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Inertia of the matrix $[(p_i + p_j)^r]$

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INERTIA OF THE MATRIX $[(p_i + p_j)^r]$

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Abstract. Let $p_1, \dots, p_n$ be positive real numbers. It is well known that for every $r < 0$ the matrix $[(p_i + p_j)^r]$ is positive definite. Our main theorem gives a count of the number of positive and negative eigenvalues of this matrix when $r > 0$. Connections with some other matrices that arise in Loewner's theory of operator monotone functions and in the theory of spline interpolation are discussed.

1. Introduction

Let $p_1, p_2, \dots, p_n$ be distinct positive real numbers. The $n \times n$ matrix $C = [\frac{1}{p_i + p_j}]$ is known as the *Cauchy matrix*. The special case $p_i = i$ gives the *Hilbert matrix* $H = [\frac{1}{i+j}]$. Both matrices have been studied by several authors in diverse contexts and are much used as test matrices in numerical analysis.

The Cauchy matrix is known to be positive definite. It possesses a stronger property: for each $r > 0$ the entrywise power $C^{\circ r} = \left[\frac{1}{(p_i+p_j)^r}\right]$ is positive definite. (See [4] for a proof.) The object of this paper is to study positivity properties of the related family of matrices

$$P_r = [(p_i + p_j)^r], \quad r \geq 0. \tag{1}$$

The *inertia* of a Hermitian matrix A is the triple

$$\operatorname{In}(A) = (\pi(A), \zeta(A), \nu(A)),$$

in which $\pi(A)$, $\zeta(A)$ and $\nu(A)$ stand for the number of positive, zero, and negative eigenvalues of A, respectively. Our main result is the following.

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Theorem 1. *Let $p_1, \ldots, p_n$ be distinct positive real numbers, and let P_r be the $n \times n$ matrix defined in* (1). *Then*

(i) *P_r is singular if and only if r is a nonnegative integer smaller than $n-1$.*

(ii) *If r is an integer and $0 \le r \le n-1$, then*

$$\mathrm{In}\ P_r = \left(\left\lceil \frac{r+1}{2} \right\rceil, n-(r+1), \left\lfloor \frac{r+1}{2} \right\rfloor \right).$$

(iii) *Suppose r is not an integer, and $0 < r < n-2$.*

$$\begin{aligned} \textit{If } \lfloor r \rfloor &= 2k \textit{ for some integer } k, \textit{then} \\ \mathrm{In}\ P_r &= (k+1, 0, n-(k+1)), \\ \textit{and if } \lfloor r \rfloor &= 2k+1 \textit{ for some integer } k, \textit{then} \\ \mathrm{In}\ P_r &= (n-(k+1), 0, k+1). \end{aligned}$$

(iv) *For every real number $r > n-2$*

$$\mathrm{In}\ P_r = \mathrm{In}\ P_{n-1}.$$

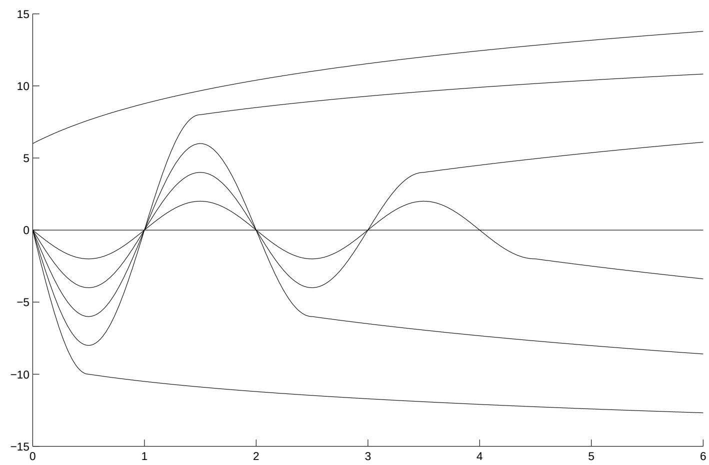

FIGURE 1

Figure 1 is a schematic representation of the eigenvalues of a 6×6 matrix P_r, when p_i have been fixed and r varies. This kind of behaviour has been observed in other problems. The *Loewner matrix* is defined as

$$L_r = \left[\frac{p_i^r - p_j^r}{p_i - p_j}\right], \quad r \geq 0.$$

It is a famous theorem of C. Loewner [2, 3] that for $0 < r < 1$, the matrix L_r is positive definite. R. Bhatia and J. Holbrook [5] showed that for $1 < r < 2$, the matrix L_r has only one positive eigenvalue. This encouraged them to speculate what might happen for other values of r. They made a conjecture that, in the light of our Theorem 1, may be rephrased as

Conjecture 1. *For all* $r > 0$, $\text{In } P_r = \text{In } L_{r+1}$.

The conjecture of Bhatia and Holbrook remains unproved, except that it was shown by R. Bhatia and T. Sano [7] that for $2 < r < 3$ the matrix L_r has only one negative eigenvalue.

The matrix

$$B_r = [|p_i - p_j|^r], \quad r \geq 0$$

has been studied widely in connection with interpolation of scattered data and spline functions. In [8] N. Dyn, T. Goodman and C. A. Micchelli establish inertia properties of this matrix as r varies. Some of our proofs can be adapted to achieve substantial simplifications of those in [8].

Closely related to Loewner matrices is the matrix

$$K_r = \left[\frac{p_i^r + p_j^r}{p_i + p_j}\right], r \geq 0.$$

M. K. Kwong [9] showed that K_r is positive definite when $0 < r < 1$. Bhatia and Sano [7] showed that K_r has only one positive eigenvalue when $1 < r < 3$. In our ongoing work [6] we have carried this analysis further, and are led to

Conjecture 2. *For all* $r > 0$, $\text{In } B_r = \text{In } K_{r+1}$.

The rest of the paper is devoted to the proof of Theorem 1 followed by some remarks.

2. The case $r \leq n - 1$, r an integer

The Sylvester Law of Inertia says that if A and X are $n \times n$ matrices, and A is Hermitian and X nonsingular, then $\text{In } X^*AX = \text{In } A$. We need a small generalisation of this given in the next proposition.

Proposition 2. *Let $n \geq r$. Let A be an $r \times r$ Hermitian matrix, and X an $r \times n$ matrix of rank r. Then*

$$\operatorname{In} X^*AX = \operatorname{In} A + (0, n-r, 0).$$

Proof. The matrix X has a singular value decomposition $X = U\Sigma V^*$, in which $U \in \mathbb{C}^{r\times r}, V \in \mathbb{C}^{n\times n}, \Sigma \in \mathbb{C}^{r\times n}$; U and V are unitary, and Σ can be partitioned as $\Sigma = [S, O]$, where S is an $r \times r$ positive diagonal matrix, and O is the null matrix of order $r \times (n-r)$. Then

$$X^*AX = V\Sigma^*U^*AU\Sigma V^*.$$

By Sylvester's Law

$$\begin{aligned} \operatorname{In} X^*AX &= \operatorname{In} \Sigma^*U^*AU\Sigma \\ &= \operatorname{In} \begin{bmatrix} SU^*AUS & O \\ O & O \end{bmatrix}. \end{aligned}$$

In this last 2×2 block matrix, the top left block is an $r \times r$ matrix. So

$$\begin{aligned} \operatorname{In} X^*AX &= \operatorname{In} (SU^*AUS) + (0, n-r, 0) \\ &= \operatorname{In} A + (0, n-r, 0). \end{aligned}$$

■

Now let W be the $(r+1) \times n$ Vandermonde matrix

$$W = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ p_1 & p_2 & p_3 & \cdots & p_n \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ p_1^r & p_2^r & p_3^r & \cdots & p_n^r \end{bmatrix},$$

let V_1 be the $(r+1)\times(r+1)$ antidiagonal matrix with entries $\binom{r}{0}, \binom{r}{1}, \ldots, \binom{r}{r}$ on its sinister diagonal and 0's elsewhere; i.e.,

$$V_1 = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & \binom{r}{0} \\ 0 & 0 & \cdots & 0 & \binom{r}{1} & 0 \\ 0 & 0 & \cdots & \binom{r}{2} & 0 & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ \binom{r}{r} & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}.$$

It can be seen that for $r \leq n-1$

$$P_r = W^*V_1W.$$

So by Proposition 2

$$\operatorname{In} P_r = \operatorname{In} V_1 + (0, n-(r+1), 0). \tag{2}$$

The inertia of V_1 can be computed as follows. When $r+1=2k$, the entries on the sinister diagonal of V_1 are

$$\left(\binom{r}{0},\binom{r}{1},\ldots,\binom{r}{k-1},\binom{r}{k-1},\ldots,\binom{r}{1},\binom{r}{0}\right).$$

The eigenvalues of V_1 are readily seen to be $\pm\binom{r}{j}$, $0\le j\le k-1$. So

$$\operatorname{In} V_1=(k,0,k)=\left(\frac{r+1}{2},0,\frac{r+1}{2}\right).$$

When $r+1=2k+1$, the entries on the sinister diagonal of V_1 are

$$\left(\binom{r}{0},\binom{r}{1},\ldots,\binom{r}{k-1},\binom{r}{k},\binom{r}{k-1},\ldots,\binom{r}{1}\binom{r}{0}\right).$$

In this case the eigenvalues of V_1 are $\pm\binom{r}{j}$, $0\le j\le k-1$, together with $\binom{r}{k}$. Thus

$$\operatorname{In} V_1=(k+1,0,k)=\left(\left\lceil\frac{r+1}{2}\right\rceil,0,\left\lfloor\frac{r+1}{2}\right\rfloor\right).$$

So, part (ii) of Theorem follows from (2).

3. The cases $0<r<1$ and $1<r<2$

Let $\mathcal{H}=\mathbb{C}^n$ and let $\mathcal{H}_1$ be its subspace $\mathcal{H}_1=\left\{x:\sum\limits_{j=1}^{n}x_j=0\right\}$. Let $e=(1,1,\ldots,1)$ and E the matrix with all its entries equal to 1. Then

$$\mathcal{H}_1=e^{\perp}=\{x:Ex=0\}.$$

A Hermitian matrix A is said to be *conditionally positive definite (cpd)* if $\langle x,Ax\rangle\ge 0$ for $x\in\mathcal{H}_1$. It is said to be *conditionally negative definite (cnd)* if $-A$ is cpd. Basic facts about such matrices can be found in [1]. A cpd matrix is nonsingular if $\langle x,Ax\rangle>0$ for all nonzero vectors x in $\mathcal{H}_1$.

If t is a positive number and $0<r<1$, then

$$t^r=\frac{\sin r\pi}{\pi}\int_0^\infty\frac{t}{\lambda+t}\lambda^{r-1}d\lambda. \tag{3}$$

See [2, p.116]. We write this briefly as

$$t^r=\int_0^\infty\frac{t}{\lambda+t}\,d\mu(\lambda), \tag{4}$$

where μ is a positive measure on $(0,\infty)$, depending on r.

Theorem 3. *The matrix P_r is cnd and nonsingular for $0 < r < 1$, and it is cpd and nonsingular for $1 < r < 2$.*

Proof. Let $0 < r < 1$ and use (4) to write

$$(p_i + p_j)^r = \int_0^\infty \frac{p_i + p_j}{p_i + p_j + \lambda} d\mu(\lambda). \tag{5}$$

Then use the identity

$$\frac{p_i + p_j}{p_i + p_j + \lambda} = 1 - \frac{\lambda}{p_i + p_j + \lambda},$$

to see that the matrix

$$G_\lambda = \left[\frac{p_i + p_j}{p_i + p_j + \lambda}\right], \ \ \lambda > 0$$

can be expressed as

$$G_\lambda = E - \lambda\, C_\lambda, \tag{6}$$

where $C_\lambda = \left[\frac{1}{p_i + p_j + \lambda}\right]$ is a Cauchy matrix. This matrix is positive definite, and $Ex = 0$ for all $x \in \mathcal{H}_1$. It follows from (6) that G_λ is cnd. so $P_r = \int\limits_0^\infty G_\lambda \, d\mu(\lambda)$ is also cnd.

Now let $1 < r < 2$. Using (4) we can express

$$t^r = \int_0^\infty \frac{t^2}{\lambda + t} \, d\mu(\lambda). \tag{7}$$

So,

$$(p_i + p_j)^r = \int_0^\infty \frac{(p_i + p_j)^2}{p_i + p_j + \lambda} \, d\mu(\lambda)$$

Use the identity

$$\frac{(p_i + p_j)^2}{p_i + p_j + \lambda} = p_i + p_j - \frac{\lambda(p_i + p_j)}{p_i + p_j + \lambda},$$

to see that the matrix

$$H_\lambda = \frac{(p_i + p_j)^2}{p_i + p_j + \lambda}, \ \ \lambda > 0$$

can be expressed as

$$H_\lambda = DE + ED - \lambda G_\lambda, \tag{8}$$

where $D = \operatorname{diag}(p_1, \ldots, p_n)$ and G_λ is the matrix in (6). If $x \in \mathcal{H}_1$, then

$$\langle x, (DE + ED)x \rangle = \langle x,\ DEx \rangle + \langle DEx, x \rangle = 0.$$

So $\langle x, H_\lambda x \rangle \geq 0$. In other words, the matrix H_λ is cpd, and hence so is P_r, $1 < r < 2$.

It remains to show that P_r is nonsingular. The Cauchy matrix C_λ is positive definite. This can be seen by writing

$$\frac{1}{p_i + p_j + \lambda} = \int_0^\infty e^{-t(p_i + p_j + \lambda)}\ dt,$$

which shows that C_λ is the Gram matrix corresponding to the vectors $u_i = e^{-t(p_i + \lambda/2)}$ in $L_2(0, \infty)$. Since p_i are distinct, the vectors u_i are linearly independent, and C_λ nonsingular. This shows that for all nonzero vectors x in $\mathcal{H}_1$, $\langle x, G_\lambda x \rangle < 0$ for all $\lambda > 0$. Hence $\langle x, P_r x \rangle < 0$ for $0 < r < 1$. So P_r is nonsingular. In the same way, we see that $\langle x, H_\lambda x \rangle > 0$ for all $x \in \mathcal{H}_1$, and $\lambda > 0$. So P_r is nonsingular for $1 < r < 2$. ■

Each entry of P_r is positive. So, P_r has at least one positive eigenvalue. For $r > 0$ the matrix P_r is not positive semidefinite as its top 2×2 subdeterminant is negative. So P_r has at least one negative eigenvalue. The space $\mathcal{H}_1$ has dimension $n - 1$. So using the minmax principle [2, Ch.III] and Theorem 3, we see that

$$\operatorname{In} P_r = (1, 0, n-1), \quad 0 < r < 1,$$

and

$$\operatorname{In} P_r = (n-1, 0, 1), \quad 1 < r < 2.$$

This establishes the statement (iii) of Theorem 1 for these values of r. (In fact, Theorem 3 says a little more, in that P_r is cnd/cpd.)

4. Nonsingularity

In this section we show that if $n \geq 2$, and $r > n - 2$, then P_r is nonsingular. This is a consequence of the following.

Theorem 4. *Let $c_1, \ldots, c_n$ be real numbers not all zero, and for $r > n - 2$ let f_r be the function defined on $(0, \infty)$ as*

$$f_r(x) = \sum_{j=1}^{n} c_j (x + p_j)^r. \tag{9}$$

Then f_r has at most $n - 1$ zeros.

Proof. We denote by $Z(f)$ the number of zeros of a function f on $(0,\infty)$, and by $V(c_1,\ldots,c_n)$ the number of sign changes in the tuple $c_1,\ldots,c_n$. (We follow the terminology and conventions of the classic [10, Part V].

Let s be any positive real number and $(c_1,\ldots,c_n)$ any n-tuple with $V(c_1,\ldots,c_n) < s+1$. We will show that

$$Z(f_s) \leq V(c_1,\ldots,c_n). \tag{10}$$

We use induction on $V(c_1,\ldots,c_n)$.

Clearly, if $V(c_1,\ldots,c_n) = 0$, then $Z(f_s) = 0$. Assume that the assertion is true for all n-tuples $(c_1,\ldots,c_n)$ with $V(c_1,\ldots,c_n) = k-1 < s$. Let $(c_1,\ldots,c_n)$ be any n-tuple with $V(c_1,\ldots,c_n) = k$, $0 < k < s+1$. Without loss of generality assume $c_i \neq 0$ for $i = 1,2,\ldots,n$. There exists an index j, $1 < j \leq n$ such that $c_{j-1}c_j < 0$. We may assume that $p_1 < p_2 < \cdots < p_n$. Choose any number u such that $p_{j-1} < u < p_j$ and let

$$\varphi(x) = \sum_{j=1}^{n} c_j(p_j - u)(x + x_j)^{s-1}.$$

Note that

$$V(c_1(p_1-u),\ldots,c_n(p_n-u)) = k-1 < s.$$

So, by the induction hypothesis $Z(\varphi) \leq k-1$. We have

$$\begin{aligned}
\varphi(x) &= \sum_{j=1}^{n} c_j(p_j-u)(x+p_j)^{s-1} \\
&= \sum_{j=1}^{n} c_j(x+p_j)^{s} - (x+u)\sum_{j=1}^{n} c_j(x+p_j)^{s-1} \\
&= \frac{-(x+u)^{s+1}}{s} \left\{ \frac{-s}{(x+u)^{s+1}}\, f_s(x) + \frac{1}{(x+u)^{s}}\, f_s'(x) \right\},
\end{aligned}$$

where f_s' is the derivative of f_s. Let

$$h(x) = \frac{f_s(x)}{(x+u)^s}.$$

Then the last equality above says

$$\varphi(x) = \frac{-(x+u)^{s+1}}{s}\, h'(x).$$

So $Z(\varphi) = Z(h')$. From the definition of h, it is clear that $Z(f_s) = Z(h)$. By Rolle's Theorem $Z(h) \leq Z(h') + 1$. Putting these relations together we see that $Z(f_s) \leq Z(\varphi) + 1 \leq k$. This establishes (10).

Now let $r > n-2$, and let $c_1, \ldots, c_n$ be any n-tuple. Then $V(c_1, \ldots, c_n) \leq n-1$. It follows that $Z(f_r) \leq n-1$, and that proves the theorem. ∎

Corollary 5. *The matrix P_r is nonsingular for all $r > n-2$.*

Proof. P_r is singular if and only if there exists a nonzero vector $c = (c_1, \ldots, c_n)$ in $\mathbb{R}^n$ such that $P_r c = 0$; i.e.,

$$\sum_{j=1}^{n} c_j (p_i + p_j)^r = 0 \quad \text{for} \quad i = 1, 2, \ldots, n.$$

That implies that the function $f_r(x)$ has at least n zeros (the distinct points $p_1, \ldots, p_n$). This is not possible by Theorem 4. ∎

As a consequence of this the inertia of P_r remains unchanged for $r > n-2$. This establishes part (iv) of Theorem 1.

5. Completing the proof of Theorem 1

We are left with the case $2 < r < n-2$, r not an integer. We will consider in detail the two cases $2 < r < 3$ and $3 < r < 4$. The essential features of the pattern given in part (iii) of the theorem, and of the proof are seen in these two cases.

Let $p = (p_1, \ldots, p_n)$. In Section 3 we introduced the space

$$\mathcal{H}_1 = \left\{ x : \sum x_j = 0 \right\} = \{x : Ex = 0\} = e^{\perp}.$$

Let

$$\mathcal{H}_2 = \left\{ x : \sum x_j = 0, \sum p_j x_j = 0 \right\}.$$

Then

$$\mathcal{H}_2 = \{x : Ex = 0, EDx = 0\} = \{e, p\}^{\perp},$$

where $D = \operatorname{diag}(p_1, \ldots, p_n)$, and $\{e, p\}^{\perp}$ stands for the orthogonal complement of the span of the vectors e and p. For $1 \leq k \leq n-1$, let $p^k = (p_1^k, \ldots, p_n^k)$ and let

$$\mathcal{H}_\ell = \left\{ x : \sum p_j^k x_j = 0, \;\; 0 \leq k \leq \ell - 1 \right\}.$$

Then

$$\begin{aligned}\mathcal{H}_\ell &= \left\{x : ED^k x = 0,\ \ 0 \le k \le \ell - 1\right\} \\ &= \left\{e, p, p^2, \ldots, p^{\ell-1}\right\}^\perp.\end{aligned}$$

Evidently, $\mathcal{H}_1 \supset \mathcal{H}_2 \supset \cdots \mathcal{H}_\ell$, and $\dim \mathcal{H}_\ell = n - \ell$.

Let m be any nonnegative integer, and let $m < r < m+1$. From (4) we have

$$(p_i + p_j)^r = \int_0^\infty \frac{(p_i + p_j)^{m+1}}{\lambda + p_i + p_j}\, d\mu(\lambda). \tag{11}$$

Let

$$G_{m,\lambda} = \left[\frac{(p_i + p_j)^{m+1}}{\lambda + p_i + p_j}\right]. \tag{12}$$

Use the identity

$$\frac{(p_i + p_j)^3}{\lambda + p_i + p_j} = (p_i + p_j)^2 - \frac{\lambda(p_i + p_j)^2}{\lambda + p_i + p_j} \tag{13}$$

to see that

$$G_{2,\lambda} = D^2E + 2DED + ED^2 - \lambda G_{1,\lambda}. \tag{14}$$

If $x \in \mathcal{H}_2$, then

$$\langle x, (D^2E + 2DED + ED^2)x\rangle = 0.$$

In Section 3, we saw that $\langle x, G_{1,\lambda}\, x\rangle > 0$ for all $x \in \mathcal{H}_1$, $x \neq 0$. (The matrix $G_{1,\lambda}$ was called H_λ there.) So it follows from (14) that

$$\langle x,\ G_{2,\lambda}\ x\rangle\ <\ 0 \ \text{ for all } \ x \in \mathcal{H}_2, x \neq 0 \quad \text{and} \quad \lambda > 0.$$

This, in turn implies that for $2 < r < 3$.

$$\langle x, P_r\ x\rangle\ <\ 0 \ \text{ for all } x \in \mathcal{H}_2, x \neq 0.$$

Since $\dim\ \mathcal{H}_2 = n-2$, the minmax principle implies that for $2 < r < 3$, P_r has at least $n-2$ negative eigenvalues. We show that its remaining two eigenvalues are positive.

Consider the matrix P_r when $n = 3$. We have established in Section 3 that when $1 < r \le 2$, P_r has two positive and one negative eigenvalue. In Section 4 we have established that this remains unchanged for $r > 2$. Now consider any $n > 3$. Any 3×3 principal submatrix of P_r has two positive eigenvalues, by what we have just said. So, by Cauchy's interlacing principle [2, Ch.III], P_r has at least two positive eigenvalues. The conclusion, then, is P_r has exactly two positive and $n-2$ negative eigenvalues, for all $2 < r < 3$.

Next consider the case $3 < r < 4$. Use the identity

$$\frac{(p_i + p_j)^4}{\lambda + p_i + p_j} = (p_i + p_j)^3 - \frac{\lambda (p_i + p_j)^3}{\lambda + p_i + p_j},$$

to see that

$$G_{3,\lambda} = D^3 E + 3D^2 ED + 3DED^2 + ED^3 - \lambda G_{2,\lambda}. \tag{15}$$

Again, if $x \in \mathcal{H}_2$, then one can see that

$$\langle x, (D^3 E + 3D^2 ED + 3DED^2 + ED^3)x \rangle = 0.$$

We have proved that for $x \in \mathcal{H}_2$, $x \neq 0$, we have $\langle x, G_{2,\lambda} x \rangle < 0$ for all $\lambda > 0$. Then (15) shows that $\langle x, G_{3,\lambda} x \rangle > 0$, and hence $\langle x, P_r\, x \rangle > 0$ for all $x \in \mathcal{H}_2$, $x \neq 0$, and $3 < r < 4$. So, P_r has at least $n - 2$ positive eigenvalues. We have to show that the remaining two of its eigenvalues are negative.

The argument we gave earlier can be modified to show that when $n = 4$, and $r > 2$, then P_r has two positive and two negative eigenvalues. So for $n > 4$, P_r has at least two negative eigenvalues. Hence, it has exactly two negative eigenvalues.

We have established the assertion of the theorem for $2 < r < 3$, and for $3 < r < 4$. The argument can be extended to the next interval. We leave this to the reader. Some remarks are in order here.

1. The proof for the cases covered in Section 3 was simpler because of the available criterion for the nonsingularity of a cpd/cnd matrix. More arguments are needed for $r > 2$.

2. The expressions (14) and (15) display $G_{2,\lambda}$ and $G_{3,\lambda}$ as $G_{2,\lambda} = S - \lambda G_{1,\lambda}$ and $G_{3,\lambda} = T - \lambda G_{2,\lambda}$. Though S and T are quite different, the first being quadratic in D and the second cubic, it is a happy coincidence that both $\langle x, Sx \rangle$ and $\langle x, Tx \rangle$ vanish for all $x \in \mathcal{H}_2$. This allows us to conclude that $\langle x, G_{2,\lambda} x \rangle$ is negative and $\langle x, G_{3,\lambda} x \rangle$ is positive on $\mathcal{H}_2$. The same argument carried to the next stage will give

$$G_{4,\lambda} = D^4 E + 4D^3 ED + 6D^2 ED^2 + 4DED^3 + ED^4 - \lambda G_{3,\lambda}. \tag{16}$$

Now $\langle x, D^2 E\, D^2 x \rangle$ need not vanish for $x \in \mathcal{H}_2$, but it does for $x \in \mathcal{H}_3$. So, we can conclude that P_r has at least $n - 3$ negative eigenvalues for $4 < r < 5$. The successor of (16) $G_{5,\lambda} = W - \lambda G_{4,\lambda}$ again has W satisfying $\langle x, Wx \rangle = 0$ for $x \in \mathcal{H}_3$. So, the inertia of P_r just changes sign when we go from $4 < r < 5$ to $5 < r < 6$. This explains some features of the theorem.

6. The matrix B_r

The arguments used in our analysis of P_r can be applied to other matrices, one of them being

$$B_r = [|p_i - p_j|^r], r \geq 0.$$

Inertias of these matrices have been computed in [8]. We summarise the results of that paper in a succint form parallel to our Theorem 1:

(i) *B_r is singular if and only if r is an even integer smaller than $n-1$.*

(ii) *Let r be an even integer $r = 2k < n$. Then*

$$\text{In } B_r = \left(\left\lceil \frac{r+1}{2} \right\rceil, n-(r+1), \left\lfloor \frac{r+1}{2} \right\rfloor\right), \quad \textit{if } k \textit{ is even}$$

and

$$\text{In } B_r = \left(\left\lfloor \frac{r+1}{2} \right\rfloor, n-(r+1), \left\lceil \frac{r+1}{2} \right\rceil\right), \quad \textit{if } k \textit{ is odd.}$$

(iii) *Suppose r is not an even integer and $0 < r < n-2$. If $2k < r < 2(k+1)$, then*

$$\text{In } B_r = (k+1,\ 0,\ n-(k+1)) \ \textit{ if } k \textit{ is even},$$

and

$$\text{In } B_r = (n-(k+1),\ 0,\ k+1) \ \textit{ if } k \textit{ is odd.}$$

(iv) *For every real number $r > n-2$,*

$$\text{In } B_r = \left(\frac{n}{2},\ 0,\ \frac{n}{2}\right) \ \textit{ if } n \textit{ is even},$$

and

$$\text{In } B_r = \left(\frac{n-1}{2},\ 0,\ \frac{n+1}{2}\right) \ \textit{ if } n \textit{ is odd.}$$

We briefly indicate how the proofs in [8] can be considerably simplified using our arguments.

1. Let $r = 2k$ be an even integer. Then

$$B_r = \left[(p_i - p_j)^{2k}\right].$$

Let W be the $(r+1)\times n$ Vandermonde matrix introduced in Section 2. Let V_2 be the $(r+1)\times(r+1)$ antidiagonal matrix obtained by multiplying V_1 on the left by the diagonal matrix $\Gamma = (1, -1, 1, -1, \ldots, -1, 1)$. Then one can see that

$$B_r = W^* V_2 W.$$

Therefore, by Proposition 2, if $r + 1 \le n$, then

$$\text{In } B_r = \text{In } V_2 + (0, n - (r+1), 0).$$

When k is even, the first k entries on the sinister diagonal of V_2 are $\binom{r}{0}, \ldots, \binom{r}{k-1}$ with alternating signs $\pm$; the $(k+1)$th entry is $\binom{r}{k}$; the next k entries are $\binom{r}{k-1}, \ldots, \binom{r}{0}$ with alternating signs $\mp$. A little argument shows that the eigenvalues of V_2 are $\binom{r}{k}$ and $\pm\binom{r}{j}$, $0 \le j \le k-1$. So

$$\text{In } V_2 = \left(\left\lceil \frac{r+1}{2} \right\rceil,\ 0,\ \left\lfloor \frac{r+1}{2} \right\rfloor \right).$$

When k is odd, the $(k+1)$th entry on the sinister diagonal of V_2 is $-\binom{r}{k}$. The eigenvalues of V_2 are $-\binom{r}{k}$ and $\pm\binom{r}{j}$, $0 \le j \le k-1$. So

$$\text{In } V_2 = \left(\left\lfloor \frac{r+1}{2} \right\rfloor,\ 0,\ \left\lceil \frac{r+1}{2} \right\rceil \right).$$

From the three equations displayed above we get statement (ii). This includes the assertion that B_r is singular when r is an even integer smaller than n.

2. Let $0 < r < 2$. Then

$$|p_i - p_j|^r = ((p_i - p_j)^2)^{r/2}.$$

So, using (4) we can write

$$|p_i - p_j|^r = \int_0^\infty \frac{(p_i - p_j)^2}{\lambda + (p_i - p_j)^2}\, d\mu(\lambda).$$

Arguing as before, we can express B_r as

$$B_r = \int_0^\infty T_{0,\lambda}\ d\mu(\lambda),$$

where $T_{0,\lambda} = E - \lambda S_{0,\lambda}$, and

$$S_{0,\lambda} = \left[\frac{1}{\lambda + (p_i - p_j)^2}\right], \quad \lambda > 0.$$

This last matrix is positive definite. A simple proof of this goes as follows

$$\begin{aligned}\frac{1}{\lambda + (p_i - p_j)^2} &= \int_0^\infty e^{-t(\lambda + (p_i - p_j)^2)}\ dt \\ &= \int_0^\infty e^{-t\lambda/2}\ e^{-t(p_i - p_j)^2}\ e^{-t\lambda/2}\ dt;\end{aligned}$$

and it is well-known that $\left[e^{-(p_i-p_j)^2}\right]$ is a positive definite matrix. See [3, p.146].

Since $S_{0,\lambda}$ is positive definite, for all $x \in \mathcal{H}_1$, $x \neq 0$, we have $\langle x, T_{0,\lambda}\ x\rangle < 0$. Hence, the same is true for B_r. So, B_r is cnd and nonsingular for $0 < r < 2$. (This is a well-known fact. We have given a proof to ease the passage to the next argument.)

3. Let $2 < r < 4$. Then $|p_i - p_j|^r = ((p_i - p_j)^2)^s$, where $1 < s < 2$. So, using (7) we can write

$$|p_i - p_j|^r = \int_0^\infty \frac{(p_i - p_j)^4}{\lambda + (p_i - p_j)^2}\ d\mu(\lambda).$$

We leave it to the reader to check that using this we have

$$B_r = \int_0^\infty T_{1,\lambda}\ d\mu(\lambda),$$

where $T_{1,\alpha} = D^2E - 2DED + ED^2 - \lambda T_{0,\lambda}$. If $x \in \mathcal{H}_2$, then $\langle x, (D^2E - 2DED + ED^2)x\rangle = 0$. So, $\langle x, T_{1,\alpha}x\rangle > 0$ for all nonzero vectors x in $\mathcal{H}_2$. Thus B_r has at least $n-2$ positive eigenvalues. Further arguments are needed to show it has two negative eigenvalues.

4. Let n be an odd integer, $n \geq 3$. It is shown in [8] that B_r is nonsingular for $r \geq n - 1$. (We have no essential simplification of this part of the proof.) This is a crucial ingredient needed for the rest of the cases.

5. Our arguments for P_r (using interlacing etc.) can be adapted to show that for $2 < r < 4$, B_r has two negative eigenvalues.

6. We can then argue in the same way the case $2k < r < 2(k+1)$, for $k = 2, 3, \ldots$. This gives statement (iii). Statement (i) is included in this.

7. As in the case of P_r, the inertia of B_r stabilises after some stage. Statement (iv) is a consequence of Remark 4 above.

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