - & - & - & - & + & - & + & - \\
- & - & - & - & + & - & + & - & - & + & + & - & - & - & - & + & - & + & - & + \\
- & - & + & - & - & + & + & - & - & + & - & - & + & + & - & - & - & - & + & - \\
- & + & - & - & - & + & - & - & + & - & - & + & + & - & - & - & - & + & - & + \\
- & - & - & + & + & - & - & + & - & + & - & + & + & + & + & - & - & + & - & - \\
- & - & - & + & + & + & - & - & + & + & - & - & - & - & + & - & + & - & + & + \\
- & - & + & - & + & + & + & + & + & - & - & + & - & - & + & + & - & - & - & - \\
- & - & + & + & - & - & - & + & + & - & - & - & - & + & - & + & - & + & + & + \\
- & - & + & + & - & + & - & - & - & - & + & - & + & - & + & + & + & + & - & - \\
- & + & + & - & - & - & - & - & - & + & - & + & - & + & + & + & + & - & - & + \\
- & + & - & - & + & - & - & - & + & - & + & - & + & + & + & + & - & - & + & - \\
- & + & - & + & - & - & + & + & + & + & - & - & + & - & - & + & + & - & - & - \\
- & + & - & + & - & + & + & + & - & - & + & - & - & + & + & - & - & - & - & + \\
- & - & + & + & + & - & + & - & + & - & + & + & + & + & - & - & + & - & - & + \\
- & + & + & + & - & - & + & - & + & + & + & + & - & - & + & - & - & + & + & - \\
- & + & + & - & + & + & - & + & + & + & + & - & - & + & - & - & + & + & - & - \\
- & + & + & - & + & - & + & + & - & - & - & - & + & - & + & - & + & + & + & + \\
- & + & - & + & + & + & + & - & - & - & - & + & - & + & - & + & + & + & + & - \\
- & + & + & + & + & + & - & + & - & + & + & + & + & - & - & + & - & - & + & + \\
- & - & - & - & - & + & + & + & + & + & + & + & + & + & + & + & + & + & + & +
\end{bmatrix}, \tag{8}$$

and let

$$H_m = H_2 \otimes \cdots \otimes H_2 \otimes H_{20}, \tag{9}$$

where H_2 appears$(k-1)$ times in (9) and $+1$ and -1 in (8), as before, stand for $+1$ and -1, respectively. The matrices $U_m, \bar{U}_m$ and L_m are defined as earlier, with the appropriate value of m in this case. We then have the following result.

Theorem 3. Starting from a Hadamard matrix of order 20 as in (8) and following (2) and (3), one can obtain a series of choice sets with $m = 5(2^{k+1}), k > 1$ attributes. These PO choice sets are $S^*_{5m/20} \subset S_{5m/20}$, $S^*_{7m/20} \subset S_{7m/20}$, $S^*_{9m/20} \subset S_{9m/20}$, $S^*_{10m/20} \subset S_{10m/20}$, $S^*_{10m/20} \subset S_{10m/20}$, $S^*_{11m/20} \subset S_{11m/20}$, $S^*_{13m/20} \subset S_{13m/20}$ and $S^*_{15m/20} \subset S_{15m/20}$ with cardinality 2, 4, 14, $2(m-20)$, 14, 4 and 2 respectively.

The proofs of Theorems 2-3 follow along the line of the proof of Theorem 1.

The next theorem presents a series of PO choice sets with $m = 4t - 1, t \geq 1$ attributes and a total of $n = 2(4t - 1)$ profiles.

Theorem 4. Let H be a Hadamard matrix of order $4t, t \geq 1$ in its normal form and B_m be a square matrix of order $m = 4t - 1$ obtained from H after deleting the first row and the first column. Define, as earlier,

$$U_m = (1/2)[B_m + J], \;\; \bar{U}_m = (1/2)[-B_m + J], \;\; L_m = \begin{pmatrix} U_m \\ \bar{U}_m \end{pmatrix}.$$

Then the set of $2m$ profiles can be partitioned into two PO choice sets $S^*_{(m-2)/2} \subset S_{(m-1)/2}$ and $S^*_{(m+1)/2} \subset S_{(m+1)/2}$ with cardinalities m each.

Proof. The proof of this theorem follows by noting that in each column of B_m, there are exactly $(m + 1)/2$ -1's and $(m - 1)/2$ +1's.

3 Efficiency

In this section, we examine the properties of the proposed designs based on the IPP criterion. Let us write the design matrix of model (1) as

$$X \;=\; [\mathbf{1}_n \; X_1] \;=\; [\mathbf{1}_n \; X_M \; X_I],$$

where X_M is the design matrix corresponding the main effects and X_I, the design matrix corresponding to the two-factor interaction effects included in the model. It is easy to see that information matrix under (1) (including μ) is given by

$$\mathcal{I} \;=\; \begin{pmatrix} n & \mathbf{1}_n' X_1 \\ X_1' \mathbf{1}_n & X_1' X_1 \end{pmatrix},$$

and the information matrix after eliminating μ is

$$\mathcal{I}^* \;=\; X_1' X_1 - n^{-1} X_1' \mathbf{1}_n \mathbf{1}_n' X_1$$

To compare designs obtained from different PO choice sets, we define along the line of Raghavarao and Wiley (1998), the Information Per Profile (θ) in the design as an optimality criterion. According to them, the Information Per Profile (IPP(θ)) is given by

$$\text{IPP}(\theta) \;=\; \frac{2m-1}{n\ \text{trace}(\mathcal{I}^{*-1})}$$

Theorem 5. The choice sets obtained through Theorems 1-3 with a total of $n = 2m$ profiles and m attributes are connected main effects plans and are capable of estimating two-attribute interaction effects that includes one attribute in common. Moreover, for these designs, we have IPP(θ) $= 1$ and thus these designs are optimal.

Proof. The proof of this theorem follows from the fact that

$$\mathcal{I}^* \;=\; nI_{2m-1}.$$

Theorem 6. The choice sets obtained through Theorem 4 with a total of $n = 2m$ profiles and $m = 4t-1, t \geq 1$ attributes are connected main effects plans and are capable of estimating two-attribute interaction effects that includes one attribute in common.

Proof. Without loss of generality, let us suppose that the model includes interaction effects with the first attribute and the rest. Let us rewrite the design matrix of model (1) as

$$X \;=\; [\mathbf{1}_n\ \boldsymbol{x}_1\ \boldsymbol{x}_2\ \cdots\ \boldsymbol{x}_m\ \boldsymbol{x}_1 \odot \boldsymbol{x}_2\ \boldsymbol{x}_1 \odot \boldsymbol{x}_3\ \ldots \boldsymbol{x}_1 \odot \boldsymbol{x}_m],$$

where for two vectors $\boldsymbol{a} = (a_1, a_2, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, b_2, \ldots, b_n)$, $\boldsymbol{a}\odot\boldsymbol{b} = (a_1b_1, a_2b_2, \ldots, a_nb_n)$. It is to be noted that

$$\begin{aligned}&\mathbf{1}_n'\boldsymbol{x}_j = 0, 1 \leq j \leq m;\ \mathbf{1}_n'(\boldsymbol{x}_1 \odot \boldsymbol{x}_j) = -2,\ 2 \leq j \leq m;\ \boldsymbol{x}_j'\boldsymbol{x}_j = n,\ 1 \leq j \leq m, \boldsymbol{x}_j'\boldsymbol{x}_k = -2,\\ &1 \leq j < k \leq m,\ \boldsymbol{x}_1'(\boldsymbol{x}_1 \odot \boldsymbol{x}_j) = 0,\ 2 \leq j \leq m,\ \boldsymbol{x}_j'(\boldsymbol{x}_1 \odot \boldsymbol{x}_k) = 0,\ 2 \leq j, k \leq m,\\ &(\boldsymbol{x}_1 \odot \boldsymbol{x}_j)'(\boldsymbol{x}_1 \odot \boldsymbol{x}_j) = n, (\boldsymbol{x}_1 \odot \boldsymbol{x}_j)'(\boldsymbol{x}_1 \odot \boldsymbol{x}_k) = -2,\ 2 \leq j < k \leq m.\end{aligned}$$

Thus we get

$$\mathcal{I} \;=\; X'X \;=\; \begin{pmatrix} n & \mathbf{0}_m' & -2\mathbf{1}_{m-1}' \\ \mathbf{0}_m & (n+2)I_m - 2J & O \\ -2\mathbf{1}_{m-1} & O & (n+2)I_{m-1} - 2J \end{pmatrix},$$

which gives

$$\mathcal{I}^* \;=\; \begin{pmatrix} (n+2)I_m - 2J & O \\ O & (n+2)I_{m-1} - 2(1+2/n)J \end{pmatrix},$$

$$
{\mathcal{I}^*}^{-1} = \begin{pmatrix} \frac{1}{n+2}\left[I_m + J\right] & O \\ O & \frac{1}{n+2}\left[I_{m-1} + J\right] \end{pmatrix},
$$

The Information Per Profile (IPP(θ)) is

$$
\mathrm{IPP}(\theta) = \frac{2m-1}{n \ \mathrm{trace}({\mathcal{I}^*}^{-1})} = \frac{n+2}{2n}.
$$

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