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Torus equivariant spectral triple for quantum quaternion sphere

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Abstract

We give an explicit description of the q-deformation of symplectic group $SP_q(2n)$ at the C^*-algebra level and find all irreducible representations of this C^*-algebra. Further we study its Stiefel manifold $SP_q(2n)/SP_q(2n-2)$ by getting its defining relations and describe its irreducible representations. We compute its K-theory by obtaining a chain of short exact sequence for the C^*-algebras underlying such manifolds. The torus group $\mathbb{T}^n$ has a canonical action on $C(SP_q(2n)/SP_q(2n-2))$. We find a non-trivial, finitely summable equivariant spectral triple associated with this action.

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1 Introduction

Quantum groups and noncommutative geometry are two areas of mathematics that have been active in recent years. The theory of quantum group was first studied in the topological setting independently by Woronowicz [23] and Vaksman & Soibelman [22] for the case of quantum $SU(2)$ group. Later Woronowicz developed the theory of compact quantum groups and their representation theory. The notion of quantum subgroups and quantum homogeneous spaces was soon introduced by Podles [17]. The most well-known example of compact quantum group is the q- deformation of $SU(n)$ group whose representation theory is obtained by Vaksman & Soibelman [22]. Quantum analogs of their Stifel manifold $SU_q(n)/SU_q(n-m)$ were introduced by Podkolzin & Vainerman [16] who precisely described the irreducible representation of the C^*-algebras underlying such manifolds.

Noncommutative differential geometry was introduced by Alain Connes in 1980s. In his interpretation, geometric data is encoded in elliptic operators or more generally, in specific unbounded K-cycles which he called spectral triples. But to put quantum groups and their homogeneous spaces into Connes' framework proved to be a difficult task. First success towards this direction was achieved by Chakraborty & Pal [2] . They constructed a nontrivial spectral triple for $SU_q(2)$ group equivariant under its own (co-) action which established quantum groups as objects of non commutative geometry. Later they found nontrivial spectral triples associated with the (co-)action of $SU_q(n)$ on quantum analog of odd dimensional spheres which is same as stiefel manifold $SU_q(n)/SU_q(n-1)$. Pal & Sundar [15] then proved the regularity of these spectal triples which is a desirable property for a spectral triple. They computed their dimension spectrum in order to be able to apply Connes-Moscovicci theorem, a major result in noncommutative geometry. Chakraborty & Sundar [6] then extended the results of [15] and gave general results on regularity of a spectral triple constructed out of a regular spectral triple for a smaller space. Chakraborty & Sundar [5] calculated K-theory of Stiefel manifolds $SU_q(n)/SU_q(n-2)$ and as a consequence they found K-theory of $SU_q(3)$.

One natural question arises, can we put q-defomation of other classical lie group into Connes' formulation of noncommutative geometry or more specifically can we extend these results to the q-deformations Lie groups of types B, C and D. Neshveyev & Tuset [13] constructed an equivariant spetral triple D_q for G_q the q- deformation of any simply connected, simple, compact Lie group G. This spectral triple is an isospectral deformation of the classical Dirac operator D on G. In [14] they proved that K-homology class of D_q is same as that of D via the KK-equivalence of $C(G_q)$ and $C(G)$ established by Nagy [11]. Further, for a closed Poisson Lie subgroup $K$of G, Neshveyev & Tuset [12] studied quantization $C(G_q/K_q)$ of algebra of continuous functions on Steifel manifold G/K. Using results of Soibelman ([19],[10]) and Dijkhuizen & Stokman ([20],[21]) they found all its irreducible representations. Extending a result of Nagy [11], they proved KK-equivalence of $C(G_q/K_q)$ and $C(G/K)$ and calculated K-goups of $C(G_q/K_q)$ via this equivalence. In this article, we start with q-deformation of symplectic group and find its representation theory. Then we consider its quantum Stiefel manifold $SP_q(2n)/SP_q(2n-2)$. We get the defining relations satisfied by the generators of the C^*-algebra underlying this manifold and describe all irreducible representations of the universal C^*-algebra satisfying those relations and prove that both are isomorphic. This describes $C(SP_q(2n)/SP_q(2n-2))$ as a universal C^*-algebra satisfying some relations as in case of odd dimensional quantum sphere. Using this fact, we get a natural $\mathbb{T}^n$ action on it and obtain a non-trivial finitely summable spectral triple equivariant under this action. Next we find a chain of short exact sequence for $C\left(SP_q(2n)/SP_q(2n-2)\right)$ and utilizing them, we compute its K-groups with explicit generators.

Here is a brief outline of the contents of this article. In the next section, we define q-deformation of symplectic group $SP_q(2n)$ at C^*-algebra level and describe its quantum group structure. In section 3 and section 4, our main aim will be to find all irreducible representations of $C(SP_q(2n))$ and obtain its faithful realisation in a Hilbert space. The key step here is to find all elmentary representations of $C(SP_q(2n))$ by imbedding quantum universal enveloping algebra of $SL(2)$ (i.e $U_q(sl(2))$) into that of $SP(2n)$ (i.e $U_q(sp(2n))$) and then use the pairing given in [9]. To make the paper readable and for lack of a good reference we include explicit computations of the Weyl group of sp_{2n}. We then apply results given in [10](page 121) to get all its irreducible representaions. In section 5, we describe how to draw diagrams of these representations. In section 6, we study quantum Stiefel manifold $SP_q(2n)/SP_q(2n-2)$ and fi nd the relations satisfied by their generators. In the next section we prove that it is the universal C^*-algebra satisfying those relations. Section 8 mainly deals with the computation of its K-groups by obtaining a chain of short exact sequence. In section 9, we study the torus action on $SP_q(2n)/SP_q(2n-2)$ and find associated equivariant spectral triple. Here the idea is exactly as in [3] for the case of odd dimensional quantum sphere. Throughout this paper, $q \in (0,1)$ and C is used to denote generic constant.

2 $C(SP_q(2n))$

We set up some notation that will be used throughout this paper. Define,

$$\begin{aligned}
i^{'} &= 2n+1-i \\
\rho_i &= n+1-i \quad \text{ if } i \leq n. \\
\rho_{i'} &= -\rho_i. \\
\epsilon_i &= \begin{cases} 1 & \text{if } 1 \leq i \leq n, \\ -1 & \text{if } n+1 \leq i \leq 2n, \end{cases} \\
C^i_j &= \epsilon_i \delta_{ij} q^{-\rho_i}. \\
\theta(i) &= \begin{cases} 0 & \text{if } i \leq 0, \\ 1 & \text{if } i > 0, \end{cases} \\
R^{ij}_{mn} &= q^{\delta_{ij}-\delta_{ij'}} \delta_{im}\delta_{jn} + (q-q^{-1})\theta(i-m)(\delta_{jm}\delta_{in} + C^i_j C^m_n).
\end{aligned}$$

Let $\mathbb{C}\left\langle u^i_j \right\rangle$ denote the free algebra with generators u^i_j, $i,j = 1,2,\cdots 2n$ and let $J(R)$ be the two sided ideal of $\mathbb{C}\left\langle u^i_j \right\rangle$ generated by the following elements,

$$I^{ij}_{st} = \sum_{k,l=1}^{2n} R^{ji}_{kl} u^k_s u^l_t - R^{lk}_{st} u^i_k u^j_l, \quad i,j,s,t = 1,2\cdots 2n.$$

Let $A(R)$ denote the quotient algebra $\mathbb{C}\left\langle u^i_j \right\rangle / J(R)$. The $2n \times 2n$ matrices $((u^i_j))$ and $((C^i_j))$ is denoted by U and C respectively. Define $J = \left\langle UCU^tC^{-1} - I, CU^tC^{-1}U - I \right\rangle$ the two sided ideal generated by entries of matrices $UCU^tC^{-1} - I$ and $CU^tC^{-1}U - I$. Let $O(SP_q(2n))$ denote the quotient algebra $A(R)/J$.

The algebra $O(SP_q(2n))$ is a Hopf-*algebra with co-multiplication Δ, co-unit ϵ, antipode S

and involution $*$ given on the generating elements by,

$$\Delta(u^i_j) = \sum_{k=1}^{N} u^i_k \otimes u^k_j$$
$$\epsilon(u^i_j) = \delta_{ij}$$
$$S(u^i_j) = \epsilon_i \epsilon_j q^{\rho_i - \rho_j} u^{j'}_{i'}$$
$$(u^i_j)^* = \epsilon_i \epsilon_j q^{\rho_i - \rho_j} u^{i'}_{j'}$$

Note that $U^* = CU^tC^{-1}$. Hence we have,

$$UU^* = U^*U = I. \tag{2.1}$$

Now to make $O(SP_q(2n))$, a normed-$*$algebra, we define,

$$\|a\| = \ \sup \ \{\|\pi(a)\| : \pi \text{ is a representation of } O(SP_q(2n))\}.$$

By (2.1), we have, $\left\|u^i_j\right\| \leq 1$, hence $\forall a \in O(SP_q(2n))$, $\|a\| < \infty$.
We denote $C(SP_q(2n))$ to be the completion of $O(SP_q(2n))$. $(C(SP_q(2n)), \Delta)$ is a compact quantum group called as q-deformation of symplectic group $SP_q(2n)$.

3 Pairing between $U_q(sp_{2n})$ and $C(SP_q(2n))$

Let $((a_{ij}))$ be the Cartan matrix of Lie algebra $sp(2n)$ given by,

$$a_{ij} = \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } i = j+1, \\ -1 & \text{if } i = j-1, i \neq n-1, \\ -2 & \text{if } i = j-1 = n-1, \\ 0 & \textit{otherwise}, \end{cases}$$

Define $q_i = q^{d_i}$, where d_i= 1 for $i = 1, 2, ..n-1$ and $d_n = 2$. The quantised universal envelopping algebra (QUEA) $U_q(sp_{2n})$ is the universal algebra generated by E_i, F_i, K_i and K_i^{-1}, $i = 1, \ldots, \ell$, satisfying the following relations

$$K_iK_j = K_jK_i, \quad K_iK_i^{-1} = K_i^{-1}K_i = 1,$$

$$K_iE_jK_i^{-1} = q_i^{a_{ij}}E_j, \quad K_iF_jK_i^{-1} = q_i^{a_{ij}}F_j,$$

$$E_iF_j - F_jE_i = \delta_{ij}\frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},$$
$$\sum_{r=0}^{1-a_{ij}} (-1)^r \binom{1-a_{ij}}{r}_{q_i} E_i^{1-a_{ij}-r}E_jE_i^r = 0 \quad \forall\, i \neq j,$$
$$\sum_{r=0}^{1-a_{ij}} (-1)^r \binom{1-a_{ij}}{r}_{q_i} F_i^{1-a_{ij}-r}F_jF_i^r = 0 \quad \forall\, i \neq j,$$

where $\binom{n}{r}_q$ denote the q-binomial coefficients. Hopf *-structure comes from the following maps:

$$\Delta(K_i) = K_i \otimes K_i, \quad \Delta(K_i^{-1}) = K_i^{-1} \otimes K_i^{-1},$$
$$\Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i, \quad \Delta(F_i) = F_i \otimes 1 + K_i^{-1} \otimes F_i,$$
$$\epsilon(K_i) = 1, \quad \epsilon(E_i) = 0 = \epsilon(F_i),$$
$$S(K_i) = K_i^{-1}, \quad S(E_i) = -E_iK_i^{-1}, \quad S(F_i) = -K_iF_i,$$
$$K_i^* = K_i, \quad E_i^* = K_iF_i, \quad F_i^* = E_iK_i^{-1}.$$

See [9] for more detail.

Dual pairing of $\boldsymbol{O}(SP_q(2n))$ and $U_q(sp_{2n})$: We refer to [9] for a proof of following theorem.

Theorem 3.1. ([9]) *There exist unique dual pairing $\langle \cdot, \cdot \rangle$ between the hopf algebras and $U_q(sl_2)$ and $\mathrm{O}(SL_q(2))$ and between $U_q(sp_{2n})$ and $\mathrm{O}(SP_q(2n))$ such that*

$$\left\langle f, u_l^k \right\rangle = t_{kl}(f) \qquad\qquad \text{for } k, l = 1, 2, ...2n.$$

where t_{kl} is the matrix element of T_1, the vector representation of $U_q(sl_2)$ in first case and that of $U_q(sp_{2n})$ in second case.

We will explicitly describe T_1 for both cases and determine the pairing. For that, let E_{ij} be the $2n \times 2n$ matrix with 1 in the $(i,j)^{th}$ position and 0 elsewhere and D_j be the diagonal matrix with q in the $(j,j)^{th}$ position and 1 elsewhere on the diagonal.

For the QUEA $U_q(sp_{2n})$, one has

for $i = 1, 2, ...n-1$,

$$T_1(K_i) = D_i^{-1}D_{i+1}D_{2n-1}^{-1}D_{2n-i+1}.$$
$$T_1(E_i) = E_{i+1,i} - E_{2n-i+1,2n-i}.$$
$$T_1(F_i) = E_{i,i+1} - E_{2n-i,2n-i+1}.$$

for $i = n$

$$T_1(K_n) = D_n^{-2}D_{n+1}^2.$$
$$T_1(E_n) = E_{n+1,n}.$$
$$T_1(F_n) = E_{n,n+1}.$$

For $U_q(sl_2)$, one has (here E_{ij} and D_j are the 2×2 matrices.)

$$\begin{aligned} T_1(K) &= D_1^{-1} D_2. \\ T_1(E) &= E_{2,1}. \\ T_1(F) &= E_{1,2}. \end{aligned}$$

We will use these two pairing to find irreducible representations of $O(SP_q(2n))$ which can be extended to $C(SP_q(2n))$ to get elementary representations of $C(SP_q(2n))$. To fix the idea
Elementary representation of $C(SP_q(2n))$: Let i be the vertex of Dynkin diagram of Lie algebra $sp_q(2n)$. Let $\varphi_i : U_{q_i}(sl_2) \longrightarrow U_q(sp(2n))$ be a $*$-homomorphism given on generators of $U_{q_i}(sl_2)$ by,

$$K \longmapsto K_i$$

$$E \longmapsto E_i$$

$$F \longmapsto F_i$$

Consider the dual epimorphism,

$$\varphi_i^* : C(SP_q(2n)) \longrightarrow C(SU_{q_i}(2))$$

such that

$$\langle f, \varphi_i^*(u_n^m) \rangle = \langle \varphi_i(f), u_n^m \rangle$$

In particular,

$$\langle K, \varphi_i^*(u_n^m) \rangle = \langle K_i, u_n^m \rangle \tag{3.1}$$

$$\langle E, \varphi_i^*(u_n^m) \rangle = \langle E_i, u_n^m \rangle \tag{3.2}$$

$$\langle F, \varphi_i^*(u_n^m) \rangle = \langle F_i, u_n^m \rangle \tag{3.3}$$

Remark 3.2. Initially φ_i will induce an $*$ epimorphism from the Hopf-$*$algebra $O(SP_q(2n))$ to $O(sl_{q_i}(2))$ which when extended to $C(SP_q(2n))$ gives the above homomorphism at C^*-algebra level. For more detail see [9] (page 327) and [10].

Let N be the number operator given by $N : e_n \mapsto ne_n$ and S be the shift operator given by $S : e_n \mapsto e_{n-1}$ on $L_2(\mathbb{N})$. Denote by π the following representation of $C(SU_q(2))$ on $L_2(\mathbb{N})$;

$$\pi(u_l^k) = \begin{cases} \sqrt{1 - q^{2N+2}} S & \text{if } k = l = 1, \\ S^* \sqrt{1 - q^{2N+2}} & \text{if } k = l = 2, \\ -q^{N+1} & \text{if } k = 1, l = 2, \\ q^N & \text{if } k = 2, l = 1, \\ \delta_{kl} & \text{otherwise }. \end{cases}$$

Define, $\pi_{s_i} = \pi \circ \varphi_i^*$. Applying (3.1), (3.2), (3.3), we have, for $i = 1, 2, \cdots, n-1$,

$$\pi_{s_i}(u_l^k) = \begin{cases} \sqrt{1-q^{2N+2}}S & \text{if } (k,l) = (i,i) \text{ or } (2n-i, 2n-i), \\ S^*\sqrt{1-q^{2N+2}} & \text{if } (k,l) = (i+1,i+1) \text{ or } (2n-i+1, 2n-i+1), \\ -q^{N+1} & \text{if } (k,l) = (i,i+1) \\ q^N & \text{if } (k,l) = (i,i+1), \\ q^{N+1} & \text{if } (k,l) = (2n-i, 2n-i+1), \\ -q^N & \text{if } (k,l) = (2n-i+1, 2n-i), \\ \delta_{kl} & \text{otherwise .} \end{cases}$$

for $i = n$,

$$\pi_{s_n}(u_l^k) = \begin{cases} \sqrt{1-q^{4N+4}}S & \text{if } (k,l) = (n,n), \\ S^*\sqrt{1-q^{4N+4}} & \text{if } (k,l) = (n+1,n+1), \\ -q^{2N+2} & \text{if } (k,l) = (n,n+1), \\ q^{2N} & \text{if } (k,l) = (n+1,n), \\ \delta_{kl} & \text{otherwise .} \end{cases}$$

$\{\pi_{s_i}\}_{i=1,2,\cdots,n}$ are irreducible representations called elementary representations of $C(SP_q(2n))$. For any two representations φ and ψ of $C(SP_q(2n))$ define, $\varphi * \psi = (\varphi \otimes \psi) \circ \Delta$. Let W be the Weyl group of sp_{2n} and $\vartheta \in W$ such that $s_{i_1}s_{i_2}...s_{i_k}$ is a reduced expression for ϑ. Then $\pi_\vartheta = \pi_{s_{i_1}} * \pi_{s_{i_2}} * \cdots * \pi_{s_{i_k}}$ is an irreducible representation which is independent of reduced expression. Now for $t = (t_1, t_2, \cdots, t_n) \in \mathbb{T}^n$
Define,

$$\tau_t : C(SP_q(2n) \longrightarrow \mathbb{C}$$

given by,

$$\tau_t(u_j^i) = \begin{cases} \overline{t_i}\delta_{ij} & \text{if } i \leq n, \\ t_{2n+1-i}\delta_{ij} & \text{if } i > n, \end{cases}$$

Then τ_t is a $*$-algebra homomorphism. For $t \in \mathbb{T}^n, \vartheta \in W$ let $\pi_{t,\vartheta} = \tau_t * \pi_\vartheta$.
We refer to [10] (page 121) for the proof of the following theorem.

Theorem 3.3. *$\{\pi_{t,\vartheta}; t \in \mathbb{T}^n, \vartheta \in W\}$ is a complete set of mutually inequivalent representations of $C(SP_q(2n))$.*

Hence, to find all irreducible representations of $C(SP_q(2n))$, we need to know the Weyl group of $sp_q(2n)$.

4 Weyl group of sp_{2n}

The Weyl group W_n of is a coxeter group of sp_{2n} generated by $s_1, s_2, ...s_n$ satisfying the following relations:

$$\begin{aligned} s_i^2 &= 1 && \text{for } i = 1, 2, ...n \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} && \text{for } i = 1, 2, ...n-1 \\ s_{n-1} s_n s_{n-1} s_n &= s_n s_{n-1} s_n s_{n-1} \end{aligned}$$

W_n can be embedded faithfully in $M_n(\mathbb{R})$ as,
for $i = 1, 2, ...n-1$

$$s_i = I - E_{i,i} - E_{i+1,i+1} + E_{i,i+1} + E_{i+1,i},$$

for $i = n$

$$s_n = I - 2E_{n,n}.$$

So, W_n is isomorphic to a subgroup of $GL(n, \mathbb{R})$ generated by $s_1, s_2, ...s_n$.

Proposition 4.1. *1. Let $\mathfrak{S}_n$ be the permutation group and H_n be the n-fold direct product of the group $\{-1, 1\}$. $\mathfrak{S}_n$ acts on H_n by permuting its co-ordinates. Then $W_n = H_n \rtimes \mathfrak{S}_n$. In other word, W_n is the set of $n \times n$ matrices having one non-zero entry in each row and each column which is either 1 or -1.*

2. *Any element of W_n can be written in the form: $\prod_{r=1}^{n} \psi_{r,k_r}^{(\epsilon_r)}$ where $\epsilon_r \in \{0, 1, 2\}$ and $r \leq k_r \leq n$ with the convention that ,*

$$\psi_{r,k_r}^{\epsilon} = \begin{cases} s_{k_r-1} s_{k_r-2} ... s_r & \text{if } \epsilon = 1, \\ s_{k_r} s_{k_r+1} \cdots ...s_{n-1} s_n s_{n-1} \cdots s_{k_r} s_{k_r-1} \cdots s_r & \text{if } \epsilon = 2, \\ \text{empty string} & \text{if } \epsilon = 0, \end{cases}$$

Also, the above expression is a reduced expression.

3. *The longest word of W_n is $-I$ which can be written as, $\prod_{r=1}^{n} \psi_{r,r}^{(2)}$. Also, $\left\{\psi_{r,r}^{(2)}\right\}_{i=1}^{n}$ commutes, hence $-I$ can be written as, $\prod_{r=1}^{n} \psi_{n+1-r,n+1-r}^{(2)}$, which is a reduced expression.*

Proof:

1. Clearly $H_n \rtimes \mathfrak{S}_n$ contains $s_1, s_2, ...s_n$ which implies $W_n \subseteq H_n \rtimes \mathfrak{S}_n$. Now, take $T \in H_n \rtimes \mathfrak{S}_n$. First get the permutaion matrix T' by replacing -1 in T with 1. T' can be generated using $s_1, s_2..., s_{n-1}$. Now to get T, multiply successively on the right by $s_k s_{k+1} ... s_{n-1} s_n s_{n-1} ... s_k$ for $k = 1, 2, ..n$, depending on which column of T has -1 (i.e if i^{th} column has -1 then muliply on the right by $s_i s_{i+1} ... s_{n-1} s_n s_{n-1} ... s_i$). This proves the claim.

2. First note that W_{n-1} can be realised as a subgroup of W_n generated by $s_2...s_n$. $W_{n-1} \ni w \mapsto \begin{pmatrix} 1 & 0 \\ 0 & w \end{pmatrix}$ gives an embedding of W_{n-1} in W_n. We use induction. For $n = 2$, it is clear. Assume the result for $n - 1$. Take $T \in W_n$. Look at the first column of T. Let i be the first integer such that $a_{i1} \neq 0$.
Case $1 : a_{i1} = 1, i = 1$. Set $T_1 = T$.
Case $2 : a_{i1} = 1, i > 1$. Let $T_1 = s_1 s_2 \cdots s_{i-1} T$.
Case $3 : a_{i1} = -1$. In this case, let $T_1 = s_1 \cdots s_{n-1} s_n s_{n-1} \cdots s_i T$.

   T_1 has 1 at $(1, 1)$ place. Hence it can be realised as an element of W_{n-1}. Now by induction hypothesis, T_1 can be written as,
$$T_1 = \prod_{r=2}^{n} \psi_{r,k_r}^{(i_r)}$$
which establish the claim. One can also show that this expression is a reduced expression by using induction in the same way as above (see Humphreys [8]).

3. Length of a word w in a Weyl group is the number of positive roots sent by w to negative roots (see Humphreys [8]). The lie algebra sp_{2n} has n^2 positive roots. So, any word in W_n can not have length greater than n^2. Now, from first part of this proposition, it follows that $-I \in W_n$. Since $-I$ sends every positive root to a negative root, length of $-I$ is n^2 which shows that $-I$ is the longest word of W_n. Remaining part of the claim follows by direct calculation.

$\square$

Denote by $\mathscr{T}$ the Toeplitz algebra. Let ϑ be a word on $s_1, s_2, ...s_n$ of length $\ell(\vartheta)$. Then the map $\mathbb{T}^n \ni t \longmapsto \pi_{t,\vartheta}(u_j^i) \in \mathscr{T}^{\otimes \ell(\vartheta)}$ is continuous. Hence we have a homomorphism $\chi_\vartheta : C(SP_q(2n)) \longrightarrow C(\mathbb{T}^n) \otimes \mathscr{T}^{\otimes \ell(\vartheta)}$ such that $\chi_\vartheta(a)(t) = \pi_{t,\vartheta}(a),$ for all $a \in C(SP_q(2n))$.

Proposition 4.2. *if ϑ' is a subword of ϑ then $\chi_{\vartheta'}$ and $\pi_{t,\vartheta'}$ factor through χ_ϑ.*

Proof: See [5]. $\square$

Theorem 4.3. *Let ϑ_n be the longest word i.e $\vartheta_n = (s_n)(s_{n-1}s_ns_{n-1})...(s_2...s_n..s_2)(s_1s_2...s_{n-1}s_ns_{n-1}...s_1)$ of the Weyl group of sp_{2n}. Then the homomorphism*

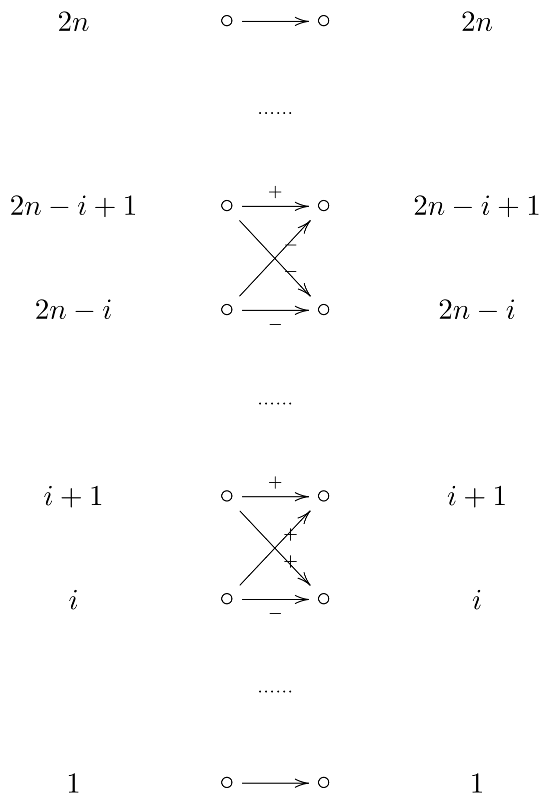
$$\chi_{\vartheta_n} : C(SP_q(2n)) \longrightarrow C(\mathbb{T}^n) \otimes \mathscr{T}^{\otimes \ell(\vartheta_n)}$$

is faithful.

Proof: Any irreducible representation of $C(SP_q(2n))$ is of the form $\pi_{t,\vartheta}$ where ϑ is a word on $s_1, s_2, ...s_n$ and $t \in \mathbb{T}^n$. From proposition (4.1), it is clear that ϑ is a subword of ϑ_n, hence $\pi_{t,\vartheta}$ factors through χ_{ϑ_n} which shows that χ_{ϑ_n} is faithful. $\square$

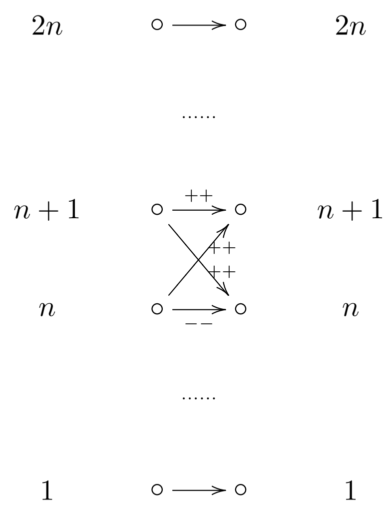
5 Diagram representation

At this point it will be useful to have some pictures of above representations in our mind. We will follow [4]. Let us describe how to use a diagram to represent the irreducible π_{s_i}, $\quad i \neq n$.



$\mathcal{H}$

For $i = n$,



$\mathcal{H}$

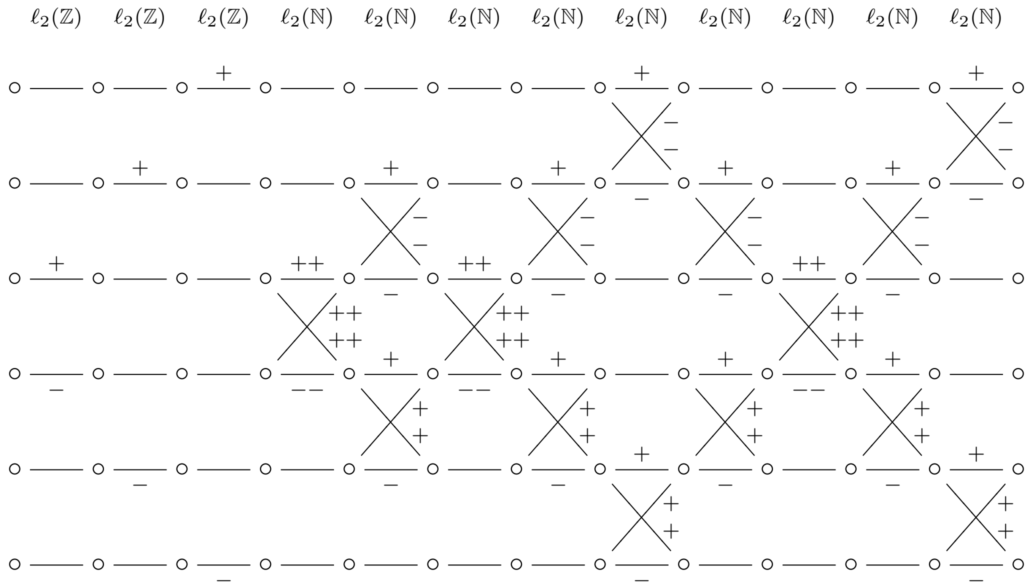
In this diagram, each path from a node k on the left to a node l on the right stands for an operator on $\mathcal{H} = L_2(\mathbb{N})$. A horizontal unlabelled line stands for the identity operator, a horizontal line labelled with $+$ or $++$ sign stands for $S^*\sqrt{I - q^{2N+2}}$ or $S^*\sqrt{I - q^{4N+4}}$ respectively and one labelled with $-$ or $--$ sign stands for $\sqrt{I - q^{2N+2}}S$ or $\sqrt{I - q^{4N+4}}S$ respectively. A diagonal line going upward labelled with $+$ or $++$ sign represents $-q^{N+1}$ or $-q^{2N+2}$ respectively and one labelled with $-$ sign represents q^{N+1}. A diagonal line going downward labelled with $+$ or $++$ sign represents q^N or q^{2N} respectively and one labelled with $-$ sign represents $-q^N$. Now $\pi_{s_i}(u_l^k)$ is the operator represented by the path from k to l, and is zero if there is no such path. Thus, for example, $\pi_{s_i}(u_1^1)$ is I, $\pi_{s_i}(u_1^2)$ is zero, whereas $\pi_{s_i}(u_{i+1}^i) = -q^{N+1}$, if $i > 1$.

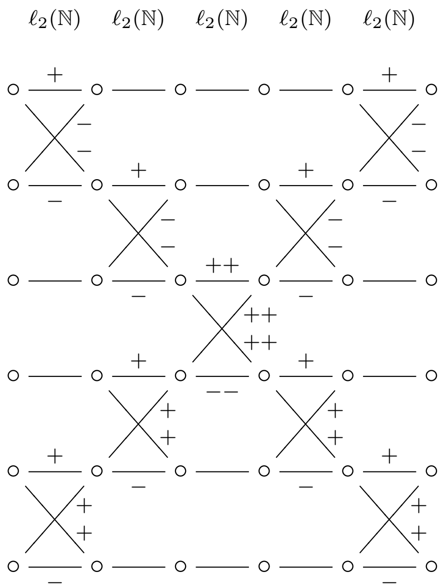
Next, let us explain how to represent $\pi_{s_i} * \pi_{s_j}$ by a diagram. Simply keep the two diagrams representing π_{s_i} and π_{s_j} adjacent to each other. Identify, for each row, the node on the right side of the diagram for π_{s_i} with the corresponding node on the left in the diagram for π_{s_j}. Now, $\pi_{s_i} * \pi_{s_j}(u_l^k)$ would be an operator on the Hilbert space $L_2(\mathbb{N}) \otimes L_2(\mathbb{N})$ determined by all the paths from the node k on the left to the node l on the right. It would be zero if there is no such path and if there are more than one paths, then it would be the sum of the operators given by each such path. In this way, we can draw diagrams for each irreducible representation of $C(SP_q(2n))$.

Next, we come to χ_ϑ. The underlying Hilbert space now is $L_2(\mathbb{Z}^n) \otimes L_2(\mathbb{N})^{\otimes \ell(\vartheta)}$. To avoid any ambiguity, we have explicitly mentioned above the diagram the space on which the operator between two hollow circle acts. For the operators on $L_2(\mathbb{Z})$, an unlabelled horizontal arrow stands for I, an arrow labelled with a '$+$' above it indicates S^* and one labelled '$-$' below it stands for S. As earlier, $\chi_\vartheta(u_l^k)$ stands for the operator on $L_2(\mathbb{Z}^n) \otimes L_2(\mathbb{N})^{\otimes \ell(\vartheta)}$ represented by the path from k on the left to l on the right. Note that we view $C(\mathbb{T}^n)$ as a subalgebra of

$\mathcal{L}(L_2(\mathbb{Z}^n))$.

The following diagrams are for the representations χ_{ϑ_3} and π_{1,ω_3} of $C(SP_q(6))$ where $\vartheta_3 = s_3s_2s_3s_2s_1s_2s_3s_2s_1$ and $\omega_3 = s_1s_2s_3s_2s_1$.





The diagram (in fact the pattern of diagram) for π_{1,ω_3} introduced above will play an important role in what follows.

6 Stiefel manifold $C(SP_q(2n)/SP_q(2n-2))$

The Weyl group W_{n-1} of sp_{2n-2} can also be realised as a subgroup of W_n of sp_{2n} generated by $s_2, s_3, \cdots s_n$ and hence, the longest word ϑ_{n-1} in W_{n-1} is a subword of the longest word ϑ_n in W_n which can easily be seen from proposition 4.1. This shows that $C(SP_q(2n-2))$ is a subgroup of $C(SP_q(2n))$ i.e. there is a C^*-epimorphism $\phi : C(SP_q(2n)) \to C(SP_q(2n-2))$ obeying $\Delta\phi = (\phi \otimes \phi)\Delta$. More precisely, let $\sigma : \mathscr{T} \to \mathbb{C}$ is the homomorphism for which $\sigma(S) = 1$. Define ϕ to be the restriction of $1^{\otimes n-1} \otimes ev_1 \otimes 1^{\otimes(n-1)^2} \otimes \sigma^{\otimes(2n-1)}$ to $\chi_{\vartheta_n}(C(SP_q(2n)))$ which is contained in $C(T^n) \otimes \mathscr{T}^{\otimes n^2}$. Here ev_1 denote the evaluation map at 1 i.e. $ev_1 : C(T) \to \mathbb{C}$ such that $ev_1(f) = f(1)$. Image of ϕ is equal to $\chi_{\vartheta_{n-1}}(C(SP_q(2n-2)))$ as,

$$\phi(\chi_{w_n}(u^i_j)) = \begin{cases} \chi_{\vartheta_{n-1}}(v^i_j), & \text{if } i \neq 1 \text{ or } 2n, \text{ or } j \neq 1 \text{ or } 2n, \\ \delta_{ij}, & \text{otherwise.} \end{cases}$$

where v^i_j are generators of $C(SP_q(2n-2))$. In such a case, one defines the quotient space $C(SP_q(2n)/SP_q(2n-2))$ by,

$$C(SP_q(2n)/SP_q(2n-2)) = \{a \in C(SP_q(2n) : (\phi \otimes id)\Delta(a) = I \otimes a\}.$$

Clearly, u^{2n}_m and hence $u^1_m (= \epsilon_m q^{\rho_1 - \rho_m}(u^{2n}_{2n-m+1})^*)$ are in $C(SP_q(2n)/SP_q(2n-2))$ for $m = 1, 2, \cdots 2n$. Neshveyev & Tuset [12] proposition 1.2 described all irreducible representations of $C(SP_q(2n)/SP_q(2n-2))$. As a consequence, one can show that $C(SP_q(2n)/SP_q(2n-2))$ is the C^*-subalgebra of $C(SP_q(2n)$ generated by $\{u^{2n}_m\}_{m=1}^{2n}$. Now, if we look at the relations I^{ij}_{st} involving u^{2n}_m and u^1_m by putting $(i,j) = (1,1)$ and $(2n, 1)$, we get the relations mentioned in next section where $z_m = u^{2n}_{2n+1-m}$. We will prove that $C(SP_q(2n)/SP_q(2n-2))$ is the universal C^*-algebra satisfying those relations.

7 Quantum quaternion sphere

Let $q \in (0,1)$. The C^*-algebra $C(H^{2n}_q)$ of continuous functions on the quantum quarternion sphere is defined as the universal C^*-algebra generated by elements z_1, z_2, ...z_{2n} satisfying the following relations

$$z_i z_j = q z_j z_i \qquad \text{for } i > j, i + j \neq 2n+1 \tag{7.1}$$

$$z_i z_{i'} = q^2 z_{i'} z_i - (1-q^2) \sum_{k>i} q^{i-k} z_k z_{k'} \qquad \text{for } i > n \tag{7.2}$$

$$z^*_i z_{i'} = q^2 z_{i'} z^*_i \tag{7.3}$$

$$z^*_i z_j = q z_j z^*_i \qquad \text{for } i + j > 2n+1, i \neq j \tag{7.4}$$

$$z_i^* z_j = q z_j z_i^* + (1-q^2)\epsilon_i\epsilon_j q^{\rho_i+\rho_j} z_{i'} z_{j'}^* \qquad \text{for } i+j<2n+1, i\neq j \tag{7.5}$$

$$z_i^* z_i = z_i z_i^* + (1-q^2)\sum_{k>i} z_k z_k^* \qquad \text{for } i>n \tag{7.6}$$

$$z_i^* z_i = z_i z_i^* + (1-q^2)q^{2\rho_i} z_{i'} z_{i'}^* + (1-q^2)\sum_{k>i} z_k z_k^* \qquad \text{for } i\leq n \tag{7.7}$$

$$\sum_{i=1}^{2n} z_i z_i^* = 1 \tag{7.8}$$

To get a faithful realisation of $C(H_q^{2n})$ in a Hilbert space, we need to find all its irreducible representations. It follows from the commutation relations that $\|z_i\| \leq 1$, for $1 \leq i \leq 2n$ and z_{2n} is normal. We denote $z_{2n}^* z_{2n}$ by ω.

Proposition 7.1. *Let π be a representation of $C(H_q^{2n})$. Then one has,*

1. $z_i\omega = q^{-2}\omega z_i$, *and* $z_i^*\omega = q^2\omega z_i^*$, $\quad \forall i \neq 1$ *or* $2n$.
   $z_1\omega = q^{-4}\omega z_1$, *and* $z_1^*\omega = q^4\omega z_1^*$.
2. $\pi(\omega) = I$, $\qquad$ *on* $\bigcap_{i=1}^{2n-1} \ker \pi(z_i^*)$.
3. $1_{(q^{2m+2},q^{2m})}(\pi(\omega)) = 0$, $\quad \forall m \in \mathbb{N}$.
4. $\ker(z_i) \subseteq \ker(z_k^*)$, *for* $k \geq i$, *and* $1 \leq i \leq 2n$.
5. *if u is a nonzero eigenvector of $\pi(\omega)$ corresponding to eigenvalue q^{2m}, then $u \notin \ker \pi(z_i)$, for $1 \leq i \leq 2n-1$.*
6. *Either* $\sigma(\pi(\omega)) = \{q^{2m} : m \in \mathbb{N}\} \bigcup \{0\}$ *or* $\sigma(\pi(\omega)) = \{0\}$.

Proof:

1. It will follow from the commutation relations (7.1), (7.3) and (7.5).
2. Easy to see from (7.8).
3. From the commutation relations, it follows that $z_1^* f(\omega) = f(q^4\omega)z_1^*$ and $z_i^* f(\omega) = f(q^2\omega)z_i^*$ for all $i \neq 1$ for all continuous functions f and hence for all L_∞ functions. Thus

   $$\pi(z_1)^* 1_{(q^{2n+2},q^{2n})}(\pi(\omega)) = 1_{(q^{2n+2},q^{2n})}(q^4\pi(\omega))\pi(z_1)^* = 1_{(q^{2n-2},q^{2n-4})}(\pi(\omega))\pi(z_1)^*,$$

   $$\pi(z_i)^* 1_{(q^{2n+2},q^{2n})}(\pi(\omega)) = 1_{(q^{2n+2},q^{2n})}(q^2\pi(\omega))\pi(z_i)^* = 1_{(q^{2n},q^{2n-2})}(\pi(\omega))\pi(z_1)^*.$$

   By repeated application, and using (7.8) and the fact that $\sigma(\omega) \subseteq [0,1]$, it follows that $1_(q^{2n+2}, q^{2n})(\pi(\omega)) = 0$.

4. Let $h \in \ker(\pi(z_i)$ and $i > n$. Using (7.6), we have,

$$
\begin{aligned}
\langle z_i^* z_i h, h\rangle &= \left\langle z_i z_i^* h + (1-q^2)\sum_{k>i} z_k z_k^* h, h \right\rangle. \\
\Rightarrow &\|z_i^* h\|^2 + (1-q^2)\sum_{k>i}\|z_k^* h\| = 0. \\
\Rightarrow &\|z_k^* h\| = 0 \qquad \text{for } k \geq i. \\
\Rightarrow &z_k^* h = 0. \\
\Rightarrow &h \in \ \ker\, \pi(z_k^*).
\end{aligned}
$$

For $i \leq n$, use (7.7) and follow similar steps.

5. From part (4), we have, $\ker(z_i) \subseteq \ker(z_{2n}^*) = \ker(z_{2n}) = \ker(\omega)$. Now, if u is a non-zero eigenvector of $\pi(\omega)$ corresponding to eigenvalue q^{2m} for some $m \in \mathbb{N}$, then $u \notin \ker(z_{2n}^*)$. Hence, $u \notin \ker(z_i)$ for $1 \leq i \leq 2n$.

6. From part (3), and the fact that $\|\omega\| \leq 1$, it follows that, $\sigma(\pi(\omega)) \subseteq \left\{q^{2m} : m \in \mathbb{N}\right\} \bigcup \{0\}$. Define,

$$A = \left\{m \in \mathbb{N} : q^{2m} \in \sigma(\pi(\omega))\right\}.$$

If $A = \phi$, we have, $\sigma(\pi(\omega)) = \{0\}$. If $A \neq \phi$, define,

$$m_0 = \ \inf\ \left\{m \in \mathbb{N} : q^{2m} \in \sigma(\pi(\omega))\right\}.$$

Let u be a nonzero eigenvector corresponding to q^{2m_0} and let $u \notin \ker(z_i^*)$ for some $i \in \{1, 2, \cdots, 2n-1\}$. Then from (7.1), it follows that $\pi(z_i^*)u$ is a nonzero eigenvector corresponding to eigenvalue q^{2m_0-2} or q^{2m_0-4} depending on whether $i \neq 1$ or $i = 1$, which contradicts the fact that m_0 is inf A. Hence $u \in \bigcap_{i=1}^{2n-1}\ker(z_i^*)$. As $\omega = I$ on $\bigcap_{i=1}^{2n-1}\ker(z_i^*)$, we get $m_0 = 0$. From part(5), it follows that $u \notin \ker(z_i)$ for any $i \in \{1, 2, \cdots 2n\}$. Now, applying (7.1), we have $\pi(z_2^m)u$ is a nonzero eigenvector corresponding to eigenvalue q^{2m}, for all $m \in \mathbb{N}$. This proves the claim.

□

Let π be a representation of $C(H_q^{2n})$ in a Hilbert space $\mathcal{H}$. From (7.1), it follows that $\ker\pi(\omega)$ is an invariant subspace for π. Therefore, if π is irreducible, then either $\pi(\omega) = 0$ or $\ker(\pi(\omega)) = 0$. Assume $\pi(\omega) \neq 0$. Then $\ker(\pi(\omega)) = 0$, and by part (6), we have, $\sigma(\pi(\omega)) = \left\{q^{2m} : m \in \mathbb{N}\right\} \bigcup \{0\}$. Hence, $\mathcal{H}$ decomposes as,

$$\mathcal{H} = \oplus_{m\in\mathbb{N}}\mathcal{H}_m.$$

where $\mathcal{H}_m$ is the eigenspace of $\pi(\omega)$ corresponding to eigenvalue q^{2m}.
It is clear from (7.1), that for $1 < i < 2n, \pi(z_i)$ sends $\mathcal{H}_m$ into $\mathcal{H}_{m+1}$ and $\pi(z_i^*)$ sends $\mathcal{H}_m$ into $\mathcal{H}_{m-1}, \pi(z_1)$ sends $\mathcal{H}_m$ into $\mathcal{H}_{m+2}$ and $\pi(z_i^*)$ sends $\mathcal{H}_m$ into $\mathcal{H}_{m-2}$. Also, $\pi(z_{2n})$ and $\pi(z_{2n}^*)$ keeps $\mathcal{H}_m$ invariant. Observe that, $\pi(z_{2n})_{|\mathcal{H}_0}$ is an unitary operator.

Proposition 7.2. *Let $u \in \bigcap_{i=1}^{2n-1}$ ker $\pi(z_i^*)$. Then,*

$$\pi(z_{2n})u \in \bigcap_{i=1}^{2n-1} \ker \pi(z_i^*),$$

$$\pi(z_{2n}^*)u \in \bigcap_{i=1}^{2n-1} \ker \pi(z_i^*).$$

Proof: We need to show that $\pi(z_i)\pi(z_{2n})u = \pi(z_i^*)\pi(z_{2n})u = 0$, for all $i \in \{1, 2, \cdots 2n-1\}$, which easily follows from $(7.1), (7.3)$ and (7.4). □

Let K be a subspace of $\bigcap_{i=1}^{2n-1} \ker \pi(z_i^*)$ such that $\pi(z_{2n})h \in K, h \in K$. Define

$$\mathcal{H}^K = \text{ linear span } \{ \pi(z_1)^{\alpha_1}\pi(z_2)^{\alpha_2} \cdots \pi(z_{2n-1})^{\alpha_{2n-1}}h : h \in K\}.$$

Lemma 7.3. *Let π be an irreducible representation of $C(H_q^{2n})$ such that $\pi(z_{2n}) \neq 0$. Then, $\mathcal{H}^K$ is an invariant subspace of π .*

Proof: Define, for $h \in K$,

$$h(\alpha_1, \alpha_2, \cdots, \alpha_{2n-1}) = \pi(z_1)^{\alpha_1}\pi(z_2)^{\alpha_2} \cdots \pi(z_{2n-1})^{\alpha_{2n-1}}h.$$

It is clear that $\pi(z_{2n})$ keeps $\mathcal{H}^K$ invariant as,

$$\pi(z_{2n})h(\alpha_1, \alpha_2, \cdots, \alpha_{2n-1}) = q^{(\sum_{l=1}^{2n-1} \alpha_l)+\alpha_1} h(\alpha_1, \alpha_2, \cdots, \alpha_{2n-1}).$$

For $1 \leq i \leq n$,

$$\pi(z_i)h(\alpha_1, \alpha_2, \cdots, \alpha_{2n-1}) = q^{\sum_{l=1}^{i-1} \alpha_l} h(\alpha_1, \cdots, \alpha_{i-1}, \alpha_i + 1, \alpha_{i+1}, \cdots, \alpha_{2n-1}) \in \mathcal{H}^K.$$

for $i = 2n - 1$,

$$\pi(z_{2n-1})h(\alpha_1, \alpha_2, \cdots, \alpha_{2n-1}) = q^{\alpha_1}\pi(z_1)^{\alpha_1}\pi(z_{2n-1})\pi(z_2)^{\alpha_2}\pi(z_3)^{\alpha_3} \cdots \pi(z_{2n-1})^{\alpha_{2n-1}}h.$$

Repeated application of (7.2) gives,

$$z_i z_{i'}^m = q^{2m} z_{i'}^m z_i - (1 - q^{2m}) \sum_{k>i} q^{i-k} z_{i'}^{m-1} z_k z_{k'}.$$

Hence we have,

$$\begin{aligned}
&\pi(z_{2n-1})h(\alpha_1,\alpha_2,\cdots,\alpha_{2n-1})\\
&= q^{\alpha_1}\pi(z_1)^{\alpha_1}\pi(q^{2\alpha_2}z_2^{\alpha_2}z_{n-1} - q(1-q^{2\alpha_2})z_2^{\alpha_2-1}z_{2n}z_1)\pi(z_3)^{\alpha_3}\cdots\pi(z_{2n-1})^{\alpha_{2n-1}}h.\\
&= q^{\alpha_1+2\alpha_2+\sum_{l=3}^{2n-3}}h(\alpha_1,\alpha_2,\cdots,\alpha_{2n-1}+1)\\
&-q^{\alpha_1+1}(1-q^{2\alpha_2})\pi(z_1)^{\alpha_1}\pi(z_2^{\alpha_2-1}z_{2n}z_1)h(0,0,\alpha_3,\cdots,\alpha_{2n-1}).
\end{aligned}$$

We have shown above that z_1, z_2, and z_{2n} keeps $\mathcal{H}^K$ invriant. Hence,

$$\pi(z_{2n-1})h(\alpha_1,\alpha_2,\cdots,\alpha_{2n-1}) \in \mathcal{H}^K.$$

Similarly, by using backward induction , we can show that $\mathcal{H}^K$ is invariant under the action of $\pi(z_1), \pi(z_2), \cdots$ and $\pi(z_n)$. Also, we have

$$\pi(z_{2n}^*)h(\alpha_1,\alpha_2,\cdots,\alpha_{2n-1}) = q^{(\sum_{l=1}^{2n-1}\alpha_l)+\alpha_1}h(\alpha_1,\alpha_2,\cdots,\alpha_{2n-1}).$$

This shows that $\pi(z_{2n}^*)$ keeps $\mathcal{H}^K$ invriant.
By applying (7.5) and (7.6) repeatedly, we get,

$$z_{2n-1}^* z_1^m = q^m z_1^m z_{2n-1}^* + mq^m(1-q^2)\epsilon_{2n-1}\epsilon_1 q^{\rho_{2n-1}+\rho_1} z_1^{m-1}z_2 z_{2n}^*$$

$$z_{2n-1}^* z_{2n-1}^m = z_{2n-1}^m z_{2n-1}^* + (1-q^{2m})z_{2n-1}^{m-1}\omega$$

Hence we have,

$$\begin{aligned}
&\pi(z_{2n-1}^*)h(\alpha_1,\alpha_2,\cdots,\alpha_{2n-1})\\
&= q^{\alpha_1}\pi(z_1^{\alpha_1}z_{2n-1}^* + \alpha_1 q^{\alpha_1}(1-q^2)\epsilon_{2n-1}\epsilon_1 q^{\rho_{2n-1}+\rho_1}z_1^{\alpha_1-1}z_2 z_{2n}^*)\pi(z_2)^{\alpha_2}\cdots\pi(z_{2n-1})^{\alpha_{2n-1}}h.\\
&= q^{(\sum_{l=1}^{2n-2}l)+\alpha_2}\pi(z_1)^{\alpha_1}\cdots\pi(z_{2n-2})^{\alpha_{2n-2}}\pi(z_{2n-1})^*\pi(z_{2n-1})^{\alpha_{2n-1}}h\\
&\quad -\alpha_1 q^{\alpha_1}(1-q^2)q^{\rho_{2n-1}+\rho_1}\pi(z_1^{\alpha_1-1})\pi(z_2)\pi(z_{2n}^*)\pi(z_2)^{\alpha_2}\cdots\pi(z_{2n-1})^{\alpha_{2n-1}}h.\\
&= (1-q^{2\alpha_1})q^{(\sum_{l=1}^{2n-2}l)+\alpha_2}\pi(z_1)^{\alpha_1}\cdots\pi(z_{2n-2})^{\alpha_{2n-2}}\pi(z_{2n-1})^{\alpha_{2n-1}-1}\pi(\omega)h\\
&\quad -\alpha_1 q^{\alpha_1}(1-q^2)q^{\rho_{2n-1}+\rho_1}\pi(z_1^{\alpha_1-1})\pi(z_2)\pi(z_{2n}^*)\pi(z_2)^{\alpha_2}\cdots\pi(z_{2n-1})^{\alpha_{2n-1}}h.
\end{aligned}$$

Since $\pi(z_1), \pi(z_2)$, and $\pi(z_{2n^*})$ keeps $\mathcal{H}^K$ invriant, $\mathcal{H}^K$ is invarint under the action of $\pi(z_{2n-1}^*)$. By using backward induction and following similar steps, we can show that $\mathcal{H}^K$ is invariant for π.

□

It follows from the lemma that if K is an invariant subspace for $\bigcap_{i=1}^{2n-1}$ ker $\pi(z_i^*)$, then $\mathcal{H}^K$ is an invariant subspace for π and is a proper invariant subspace for π if K is proper subspace of $\bigcap_{i=1}^{2n-1}$ ker $\pi(z_i^*)$. Therefore, if π is an irreducible representation, then the space $\bigcap_{i=1}^{2n-1}\ker\pi(z_i^*)$ is one dimensional.

Corollary 7.4. *Let π be an irreducible representation such that $\pi(z_{2n}) \neq 0$. Then,*

$$\mathcal{H}_m = \text{ linear span } \left\{ \pi(z_1)^{\alpha_1}\pi(z_2)^{\alpha_2}\cdots\pi(z_{2n-1})^{\alpha_{2n-1}}u : (\sum_{l=1}^{2n-1}\alpha_l) + \alpha_1 = m \right\}.$$

Now, pick a unit vector u in $\bigcap_{i=1}^{2n-1} \ker \pi(z_i^*)$. Define,

$$u_{\alpha_2,\cdots,\alpha_{2n-1},\alpha_0} = \pi(z_{2n-1})^{\alpha_{2n-1}}\pi(z_{2n-2})^{\alpha_{2n-2}}\cdots\pi(z_2)^{\alpha_2}[\pi(z_n),\pi(z_{n+1})]^{\alpha_0}u.$$

where $\alpha_i \in \mathbb{N}$. Now, we develop some tools by analysing the defining relations more closely.

Proposition 7.5. *Let π be an irreducible representation of $C(H_q^{2n})$ such that $\pi(z_{2n}) \neq 0$.*

1. *for $i > n$,*

$$\pi(z_i)^*\pi(z_i)^m = \pi(z_i)^m\pi(z_i)^* + (1-q^{2m})\sum_{k>i}\pi(z_i)^{m-1}\pi(z_k)\pi(z_k)^*.$$

2. *for $i \leq n$,*

$$\begin{aligned}\pi(z_i)^*\pi(z_i)^m &= \pi(z_i)^m\pi(z_i)^* + q^{2\rho_i}(1-q^{2m})\pi(z_i)^{m-1}\pi(z_{i'})\pi(z_{i'})^* \\ &+(1-q^{2m})\sum_{k>i}\pi(z_i)^{m-1}\pi(z_k)\pi(z_k)^*.\end{aligned}$$

3. *for $i+j < 2n+1, i \neq j$,*

$$\pi(z_i)^*\pi(z_j)^m = q^m\pi(z_j)^m\pi(z_i)^* + mq^m(1-q^2)\epsilon_i\epsilon_j q^{\rho_i+\rho_j}\pi(z_j)^{m-1}\pi(z_{i'})\pi(z_{j'})^*.$$

4. *for $i > n$,*

$$\pi(z_i)^*[\pi(z_n),\pi(z_{n+1})]^m = q^{2m}[\pi(z_n),\pi(z_{n+1})]^m\pi(z_i)^*.$$

5. *for $i > n$,*

$$\pi(z_i)^*u_{\alpha_2,\cdots,\alpha_{2n-1},\alpha_0} = Cu_{\alpha_2,\cdots\alpha_{i-1},\alpha_i-1,\alpha_{i+1}\cdots,\alpha_{2n-1},\alpha_0}$$

   where C is some non-zero constant.

6. *for $n < i < 2n$,*

$$\pi(z_i)^*u_{\alpha_2,\cdots\alpha_{i-1},0,\alpha_{i+1}\cdots,\alpha_{2n-1},\alpha_0} = 0$$

7.

$$\pi(z_n)^*[\pi(z_n),\pi(z_{n+1})]^m = q^{2m}[\pi(z_n),\pi(z_{n+1})]^m\pi(z_n)^*$$

$$+((1-q^4)(1-q^2)\sum_{l=0}^{k-1} q^{4l}[\pi(z_n),\pi(z_{n+1})]^{k-1-l}\pi(z_{n+1})[\pi(z_n),\pi(z_{n+1})]^l)\sum_{k>n+1}\pi(z_k)\pi(z_k)^*.$$

8. *for* $1 < i \leq n$,

$$\pi(z_i)^*\pi(z_{i-1})^{\alpha_{i-1}}\cdots\pi(z_2)^{\alpha_2}[\pi(z_n),\pi(z_{n+1})]^{\alpha_0}u = C\pi(z_{i-1})^{\alpha_{i-1}}\cdots\pi(z_2)^{\alpha_2}\pi(z_i)^*[\pi(z_n),\pi(z_{n+1})]^{\alpha_0}u.$$

where C *is some non-zero constant.*

9.

$$[\pi(z_n),\pi(z_{n+1})]^*[\pi(z_n),\pi(z_{n+1})]^m u = C[\pi(z_n),\pi(z_{n+1})]^{m-1}u.$$

where C *is some non-zero constant.*

10. *for* $1 < i \leq n$,

$$\pi(z_i)\pi(z_i^{\prime m}) = q^{2m}\pi(z_{i'}^m)\pi(z_i) - \sum_{k>i}(1-q^{2m})q^{i-k}\pi(z_k)\pi(z_{k'})\pi(z_i^{m-1}).$$

11. *for* $1 \leq i < n$,

$$\pi(z_i^*)[\pi(z_n),\pi(z_{n+1})]^m u = C\pi(z_i^{'})[\pi(z_n),\pi(z_{n+1})]^{m-1}u.$$

where C *is some constant.*

12.

$$\pi(z_1)^* u_{\alpha_2,\cdots,\alpha_n,0,\cdots,0,\alpha_0} = Cu_{\alpha_2,\cdots\alpha_n,0,\cdots,0,\alpha_0-1}$$

Proof: We will prove part(4) and part(9) of this proposition. Other parts will follow by direct calculation using commutation relations.

1. For $i > n+1$, It follows from (7.1). For $i = n+1$, it is enough to show for $m = 1$.

$$\begin{aligned}
&\pi(z_{n+1})^*[\pi(z_n),\pi(z_{n+1})] \\
&= \pi(z_{n+1})^*\pi(z_n)\pi(z_{n+1}) - \pi(z_{n+1})^*\pi(z_{n+1})\pi(z_n). \\
&= q^2\pi(z_n)\pi(z_{n+1})^*\pi(z_{n+1}) - \pi(z_{n+1})\pi(z_{n+1})^*\pi(z_n) - (1-q^2)\sum_{k>n+1}\pi(z_k)\pi(z_k^*)\pi(z_n). \\
&= q^2\pi(z_n)\pi(z_{n+1})\pi(z_{n+1})^* + q^2(1-q^2)\sum_{k>n+1}\pi(z_n)\pi(z_k)\pi(z_k^*) \\
&\quad -q^2\pi(z_{n+1})\pi(z_n)\pi(z_{n+1})^* - q^2(1-q^2)\sum_{k>n+1}\pi(z_n)\pi(z_k)\pi(z_k^*). \\
&= q^2[\pi(z_n),\pi(z_{n+1})]\pi(z_{n+1})^*
\end{aligned}$$

2.

$$
\begin{aligned}
&[\pi(z_n),\pi(z_{n+1})]^*[\pi(z_n),\pi(z_{n+1})]^m u\\
&=\pi(z_n^*)\pi(z_{n+1})^*[\pi(z_n),\pi(z_{n+1})]^m u+\pi(z_{n+1}^*)\pi(z_n^*)[\pi(z_n),\pi(z_{n+1})]^m u.\\
&=C\pi(z_{n+1}^*)\sum_{l=0}^{m-1}q^{4l}[\pi(z_n),\pi(z_{n+1})]^{m-1-l}\pi(z_{n+1})[\pi(z_n),\pi(z_{n+1})]^l,\\
&\qquad\qquad\qquad\qquad\qquad\qquad\text{from part(7) of the proposition 7.5 .}\\
&=C\sum_{l=0}^{m-1}q^{4l}[\pi(z_n),\pi(z_{n+1})]^{m-1-l}\pi(z_{n+1}^*)\pi(z_{n+1})[\pi(z_n),\pi(z_{n+1})]^l,\qquad\text{by (7.1).}\\
&=C\sum_{l=0}^{m-1}q^{4l}[\pi(z_n),\pi(z_{n+1})]^{m-1-l}(\pi(z_{n+1})\pi(z_{n+1})^*+(1-q^2)\sum_{k>i}\pi(z_k)\pi(z_k)^*)[\pi(z_n),\pi(z_{n+1})]^l,\\
&=C[\pi(z_n),\pi(z_{n+1})]^{m-1}.
\end{aligned}
$$

□

From part (9) of proposition 7.5, $[\pi(z_n),\pi(z_{n+1})]^{\alpha_0}u\neq 0$ and since $\ker\pi(z_i)\subseteq\ker\pi(z_{2n}^*)=\{0\}$, we have $u_{\alpha_2,\alpha_2,\cdots,\alpha_{2n-1},\alpha_0}\neq 0,\quad(\alpha_2,\alpha_2,\cdots\alpha_{2n-1},\alpha_0)\in\mathbb{N}^{2n-1}$ and we can define,

$$e_{\alpha_2,\cdots,\alpha_{2n-1},\alpha_0}=\frac{u_{\alpha_2,\cdots,\alpha_{2n-1},\alpha_0}}{\left\|u_{\alpha_2,\cdots,\alpha_{2n-1},\alpha_0}\right\|}$$

Proposition 7.6. *Assume,* $\left\{e_{\alpha_2,\cdots,\alpha_{2n-1},\alpha_0}:(\sum_{l=2}^{2n-1}\alpha_l)+2\alpha_0\leq L\right\}$ *form an orthonormal basis for* $\mathcal{H}_{\leq L}=\mathcal{H}_0\oplus\mathcal{H}_1\oplus\cdots\oplus\mathcal{H}_L$*. If* $2(r+s)+1\leq L$*, then*
$[\pi(z_n),\pi(z_{n+1})]^r\,\pi(z_{n+1})\,[\pi(z_n),\pi(z_{n+1})]^s\,u=C\pi(z_{n+1})\,[\pi(z_n),\pi(z_{n+1})]^{r+s}\,u.$
where C *is a non-zero constant.*

Proof: It is enough to prove the statement for $r=1$.
The condition ensures that, $[\pi(z_n),\pi(z_{n+1})]\,\pi(z_{n+1})\,[\pi(z_n),\pi(z_{n+1})]^s\,u\in\mathcal{H}_{\leq L}$. Hence,

$$[\pi(z_n),\pi(z_{n+1})]\,\pi(z_{n+1})\,[\pi(z_n),\pi(z_{n+1})]^s\,u=\sum_{(\alpha_2\cdots\alpha_{2n-1},\alpha_0):(\sum_{i=2}^{2n-1}\alpha_i)+2\alpha_0\leq L}C(\alpha_2,\cdots,\alpha_0)e_{\alpha_2,\cdots,\alpha_0}.$$

where $C(\alpha_2,\cdots,\alpha_{2n-1},\alpha_0)=\left\langle[\pi(z_n),\pi(z_{n+1})]\,\pi(z_{n+1})\,[\pi(z_n),\pi(z_{n+1})]^s\,u,e_{\alpha_2,\cdots,\alpha_{2n-1},\alpha_0}\right\rangle$.
We will show that $C(\alpha_2,\cdots,\alpha_{2n-1},\alpha_0)=0$ if $\alpha_{n+1}=1$ and $\alpha_0=s+1$. Now,
Case $1:\alpha_i\neq 0$ for some $i>n+1$.
Applying part (4) of the proposition 7.5 we get,

$$\pi(z_i)^*\,[\pi(z_n),\pi(z_{n+1})]\,\pi(z_{n+1})\,[\pi(z_n),\pi(z_{n+1})]^s\,u=0.$$

This shows that if $\alpha_i\neq 0$ for any $i\in\{n+2,n+3,\cdots,2n-1\}$, $\quad C(\alpha_2,\cdots,\alpha_{2n-1},\alpha_0)=0.$

Case 2 : $\alpha_{n+1} \geq 1$ and $\alpha_i = 0$ for all $i > n+1$.

$$\begin{aligned}
&\pi(z_{n+1})^* \left[\pi(z_n), \pi(z_{n+1})\right] \pi(z_{n+1}) \left[\pi(z_n), \pi(z_{n+1})\right]^s u \\
&= q^2 \left[\pi(z_n), \pi(z_{n+1})\right] \pi(z_{n+1}^* z_{n+1}) \left[\pi(z_n), \pi(z_{n+1})\right]^s u. \\
&= q^2 \left[\pi(z_n), \pi(z_{n+1})\right] \pi(z_{n+1} z_{n+1}^*) \left[\pi(z_n), \pi(z_{n+1})\right]^s u \\
&\quad\quad + \sum_{k>n+1} q^2(1-q^2) \left[\pi(z_n), \pi(z_{n+1})\right] \pi(z_k z_k^*) \left[\pi(z_n), \pi(z_{n+1})\right]^s u. \\
&= q^{4s+2}(1-q^2) \left[\pi(z_n), \pi(z_{n+1})\right]^{s+1} u. \qquad \left(\text{ as } u \in \bigcap_{i=1}^{2n-1} \ker \pi(z_i^*)\right).
\end{aligned}$$

Now,

$$\begin{aligned}
&\left\langle u_{\alpha_2,\cdots,\alpha_{n+1},0,\cdots,0,\alpha_0}, \left[\pi(z_n), \pi(z_{n+1})\right] \pi(z_{n+1}) \left[\pi(z_n), \pi(z_{n+1})\right]^s u \right\rangle \\
&= \left\langle u_{\alpha_2,\cdots,\alpha_{n+1}-1,0,\cdots,0,\alpha_0}, \pi(z_{n+1}^*) \left[\pi(z_n), \pi(z_{n+1})\right] \pi(z_{n+1} \left[\pi(z_n), \pi(z_{n+1})\right]^s u \right\rangle. \\
&= \left\langle u_{\alpha_2,\cdots,\alpha_{n+1}-1,0,\cdots,0,\alpha_0}, q^{4s+2}(1-q^2) \left[\pi(z_n), \pi(z_{n+1})\right]^{s+1} u \right\rangle. \\
&\begin{cases} \neq 0 & \text{if } \alpha_{n+1} = 1, \alpha_0 = s+1, \alpha_{n-1} = \cdots \alpha_1 = 0, \\ = 0 & \text{otherwise }. \end{cases}
\end{aligned}$$

Case 3 : $\alpha_i = 0$ for all $i \geq n+1$.
By using commutation relations, we have,

$$\begin{aligned}
&\pi(z_n)^* u_{\alpha_2,\cdots,\alpha_n,0,\cdots,0,\alpha_0} \\
&= \pi(z_n)^{\alpha_n} \pi(z_n)^* \pi(z_{n-1})^{\alpha_{n-1}} \cdots, \left[\pi(z_n), \pi(z_{n+1})\right]^{\alpha_0} u \\
&\quad + (1-q^{2\alpha_n}) \sum_{k>n} \pi(z_n)^{\alpha_n - 1} \pi(z_k) \pi(z_k)^* \pi(z_{n-1})^{\alpha_{n-1}} \cdots, \left[\pi(z_n), \pi(z_{n+1})\right]^{\alpha_0} u. \\
&= C \pi(z_n)^{\alpha_n} \pi(z_{n-1})^{\alpha_{n-1}} \cdots, \pi(z_2)^{\alpha_2} \pi(z_n)^* \left[\pi(z_n), \pi(z_{n+1})\right]^{\alpha_0} u \\
&\quad + (1-q^{2\alpha_n}) \sum_{k>n} \pi(z_n)^{\alpha_n - 1} \pi(z_k) \pi(z_k)^* \pi(z_{n-1})^{\alpha_{n-1}} \cdots, \left[\pi(z_n), \pi(z_{n+1})\right]^{\alpha_0} u.
\end{aligned}$$

for some non-zero constant C.

$$\begin{aligned}
&\text{second term of R.H.S } = C u_{\alpha_2,\cdots,\alpha_{n-1},\alpha_n - 1,0,\cdots,0,\alpha_0} \qquad \text{(from part (5) of the proposition 7.5)}. \\
&\text{first term of R.H.S } = C \pi(z_n)^{\alpha_n} \cdots, \pi(z_2)^{\alpha_2} [\pi(z_n), \pi(z_{n+1})]^{\alpha_0} \pi(z_n)^* \\
&\qquad + C \pi(z_n)^{\alpha_n} \cdots, \pi(z_2)^{\alpha_2} \left(\sum_{l=0}^{\alpha-1} q^{4l} [\pi(z_n), \pi(z_{n+1})]^{\alpha_0 - 1 - l} \pi(z_{n+1}) [\pi(z_n), \pi(z_{n+1})]^l\right) u.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\pi(z_{n+1})^* \pi(z_n)^* u_{\alpha_2,\cdots,\alpha_n,0,\cdots,0,\alpha_0} \\
&= C \pi(z_n)^{\alpha_n} \cdots \pi(z_2)^{\alpha_2} \pi(z_{n+1}^*) \left(\sum_{l=0}^{\alpha_0 - 1} q^{4l} [\pi(z_n), \pi(z_{n+1})]^{\alpha_0 - 1 - l} \pi(z_{n+1}) [\pi(z_n), \pi(z_{n+1})]^l\right) u.
\end{aligned}$$

$$= C\pi(z_n)^{\alpha_n}\cdots\pi(z_2)^{\alpha_2}(\sum_{l=0}^{\alpha_0-1} q^{4l}[\pi(z_n),\pi(z_{n+1})]^{\alpha_0-1-l}\pi(z_{n+1}^*)\pi(z_{n+1})[\pi(z_n),\pi(z_{n+1})]^l)u.$$
$$= C\pi(z_n)^{\alpha_n}\cdots\pi(z_2)^{\alpha_2}(\sum_{l=0}^{\alpha_0-1} q^{4l}[\pi(z_n),\pi(z_{n+1})]^{\alpha_0-1-l}(\pi(z_{n+1})\pi(z_{n+1}^*)$$
$$+\sum_{k>n+1}\pi(z_k)\pi(z_k^*))[\pi(z_n),\pi(z_{n+1})]^l)u.$$
$$= C\pi(z_n)^{\alpha_n}\cdots\pi(z_2)^{\alpha_2}[\pi(z_n),\pi(z_{n+1})]^{\alpha_0-1}u.$$

By above calculation and by proposition 7.5 , we have,

$$\begin{aligned}
&\langle u_{\alpha_2,\cdots,\alpha_n,0.\cdots,0,\alpha_0}, [\pi(z_n),\pi(z_{n+1})]\pi(z_{n+1})[\pi(z_n),\pi(z_{n+1})]^s u\rangle\\
&= \langle[\pi(z_n),\pi(z_{n+1})]^* u_{\alpha_2,\cdots,\alpha_n,0.\cdots,0,\alpha_0}, \pi(z_{n+1})[\pi(z_n),\pi(z_{n+1})]^s u\rangle,\\
&= \left\langle \pi(z_{n+1}^* z_n^* - z_n^* z_{n+1}^*) u_{\alpha_2,\cdots,\alpha_n,0.\cdots,0,\alpha_0}, \pi(z_{n+1})[\pi(z_n),\pi(z_{n+1})]^s u\right\rangle,\\
&= \left\langle \pi(z_{n+1}^* z_n^*) u_{\alpha_2,\cdots,\alpha_n,0.\cdots,0,\alpha_0}, \pi(z_{n+1})[\pi(z_n),\pi(z_{n+1})]^s u\right\rangle,\\
&= \left\langle C\pi(z_n)^{\alpha_n}\pi(z_{n-1})^{\alpha_{n-1}}\cdots,\pi(z_2)^{\alpha_2}[\pi(z_n),\pi(z_{n+1})]^{\alpha_0-1}u, \pi(z_{n+1})[\pi(z_n),\pi(z_{n+1})]^s u\right\rangle,\\
&= \left\langle C\pi(z_{n+1}^*)\pi(z_n)^{\alpha_n}\pi(z_{n-1})^{\alpha_{n-1}}\cdots,\pi(z_2)^{\alpha_2}[\pi(z_n),\pi(z_{n+1})]^{\alpha_0-1}u, [\pi(z_n),\pi(z_{n+1})]^s u\right\rangle,\\
&= 0
\end{aligned}$$

It proves the claim. □

Lemma 7.7. *Let π be an irreducible representation on a Hilbert space $\mathcal{H}$ with $\pi(z_{2n}) \neq 0$. Then, $\left\{e_{\alpha_2,\alpha_2,\cdots,\alpha_{2n-1},\alpha_0},: (\alpha_2,\alpha_3,\cdots\alpha_{2n-1},\alpha_0)\in\mathbb{N}^{2n-1}\right\}$ defined as above, form an orthonormal basis for $\mathcal{H}$.*

Proof: From corollary 7.4, it is enough to show that $\alpha\neq\beta$, u_α is orthogonal to u_β. We apply induction on L_α defined as $(\sum_{i=2}^{2n-1}\alpha_i)+2\alpha_0$. For $L_\alpha=0$, claim is true as $u\neq 0$. Assume the hypothesis for $L_\alpha\le N-1$. Note that, $u_{\alpha_2,\cdots,\alpha_{2n-1},\alpha_0}\in\mathcal{H}_{L_\alpha}$. Hence, by induction hypothesis and corollary 7.4, it follows that $\left\{e_{\alpha_2,\cdots,\alpha_{2n-1},\alpha_0}/(\sum_{l=2}^{2n-1}\alpha_l)+2\alpha_0=m\right\}$ form an orthonormal basis of $\mathcal{H}_m$ for $m\le N-1$.
If α and β are such that $L_\alpha\neq L_\beta$, then $u_\alpha\in\mathcal{H}_{L_\alpha}$ and $u_\beta\notin\mathcal{H}_{L_\alpha}$ which shows that u_α and u_β are orthogonal. Take α and β such that $L_\alpha=L_\beta=N$. Assume for some $i>n,\alpha_i\neq 0$. Choose maximum such i. From part(6) of proposition 7.5, it follows that,

$$\begin{aligned}
\langle u_\alpha,u_\beta\rangle &= \left\langle u_{\alpha_2,\cdots,\alpha_{i-1},\alpha_i-1,0,\cdots,0,\alpha_0}, \pi(z_i^*)u_\beta\right\rangle\\
&= \left\langle u_{\alpha_2,\cdots,\alpha_{i-1},\alpha_i-1,0,\cdots,0,\alpha_0}, Cu_{\beta_2,\cdots,\beta_{i-1},\beta_i-1,\beta_{i+1},\cdots,\beta_{2n-1},\beta_0}\right\rangle
\end{aligned}$$

where C is a non zero constant. Now, by using induction we get, $\langle u_\alpha,u_\beta\rangle\neq 0$ if and only if $\alpha=$

β. Hence, it is enough to consider α and β such that $\alpha_i = \beta_i = 0$ for $i > n$. Let $\alpha_n \neq 0$.

$$\begin{aligned}
\pi(z_n^*)u_{\beta_2,\cdots,\beta_n,0,\cdots,0,\beta_0} &= (\pi(z_n^{\beta_n})\pi(z_n^*) + \sum_{k>i} C\pi(z_n^{\beta_n-1})\pi(z_k)\pi(z_k^*))u_{\beta_2,\cdots,\beta_{n-1},0,\cdots,0,\beta_0} \\
&= C\pi(z_n^{\beta_n})\pi(z_{n+1})u_{\beta_2,\cdots,\beta_{n-1},0,\cdots,0,\beta_0-1} + Cu_{\beta_2,\cdots,\beta_n-1,0,\cdots,0,\beta_0} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(by proposition 7.6)} \\
&= (C\pi(z_{n+1})\pi(z_n^{\beta_n}) + \sum_{k>n+1} C\pi(z_k)\pi(z_k^{'})\pi(z_n^{\beta_n-1}))u_{\beta_2,\cdots,\beta_{n-1},0,\cdots,0,\beta_0-1} \\
&\quad + Cu_{\beta_2,\cdots,\beta_n-1,0,\cdots,0,\beta_0}
\end{aligned}$$

Hence,

$$\begin{aligned}
\langle u_\alpha, u_\beta\rangle &= \left\langle u_{\alpha_2,\cdots,\alpha_{n-1},\alpha_n-1,0,\cdots,0,\alpha_0}, \pi(z_n^*)u_\beta\right\rangle \\
&= \left\langle u_{\alpha_2,\cdots,\alpha_{n-1},\alpha_n-1,0,\cdots,0,\alpha_0}, Cu_{\beta_2,\cdots,\beta_n-1,0,\cdots,0,\beta_0} + C\pi(z_{2n})\pi(z_1)u_{\beta_2,\cdots,\beta_n-1,0,\cdots,0,\beta_0-1}\right\rangle \\
&= \left\langle u_{\alpha_2,\cdots,\alpha_{n-1},\alpha_n-1,0,\cdots,0,\alpha_0}, Cu_{\beta_2,\cdots,\beta_n-1,0,\cdots,0,\beta_0}\right\rangle \\
&\quad + \left\langle \pi(z_1^*)u_{\alpha_2,\cdots,\alpha_{n-1},\alpha_n-1,0,\cdots,0,\alpha_0}, Cu_{\beta_2,\cdots,\beta_n-1,0,\cdots,0,\beta_0-1}\right\rangle \\
&= \left\langle u_{\alpha_2,\cdots,\alpha_{n-1},\alpha_n-1,0,\cdots,0,\alpha_0}, Cu_{\beta_2,\cdots,\beta_n-1,0,\cdots,0,\beta_0}\right\rangle \\
&\quad + \left\langle u_{\alpha_2,\cdots,\alpha_{n-1},\alpha_n-1,0,\cdots,0,\alpha_0-1}, Cu_{\beta_2,\cdots,\beta_n-1,0,\cdots,0,\beta_0-1}\right\rangle
\end{aligned}$$

Again induction proves the claim. So, we will consider α and β such that $\alpha_i = \beta_i = 0$ for $i \geq n$. Assume that for some $i \in \{2, 3, \cdots, n-1\}, \alpha_i \neq 0$ or $\beta_i \neq 0$. Choose maximum such i. Without loss of generality, we assume that $\alpha_i \neq 0$.

$$\begin{aligned}
&\pi(z_i^*)u_{\beta_2,\cdots,\beta_i,0,\cdots,0,\beta_0} \\
&= (\pi(z_i)^{\beta_i}\pi(z_i)^* + q^{2\rho_i}(1-q^{2\beta_i})\pi(z_i)^{\beta_i-1}\pi(z_{i'})\pi(z_{i'})^* \\
&\qquad + (1-q^{2\beta_i})\sum_{k>i}\pi(z_i)^{\beta_i-1}\pi(z_k)\pi(z_k)^*)\pi(z_{i-1})^{\beta_{i-1}}\cdots,\pi(z_2)^{\beta_2}[\pi(z_n),\pi(z_{n+1})]^{\beta_0}u. \\
&= C\pi(z_i)^{\beta_i}\pi(z_i^{'})\pi(z_{i-1})^{\beta_{i-1}}\cdots,\pi(z_2)^{\beta_2}[\pi(z_n),\pi(z_{n+1})]^{\beta_0-1}u \\
&+ \sum_{i<k\leq n} C\pi(z_i)^{\beta_i-1}\pi(z_k)\pi(z_k)^*)\pi(z_{i-1})^{\beta_{i-1}}\cdots,\pi(z_2)^{\beta_2}[\pi(z_n),\pi(z_{n+1})]^{\beta_0-1}u + Cu_{\beta_2,\cdots,\beta_i-1,0,\cdots,0,\beta_0}. \\
&= C\pi(z_i^{'})\pi(z_i)^{\beta_i}\pi(z_{i-1})^{\beta_{i-1}}\cdots,\pi(z_2)^{\beta_2}[\pi(z_n),\pi(z_{n+1})]^{\beta_0-1}u \\
&+ \sum_{k>i'} C\pi(z_k))\pi(z_{k'})\pi(z_i)^{\beta_i-1}\pi(z_{i-1})^{\beta_{i-1}}\cdots,\pi(z_2)^{\beta_2}[\pi(z_n),\pi(z_{n+1})]^{\beta_0-1}u. \\
&+ \sum_{i<k\leq n} C\pi(z_i)^{\beta_i-1}\pi(z_{k'}))\pi(z_k)\pi(z_{i-1})^{\beta_{i-1}}\cdots,\pi(z_2)^{\beta_2}[\pi(z_n),\pi(z_{n+1})]^{\beta_0-1}u + Cu_{\beta_2,\cdots,\beta_i-1,0,\cdots,0,\beta_0} \\
&= Cu_{\beta_2,\cdots,\beta_i-1,0,\cdots,0,\beta_0} \\
&+ \sum_{n<k\leq 2n} C\pi(z_k))\pi(z_{k'})\pi(z_i)^{\beta_i-1}\pi(z_{i-1})^{\beta_{i-1}}\cdots,\pi(z_2)^{\beta_2}[\pi(z_n),\pi(z_{n+1})]^{\beta_0-1}u.
\end{aligned} \tag{7.9}$$

Hence,

$$\begin{aligned}
\langle u_\alpha, u_\beta\rangle &= \left\langle u_{\alpha_2,\cdots,\alpha_{i-1},\alpha_i-1,0,\cdots,0,\alpha_0}, \pi(z_i^*)u_{\beta_2,\cdots,\beta_i,0,\cdots,0,\beta_0}\right\rangle \\
&= \left\langle u_{\alpha_2,\cdots,\alpha_{i-1},\alpha_i-1,0,\cdots,0,\alpha_0}, Cu_{\beta_2,\cdots,\beta_i-1,0,\cdots,0,\beta_0} + C\pi(z_{2n})\pi(z_1)u_{\beta_2,\cdots,\beta_i-1,0,\cdots,0,\beta_0-1}\right\rangle \\
&= \left\langle u_{\alpha_2,\cdots,\alpha_{i-1},\alpha_i-1,0,\cdots,0,\alpha_0}, Cu_{\beta_2,\cdots,\beta_i-1,0,\cdots,0,\beta_0}\right\rangle \\
&\quad + \left\langle \pi(z_1^*)u_{\alpha_2,\cdots,\alpha_{i-1},\alpha_i-1,0,\cdots,0,\alpha_0}, Cu_{\beta_2,\cdots,\beta_i-1,0,\cdots,0,\beta_0-1}\right\rangle \\
&= \left\langle u_{\alpha_2,\cdots,\alpha_{i-1},\alpha_i-1,0,\cdots,0,\alpha_0}, Cu_{\beta_2,\cdots,\beta_i-1,0,\cdots,0,\beta_0}\right\rangle \\
&\quad + \left\langle u_{\alpha_2,\cdots,\alpha_{n-1},\alpha_i-1,0,\cdots,0,\alpha_0-1}, Cu_{\beta_2,\cdots,\beta_i-1,0,\cdots,0,\beta_0-1}\right\rangle
\end{aligned}$$

Again induction will settle the claim. Now, we take α and β such that $\alpha_i = \beta_i = 0,$ for all $i \neq 0$.

$$\begin{aligned}
&[\pi(z_n), \pi(z_{n+1})]^*[\pi(z_n), \pi(z_{n+1})]^{\beta_0}u \\
&= C[\pi(z_n), \pi(z_{n+1})]^{\beta_0-1}. \qquad \text{from part(9) of the proposition 7.5 .}
\end{aligned}$$

Hence,

$$\begin{aligned}
\langle u_\alpha, u_\beta\rangle &= \left\langle [\pi(z_n), \pi(z_{n+1})]^{\alpha_0}u, [\pi(z_n), \pi(z_{n+1})]^{\beta_0}u\right\rangle \\
&= \left\langle [\pi(z_n), \pi(z_{n+1})]^{\alpha_0-1}u, [\pi(z_n), \pi(z_{n+1})]^*[\pi(z_n), \pi(z_{n+1})]^{\beta_0}u\right\rangle \\
&= \left\langle [\pi(z_n), \pi(z_{n+1})]^{\alpha_0-1}u, C[\pi(z_n), \pi(z_{n+1})]^{\beta_0-1}u\right\rangle
\end{aligned}$$

This completes the proof. □

Corollary 7.8. $\|u_\alpha\|$ *is a fixed constant which is independent of the representation. More precisely, if* π *and* π' *are two irreducible representations with* $\pi(z_{2n}) \neq 0$ *and* $\pi'(z_{2n}) \neq 0$, *then,*

$$\|u_\alpha\| = \left\|u'_\alpha\right\|, \qquad \forall \alpha \in \mathbb{N}^{2n-1}.$$

where u_α *and* u'_α *are defined as above.*

Now, we aim to find all irreducible representation of $C(H_q^{2n})$. One way is to do explicit calculation to determine the operators $z_1, z_2, \cdots, z_{2n}$ as done in case of odd dimensional quantum sphere. But in this case, calculations are more complicted. So, to avoid complicated calculations, we show that one can completely determine an irreducible representation π of $C(H_q^{2n})$ given that $\pi(\omega) \neq 0$ and $\pi(z_{2n})u = tu$ for some fixed $t \in T$. Then, we use representation of Stiefel manifold $C(SP_q(2n)/C(SP_q(2n-2)$ to get explicit description of the representation.

Theorem 7.9. *Let π and $\pi^{'}$ be irreducible representations of $C(H_q^{2n})$ on a Hilbert space $\mathcal{H}$ and $\mathcal{H}^{'}$ respectively such that $\pi(z_{2n})|_{\bigcap_{i=1}^{2n-1}\ker\pi(z_i^*)} = tI = \pi^{'}(z_{2n})|_{\bigcap_{i=1}^{2n-1}\ker\pi^{'}(z_i^*)}$ for $t \in \mathbb{T}$. Then π and $\pi^{'}$ are equivalent.*

Proof: Without loss of generality, we can take $t = 1$. Let u and $u^{'}$ are unit vectors in $\bigcap_{i=1}^{2n-1}\ker\pi(z_i^*)$ and $\bigcap_{i=1}^{2n-1}\ker\pi^{'}(z_i^*)$ respectively. From the lemma 7.7, we have the canonical orthonormal bases for $\mathcal{H}$ and $\mathcal{H}^{'}$ given by, $\left\{e_{\alpha_2,\alpha_3,\cdots,\alpha_{2n-1},\alpha_0}, (\alpha_2,\alpha_3,\cdots\alpha_{2n-1},\alpha_0) \in \mathbb{N}^{2n-1}\right\}$ and
$\left\{e^{'}_{\alpha_2,\alpha_3,\cdots,\alpha_{2n-1},\alpha_0}, (\alpha_2,\alpha_3,\cdots\alpha_{2n-1},\alpha_0) \in \mathbb{N}^{2n-1}\right\}$ respectively. Define,

$$U : \mathcal{H} \longleftrightarrow \mathcal{H}^{'}.$$

$$e_{\alpha_2,\alpha_3,\cdots,\alpha_{2n-1},\alpha_0} \longmapsto e^{'}_{\alpha_2,\alpha_3,\cdots,\alpha_{2n-1},\alpha_0}$$

From corollary 7.8, $U(u_{\alpha_2,\alpha_3,\cdots,\alpha_{2n-1},\alpha_0}) = u^{'}_{\alpha_2,\alpha_3,\cdots,\alpha_{2n-1},\alpha_0}$. We know,

$$\mathcal{H} = \oplus_{m\in\mathbb{N}}\mathcal{H}_m.$$

$$\mathcal{H}^{'} = \oplus_{m\in\mathbb{N}}\mathcal{H}^{'}_m.$$

where $\mathcal{H}_m$ and $\mathcal{H}^{'}_m$ are the eigenspaces of $\pi(\omega)$ and $\pi^{'}(\omega)$ repectively, corresponding to eigenvalue q^{2m}. Clearly $U(\mathcal{H}_m) = \mathcal{H}^{'}_m$. We need to show that, $U\pi(z_i)U^* = \pi^{'}(z_i)$, or equivalently $U\pi(z_i^*)U^* = \pi^{'}(z_i^*) \qquad \forall i \in \{1,2,\cdots,2n\}$.
For $i = 2n$,

$$\pi(z_{2n}^*)u_{\alpha_2,\cdots,\alpha_{2n-1},\alpha_0} = q^{(\sum_{l=2}^{2n-1}\alpha_l)+2\alpha_0}u_{\alpha_2,\cdots,\alpha_{2n-1},\alpha_0}.$$
$$\pi^{'}(z_{2n}^*)u_{\alpha_2,\cdots,\alpha_{2n-1},\alpha_0} = q^{(\sum_{l=2}^{2n-1}\alpha_l)+2\alpha_0}u^{'}_{\alpha_2,\cdots,\alpha_{2n-1},\alpha_0}.$$

For $i > n$,

$$\pi(z_i^*)u_{\alpha_2,\cdots,\alpha_{2n-1},\alpha_0} = Cu_{\alpha_2,\cdots,\alpha_{i-1},\alpha_i,\cdots,\alpha_{2n-1},\alpha_0}$$
$$\pi^{'}(z_i^*)u^{'}_{\alpha_2,\cdots,\alpha_{2n-1},\alpha_0} = Cu^{'}_{\alpha_2,\cdots,\alpha_{i-1},\alpha_i,\cdots,\alpha_{2n-1},\alpha_0}.$$

Note that constant C is same in both equations. Hence we have,

$$U\pi(z_i^*)U^* = \pi^{'}(z_i) \qquad \text{for } n < i \leq 2n.$$

For $i = n$, we will use induction on the dimension of eigenspaces of $\pi(\omega)$. For $m = 0, \pi(z_i^*)u = 0 = \pi'(z_i^*)u^{'}$. Assume that $U\pi(z_i)U^*_{|\mathcal{H}_{\leq m}} = \pi^{'}(z_i)_{|}\mathcal{H}^{'}_{\leq m}$. Take $u_{\alpha_2,\cdots,\alpha_{2n-1},\alpha_0} \in \mathcal{H}_{m+1}$.
Case $1 : \alpha_j \neq 0,$ for some $j > n,$ and $\alpha_k = 0, \forall k > j$.

$$\begin{aligned}
\pi(z_n^*)u_{\alpha_2,\cdots,\alpha_j,0,\cdots,0,\alpha_0} &= C\pi(z_j)\pi(z_n^*)u_{\alpha_2,\cdots,\alpha_j-1,0,\cdots,0,\alpha_0}, && \text{by (7.1)} \\
&= C\pi(z_j)U^*\pi^{'}(z_n^*)u^{'}_{\alpha_2,\cdots,\alpha_j-1,0,\cdots,0,\alpha_0}, && \text{(by induction)} \\
&= CU^*\pi'(z_j)UU^*\pi^{'}(z_n^*)u^{'}_{\alpha_2,\cdots,\alpha_j-1,0,\cdots,0,\alpha_0}, \\
&= CU^*\pi'(z_j)\pi^{'}(z_n^*)u^{'}_{\alpha_2,\cdots,\alpha_j-1,0,\cdots,0,\alpha_0}, \\
&= \pi'(z_n)u^{'}_{\alpha_2,\cdots,\alpha_j,0,\cdots,0,\alpha_0}.
\end{aligned}$$

Case $2: \alpha_j = 0, \forall j > n$. From part(7), (8) of the proposition 7.5 and proposition 7.6, we have,

$$\begin{aligned}
\pi(z_n^*)u_{\alpha_2,\cdots,\alpha_n,0,\cdots,0,\alpha_0} &= Cu_{\alpha_2,\cdots,\alpha_n-1,0,\cdots,0,\alpha_0} + Cu_{\alpha_2,\cdots,\alpha_n,1,0,\cdots,0,\alpha_0-1}.\\
\pi^{'}(z_n^*)u^{'}_{\alpha_1,\cdots,\alpha_n,0,\cdots,0,\alpha_0} &= Cu^{'}_{(\alpha_2,\cdots,\alpha_n-1,0,\cdots,0,\alpha_0} + Cu^{'}_{(\alpha_2,\cdots,\alpha_n,1,0,\cdots,0,\alpha_0-1)}.
\end{aligned}$$

Hence, we get,

$$U\pi(z_n)^*U^* = \pi^{'}(z_n).$$

For $1 < i < n$,
Case $1: \alpha_j \neq 0,$ for some $j > i,$ and $\alpha_k = 0, \forall k > j$. This follows exactly as in $i = n$.
Case $2: \alpha_j = 0, \quad \forall j \geq i$. It follows from part(2) of the proposition 7.5 and by using the fact $U\pi(z_k)^*U^* = \pi^{'}(z_k)$ for all $k > i$.
Case $3: \alpha_j = 0, \quad \forall j \geq i$. From part (8) and part(11) of the proposition 7.5, we have

$$\begin{aligned}
\pi(z_i)^*u_{\alpha_2,\cdots,\alpha_{i-1},0,\cdots,0,\alpha_0} &= Cu_{\alpha_2,\cdots,\alpha_{i-1},0,\cdots,0,\underbrace{1}_{i'-1 \text{ th place}},0,\cdots,0,\alpha_0}\\
\pi^{'}(z_i)^*u_{\alpha_2,\cdots,\alpha_{i-1},0,\cdots,0,\alpha_0} &= Cu^{'}_{\alpha_2,\cdots,\alpha_{i-1},0,\cdots,0,\underbrace{1}_{i'-1 \text{ th place}},0,\cdots,0,\alpha_0}.
\end{aligned}$$

which settles the claim for $1 < i < n$.
For $i = 1$, we again use induction. Take u_α such that $\alpha_j \neq 0$ for some $j \neq 0$. Choose j to be $\max\{j : \alpha_j \neq 0\}$.

$$\begin{aligned}
\pi(z_1)^*u_{\alpha_2\cdots,\alpha_j,0,\cdots,0,\alpha_0} &= C\pi(z_j)\pi(z_1)^*u_{\alpha_2,\cdots,\alpha_j-1,0,\cdots,0,\alpha_0},\\
&= CU^*\pi'(z_j)UU^*\pi'(z_1)^*u^{'}_{\alpha_2,\cdots,\alpha_j-1,0,\cdots,0,\alpha_0},\\
&= CU^*\pi'(z_j)\pi'(z_1)^*u^{'}_{\alpha_2,\cdots,\alpha_j-1,0,\cdots,0,\alpha_0},\\
&= \pi^{'}(z_1)^*u^{'}_{\alpha_2\cdots,\alpha_j,0,\cdots,0,\alpha_0}.
\end{aligned}$$

For α such that $\alpha_j = 0, \forall j \neq 0$, it follows from part(12) of the proposition 7.5 and induction. Hence, we have, $U\pi(z_i^*)U^* = \pi^{'}(z_i^*), \quad 1 \leq i \leq 2n$, which proves the claim. □

We will now discuss general case. Let π be an irreducible representation of $C(H_q^{2n})$ on a Hilbert space $\mathcal{H}$ such that $\pi(z_{2n}) = \pi(z_{2n-1}) = \cdots = \pi(z_{k+1}) = 0$, and $\pi(z_k) \neq 0$. It follows from (7.8), that z_k is normal. Denote $z_k^*z_k$ by ω. By the same reasoning, $\mathcal{H}$ decomposes as,

$$\mathcal{H} = \oplus_{m\in\mathbb{N}}\mathcal{H}_m.$$

where $\mathcal{H}_m$ is the eigenspace of $\pi(\omega)$ corresponding to eigenvalue q^{2m}.
Let K be a subspace of $\bigcap_{i=1}^{k-1}$ ker $\pi(z_i^*)$ such that $\pi(z_k)h \in K, \forall h \in K$. Define

$$\mathcal{H}^K = \text{ linear span } \left\{ \pi(z_1)^{\alpha_1}\pi(z_2)^{\alpha_2}\cdots\pi(z_{k-1})^{\alpha_{k-1}}h : h \in K \right\}.$$

In the same way, one can show that, $\mathcal{H}^K$ is an invariant subspace of a representation π and hence, by irreducibility of the representation, $\bigcap_{i=1}^{k-1}$ ker $\pi(z_i^*)$ is one dimensional. Pick $u \in \bigcap_{i=1}^{k-1}$ ker $\pi(z_i^*) = \mathcal{H}_0$. As, $\pi(z_k)$ keeps $\mathcal{H}_0$ invariant and $\pi(z_k)_{|\mathcal{H}_0}$ is an unitary operator, we get $\pi(z_k)u = tu$ for some $t \in \mathbb{T}$.

Theorem 7.10. *Let $1 \leq k \leq 2n$, and π be an irreducible representation of $C(H_q^{2n})$ on a Hilbert space $\mathcal{H}$ such that $\pi(z_{2n}) = \pi(z_{2n-1}) = \cdots = \pi(z_{k+1}) = 0$, and $\pi(z_k)u = tu, t \in \mathbb{T}$, where u is defined as above. Then π is unique representation upto equivalence which satisfies these conditions.*

Proof: This is essentially the previous proof with some minor modifications.
Case $1 : k > n$. Define,

$$\begin{aligned} &u_{\alpha_1,\alpha_2,\cdots,\alpha_{2n-k},\alpha_{2n-k+2},\cdots,\alpha_{k-1},\alpha_0} \\ &= \pi(z_1)^{\alpha_1}\cdots\pi(z_{2n-k})^{\alpha_{2n-k}}\pi(z_{k-1})^{\alpha_{k-1}}\cdots\pi(z_{2n-k+2})^{\alpha_{2n-k+2}}[\pi(z_n),\pi(z_{n+1})]^{\alpha_0}. \end{aligned} \tag{7.10}$$

where $\alpha_i \in \mathbb{N}$. (Note the changes occured in the definition of u_α.)

$$e_{\alpha_1,\alpha_2,\cdots,\alpha_{2n-k},\alpha_{2n-k+2},\cdots,\alpha_{k-1},\alpha_0} = \frac{u_{\alpha_1,\alpha_2,\cdots,\alpha_{2n-k},\alpha_{2n-k+2},\cdots,\alpha_{k-1},\alpha_0}}{\left\|u_{\alpha_1,\alpha_2,\cdots,\alpha_{2n-k},\alpha_{2n-k+2},\cdots,\alpha_{k-1},\alpha_0}\right\|}$$

Similar calculation will prove that
$\left\{e_{\alpha_1,\cdots,\alpha_{2n-k},\alpha_{2n-k+2},\cdots,\alpha_{k-1},\alpha_0}, : (\alpha_1,\cdots,\alpha_{2n-k},\alpha_{2n-k+2},\cdots,\alpha_{k-1},\alpha_0) \in \mathbb{N}^{k-1}\right\}$ defined as above, form an orthonormal basis for $\mathcal{H}$. By same argument as done in previous theorem will prove the uniqueness of the representation satisfying $\pi(z_{2n}) = \pi(z_{2n-1}) = \cdots = \pi(z_{k+1}) = 0$, and $\pi(z_k)u = tu$.
Case $2 : k \leq n$. First observe that the relations satisfied by $\pi(z_1), \pi(z_2), \cdots, \pi(z_k)$ are same as the defining relations of odd dimesional quantum sphere S_q^{2k+1} for which we know that the claim holds. (Note that we can proceed as in previous case also and get the claim.) □

We have shown so far, that if there exists an irreducible representation π such that $\pi(z_{2n}) = \pi(z_{2n-1}) = \cdots = \pi(z_{k+1}) = 0$, and $\pi(z_k)u = tu$ for $t \in T$, then it is unique. Existence of these representations still needs to be shown. For that, denote by ω_k the following word of Weyl group of sp_{2n},

$$\omega_k = \begin{cases} I & \text{if } k = 1, \\ s_1 s_2 \cdots s_{k-1} & \text{if } 2 \leq k \leq n, \\ s_1 s_2 \cdots s_{n-1} s_n s_{n-1} \cdots s_{2n-k+1} & \text{if } n < k \leq 2n. \end{cases}$$

Since $\left\{u_j^{2n} : j \in 1, 2, \cdots, 2n\right\}$ satisfies the defining relations of $C(H_q^{2n})$ with $z_j = u_{2n+1-j}^{2n}$, from the universal property of $C(H_q^{2n})$, there exist a map $\eta : C(H_q^{2n}) \longrightarrow C(SP_q(2n)/SP_q(2n-2))$ such that $\eta(z_j) = u_{2n+1-j}^{2n}, \quad \forall j \in 1, 2, \cdots, 2n$. Let $\eta_{t,\omega_k} = \pi_{t,\omega_k} \circ \eta$. Hence, we have an irreducible representation η_{t,ω_k} of $C(H_q^{2n})$ such that $\pi(z_{2n}) = \pi(z_{2n-1}) = \cdots = \pi(z_{k+1}) = 0$, and $\pi(z_k)u = tu$ where $1 < k \leq 2n$. This gives an explicit description of the irreducible representations satifying these conditions.
For $k = 1$, define, $\eta_{t,I} : C(H_q^{2n}) \to \mathbb{C}$ such that $\eta_{t,I}(z_j) = t\delta_{1j}$. The set $\{\eta_{t,I} : t \in T\}$ gives all one dimensional irreducible representations of $C(H_q^{2n})$. Also, it satisfies $\pi(z_{2n}) = \pi(z_{2n-1}) = \cdots = \pi(z_2) = 0$, and $\pi(z_1)u = tu$.

Corollary 7.11. *The set, $\{\eta_{t,\omega_k} : 1 \leq k \leq 2n, t \in \mathbb{T}\}$, gives a complete list of irreducible representations of $C(H_q^{2n})$.*

To get a faithful representation of $C(H_q^{2n})$, define, $\eta_{\omega_k} : C(H_q^{2n}) \to C(\mathbb{T}) \otimes \mathscr{T}^{\otimes k-1}$, such that $\eta_{\omega_k}(a)(t) = \eta_{\omega_{t,\omega_k}}(a), \quad \forall a \in C(H_q^{2n})$.

Corollary 7.12. *$\eta_{\omega_{2n}}$ gives a faithful representation of $C(H_q^{2n})$.*

Proof: It is easy to see that any irreducible representation factors through $\eta_{\omega_{2n}}$ as, ω_k is a subword of ω_{2n}. This proves the claim. $\square$

Corollary 7.13. *The homomorphism, $\eta : C(H_q^{2n}) \to C(SP_q(2n)/SP_q(2n-2))$ is an isomorphism.*

Remark 7.14. Neshveyev & Tuset [12] obtained all irreducible representations of $C(SP_q(2n)/SP_q(2n-2))$. Using that one can get a faithful represenation of $C(SP_q(2n)/SP_q(2n-2))$. Here we first get the relations satisfied the generators of $C(SP_q(2n)/SP_q(2n-2))$. We then obtain all irreducible representations of the universal C^*-algebra satisfying those relations and show that it is isomorphic to $C(SP_q(2n)/SP_q(2n-2))$. This represents $C(SP_q(2n)/SP_q(2n-2))$ as a universal C^*-algebra satisfying some relations. In section (9), we use this fact to give a natural $\mathbb{T}^n$ action on it and then get an equivariant spectral triple for this C^*- algebra.

We separate out some important facts which will be useful in determining the K- groups of $C(H_q^{2n})$.

Corollary 7.15. *Let $C_1 = C(\mathbb{T})$ and for $2 \leq k \leq 2n, \quad C_k = \eta_{\omega_k}(C(H_q^{2n}))$. Then, $\{\eta_{t,\omega_l} : 1 \leq l \leq k, t \in \mathbb{T}\}$ gives a complete list of irreducible representations of C_k.*

Corollary 7.16. *Let $\pi = \eta_{t,\omega_k}$.*

1. *for* $1 \leq k \leq n$,

$$\pi(z_1)^{\alpha_1} \cdots \pi(z_{k-1})^{\alpha_{k-1}} 1_{\{1\}} \pi(z_k^* z_k) = C p_{\alpha_1,0} \otimes p_{\alpha_2,0} \otimes \ldots \otimes p_{\alpha_{k-1},0}.$$

2. *for* $n < k \leq 2n$,

$$\pi(z_1)^{\alpha_1} \cdots \pi(z_{2n-k})^{\alpha_{2n-k}} \pi(z_{k-1})^{\alpha_{k-1}} \pi(z_{k-2})^{\alpha_{k-2}} \cdots \pi(z_{2n-k+2})^{\alpha_{2n-k+2}} [\pi(z_n), \pi(z_{n+1})]^{\alpha_0} 1_{\{1\}} \pi(z_k^* z_k)$$
$$= C p_{\alpha_1,0} \otimes \cdots p_{\alpha_{2n-k},0} \otimes p_{\alpha_{2n-k+2},0} \otimes \cdots p_{\alpha_n,0} \otimes p_{\alpha_0,0} \otimes p_{\alpha_{n+1},0} \otimes \cdots \otimes p_{\alpha_{k-1},0}.$$

where $p_{i,j}$ *be the rank one operator on* $L_2(\mathbb{N})$ *sending basis element* e_j *to* e_i *and* C *is some non-zero constant.*

8 K-group of $C(H_q^{2n})$

In this section, we derive certain exact sequences analogous to that for quantum sphere (see [18]). We then apply six-term sequence in K-theory to compute the K- groups of $C(SP_q(2n)/SP_q(2n-2))$. Let us introduce some notation. Let $p_{i,j}$ be the rank one operator on $L_2(\mathbb{N})$ sending basis element e_j to e_i and p be the operator $p_{0,0}$.

Lemma 8.1. *Let* $C_1 = C(\mathbb{T})$ *and for* $2 \leq k \leq 2n$, $C_k = \eta_{\omega_k}(C(H_q^{2n}))$. *Then* $C(\mathbb{T}) \otimes \mathcal{K}(L_2(\mathbb{N}))^{\otimes(k-1)}$ *is contained in* C_k. *Moreover, for* $2 \leq k \leq 2n$, *we have the exact sequence,*

$$0 \longrightarrow C(\mathbb{T}) \otimes \mathcal{K}(L_2(\mathbb{N}))^{\otimes(k-1)} \longrightarrow C_k \xrightarrow{\sigma_k} C_{k-1} \longrightarrow 0.$$

where σ_k *is the restriction of* $(1^{\otimes(k-1)} \otimes \sigma)$ *to* C_k *and* $\sigma : \mathscr{T} \to \mathbb{C}$ *is the homomorphism such that* $\sigma(S) = 1$.

Proof: First we prove that $C(\mathbb{T}) \otimes \mathcal{K}(L_2(\mathbb{N}))^{\otimes(k-1)}$ is contained in C_k. For $k \leq n$, it follows from Sheu [18], Theorem 4 as C_k is isomorphic to $C(S_q^{2k-1})$. For $k > n$, and $m \geq 0$,
$\eta_{\omega_k}(z_k^m 1_{\{1\}}(z_k^* z_k)) = t^m \otimes \underbrace{p \otimes p \otimes \ldots \otimes p}_{k-1}$ and $\eta_{\omega_k}(z_k^{*m} 1_{\{1\}}(z_k^* z_k)) = t^{-m} \otimes \underbrace{p \otimes p \otimes \ldots \otimes p}_{k-1}$.
Also from corollary 7.16, it follows that

$$\eta_{\omega_k}((z_1)^{m_1} \cdots (z_{2n-k})^{m_{2n-k}} (z_{k-1})^{m_{k-1}} (z_{k-2})^{m_{k-2}} \cdots (z_{2n-k+2})^{m_{2n-k+2}} [(z_n), (z_{n+1})]^{m_0} 1_{\{1\}}(z_k^* z_k))$$
$$= C t^{(\sum_{i=0, i \neq 2n-k+1}^{k-1} m_i) + m_0} p_{m_1,0} \otimes \cdots p_{m_{2n-k},0} \otimes p_{m_{2n-k+2}} \otimes \cdots p_{m_n,0} \otimes p_{m_0,0} \otimes p_{m_{n+1},0} \otimes \cdots \otimes p_{m_{k-1},0}.$$

which shows that $t \otimes p_{m_1,0} \otimes p_{m_2,0} \otimes \ldots \otimes p_{m_{k-1},0}$ and $1 \otimes p_{m_1,0} \otimes p_{m_2,0} \otimes \ldots \otimes p_{m_{k-1},0}$ is contained in C_k. Hence, C_k contains $C(\mathbb{T}) \otimes \mathcal{K}(L_2(\mathbb{N}))^{\otimes(k-1)}$.
Clearly, σ_k vanishes on $C(\mathbb{T}) \otimes \mathcal{K}(L_2(\mathbb{N}))^{\otimes(k-1)}$. Also, any irreducible representation of C_k is of the form η_{t,ω_l} where $l \leq k$ and $t \in \mathbb{T}$. Hence an irreducible representation of C_k which vanishes

on $C(\mathbb{T}) \otimes \mathcal{K}(L_2(\mathbb{N}))^{\otimes(k-1)}$ is of the form η_{t,ω_l} where $l \leq k-1$ and $t \in \mathbb{T}$ which factors through σ_k. This completes the proof. $\square$

Remark 8.2. Neshveyev & Tuset [12] obtained a composition series for $C(G_q/K_q)$ for any Poisson-Lie closed subgroup K of G. In particular, when K is $C(SP_q(2n-2))$, we get a composition series for $C(SP_q(2n)/SP_q(2n-2))$. Note that the series of exact sequence derived in the lemma 8.1 is different from that given in [12].

Define, for $1 \leq k \leq 2n$, $u_k = t \otimes \underbrace{p \otimes p \otimes \ldots \otimes p}_{k-1} + 1 - 1 \otimes \underbrace{p \otimes p \otimes \ldots \otimes p}_{k-2} \otimes 1$. u_k is an unitary operator which is contained in C_k as $u_k = \eta_{\omega_k}(z_k 1_{\{1\}}(z_k^* z_k) + 1 - 1_{\{1\}}(z_k^* z_k))$.

Theorem 8.3. *Let $1 \leq k \leq 2n$.The K-groups $K_0(C_k)$ and $K_1(C_k)$ are both isomorphic to $\mathbb{Z}$ and, in particular, $[u_k]$ form a $\mathbb{Z}$-basis for $K_1(C_k)$ and $[1]$ form a $\mathbb{Z}$-basis for $K_0(C_k)$.*

Proof: We apply induction on k. For $k = 1$, this is clear. Assume the result to be true for $k-1$. From the lemma 8.1, we have the short exact sequence,

$$0 \longrightarrow C(\mathbb{T}) \otimes \mathcal{K}(L_2(\mathbb{N}))^{\otimes(k-1)} \longrightarrow C_k \stackrel{\sigma_k}{\to} C_{k-1} \longrightarrow 0,$$

which gives rise to the following six-term sequence in K-theory.

$$\begin{tikzcd}
K_0(C(\mathbb{T}) \otimes \mathcal{K}(L_2(\mathbb{N}))^{\otimes(k-1)}) \arrow[r] & K_0(C_k) \arrow[r, "K_0(\sigma_k)"] & K_0(C_{k-1})) \arrow[d, "\delta"] \\
K_1(C_{k-1}) \arrow[u, "\partial"] & K_1(C_k) \arrow[l, "K_1(\sigma_k)"'] & K_1(C(T) \otimes \mathcal{K}(L_2(\mathbb{N}))^{\otimes(k-1)}) \arrow[l]
\end{tikzcd}$$

To compute six term sequence, we determine δ and ∂. Since $\sigma_k(1) = 1$, it follows that $\delta([1]) = 0$. Also, the operator $\widetilde{X} = t \otimes \underbrace{q^N \otimes q^N \otimes \ldots}_{k-2} \otimes S^*$ is in C_k as $\eta_{\omega_k}(z_{k-1}) - \widetilde{X}$ lies in $C(\mathbb{T}) \otimes \mathcal{K}(L_2(\mathbb{N}))^{\otimes(k-1)}$. Let,

$$X = 1_{\{1\}}(\widetilde{X}^* \widetilde{X})\widetilde{X} + 1 - 1_{\{1\}}(\widetilde{X}^* \widetilde{X}).$$

Then X is an isometry such that $\sigma_k(X) = u_{k-1}$ and hence

$$\partial([u_{k-1}]) = [1 - X^*X] - [1 - XX^*] = [1 \otimes \underbrace{p \otimes p \otimes \ldots \otimes p}_{k-1}].$$

Now by the Kunneth theorem for the tensor product of C^*-algebra(see [1]), it follows that $C(\mathbb{T}) \otimes \mathcal{K}(L_2(\mathbb{N}))^{\otimes(k-1)}$ has K_0 group isomorphic to $\mathbb{Z}$ generated by $[1 \otimes \underbrace{p \otimes p \otimes \ldots \otimes p}_{k-1}]$ and K_1

group isomorphic to $\mathbb{Z}$ generated by $[u_k]$. Induction hypothesis and the above calculation shows that ∂ is an isomorphism and hence $K_0(i)$ is zero map which shows that $K_0(\sigma_k)$ is injective. Since δ is zero, $K_0(\sigma_k)$ is surjective which shows that $K_0(C_k)$ are isomorphic to $\mathbb{Z}$ generated by [1]. Similarly $K_1(i)$ is injective as δ is a zero map. Also, since ∂ is an isomorphism, $K_1(\sigma_k)$ is zero map which shows that $K_1(i)$ is surjective. Hence $K_1(C_k)$ are isomorphic to $\mathbb{Z}$ generated by $[u_k]$. This establish the claim. □

Remark 8.4. Neshveyev & Tuset [12] proved KK-equivalence of $C(G/K)$ and $C(G_q/K_q)$ and determined K-groups of $C(G_q/K_q)$ from that of $C(G/K)$ via the equivalence. As a consequence, generators of K-groups of $C(G_q/K_q)$ were images of generators of K-groups of $C(G/K)$ under the equivalence. Here we obtain K-groups of $C(SP_q(2n)/SP_q(2n-2))$ in more direct way and give more explicit description of generators of K-groups of $C(SP_q(2n)/SP_q(2n-2))$.

9 Equivariant spectral triple

The group $\mathbb{T}^n$ has an action on $C(H_q^{2n})$ given on the generating elements by

$$\tau_{\mathbf{w}}(z_i) = \begin{cases} w_i z_i & \text{if } i \leq n, \\ \overline{w_{2n-i+1}} z_i & \text{if } i > n, \end{cases}$$

where $\mathbf{w} = (w_1, w_2, \ldots, w_n) \in \mathbb{T}^n$. $\tau_{\mathbf{w}}$ is an action of $\mathbb{T}^n$ as $\{\tau_{\mathbf{w}}(z_i)\}_{i=1}^{2n}$ satisfies the defining relations of $C(H_q^{2n})$. Define

$$\mathcal{H} = L_2(\mathbb{Z}) \otimes \underbrace{L_2(\mathbb{N}) \otimes \cdots \otimes L_2(\mathbb{N})}_{2n-1 \text{ copies}}.$$

Denote by π the faithful representation of $C(H_q^{2n})$ on the space $\mathcal{H}$. If $U_{\mathbf{w}}$ denotes the unitary,

$$\overline{w_1^N} \otimes w_2^N w_1^N \otimes w_3^N w_1^N \otimes \cdots \otimes w_n^N w_1^N \otimes \underbrace{w_1^{2N}}_{n+1^{th}\text{place}} \otimes \overline{w_n^N} w_1^N \otimes \overline{w_{n-1}^N} w_1^N \otimes \cdots \otimes \overline{w_2^N} w_1^N$$

on $\mathcal{H}$, then one has $\pi(\tau_{\mathbf{w}}(a)) = U_{\mathbf{w}} \pi(a) U_{\mathbf{w}}^*$ for all $a \in C(H_q^{2n})$. Thus (π, U) is a covariant representation of $(C(H_q^{2n}), \mathbb{T}^n, \tau)$ on $\mathcal{H}$. Let $\Gamma = \mathbb{Z} \times \underbrace{\mathbb{N} \times \cdots \times \mathbb{N}}_{2n-1 \text{ copies}}$, so that $L_2(\Gamma) = \mathcal{H}$. For $\gamma = (\gamma(1), \gamma(2), \cdots, \gamma(2n)) \in \Gamma$, e_γ denotes the basis element of $\mathcal{H}$ given by $e_{\gamma(1)} \otimes \cdots \otimes e_{\gamma(2n)}$. ϵ_k stands for the vector whose $\mathrm{k^{th}}$ coordinate is 1 and all other coordinates are 0.

Theorem 9.1. *Let D be a self-adjoint, diagonal operator with compact resolvent on $\mathcal{H}$ sending e_γ to $d(\gamma)e_\gamma$. D will have bounded commutators with elements from the *-subalgebra of $C(H_q^{2n})$*

generated by the z_i's if the $d(\gamma)$'s obey the following condition:

$$|d(\gamma) - d(\gamma + \epsilon_1 + \epsilon_i + \epsilon_{2n-i+2})| = O(q^{-\sum_{l=2}^{i-1}\lambda(l) - \sum_{l=2n+2-i}^{2n}\lambda(l)}). \quad \text{for } 2 \le i \le n. \tag{9.1}$$

$$|d(\gamma) - d(\gamma + \epsilon_1 + \epsilon_{n+1})| = O(q^{-\sum_{l=2}^{n}\lambda(l) - \sum_{l=n+2}^{2n}\lambda(l)}). \tag{9.2}$$

$$|d(\gamma) - d(\gamma + \epsilon_1 + \epsilon_i + \epsilon_{2n-i+2} - \epsilon_{2n-k+2})| = O(q^{-\sum_{l=2}^{i-1}\lambda(l) - \sum_{l=2n+3-i}^{2n-k+1}\lambda(l)}), \quad \text{for } 2 \le k \le n, k+1 \le i \le n. \tag{9.3}$$

$$|d(\gamma) - d(\gamma + \epsilon_1 + \epsilon_{n+1} - \epsilon_{2n-k+2})| = O(q^{-\sum_{l=2}^{n}\lambda(l) - \sum_{l=n+2}^{2n-k+1}\lambda(l)}), \quad \text{for } 2 \le k \le n. \tag{9.4}$$

$$|d(\gamma) - d(\gamma + \epsilon_1 + \epsilon_k)| = O(q^{-(\sum_{l=2}^{k-1}\lambda(l)) - \lambda(2n+2-k)}), \quad \text{for } 2 \le k \le n. \tag{9.5}$$

$$|d(\gamma) - d(\gamma + \epsilon_1 + \epsilon_k)| = O(q^{-(\sum_{l=2}^{k-1}\lambda(l)) - \lambda(n+1)}), \quad \text{for } n+2 \le k \le 2n. \tag{9.6}$$

$$|d(\gamma) - d(\gamma + \epsilon_1)| = O(q^{-(\sum_{l=2}^{2n}\lambda(l)) - \lambda(n+1)}). \tag{9.7}$$

Proof: Follow from direct calculation. □

By a compact perturbation, one can ensure that all the $d(\gamma)$'s are nonzero in the above theorem. We will assume from now on that $d(\gamma) \neq 0$ for all γ. Using above equations, we get a constant c such that the ratios of left hand side and right hand side of above equations are bounded by c. Now join two elements γ and γ' in Γ by an edge if $|d(\gamma) - d(\gamma')| \le c$. Call the resulting graph $\mathcal{G}$ the growth graph for D. We say $\gamma \overset{P}{\leftrightarrow} \gamma'$ if γ and γ' are connected by a path P.

Lemma 9.2. *Let k be an integer with $n+2 \le k \le 2n$. Let*

$$\gamma = (0, 0, \ldots, 0), \quad \gamma' = (i_1, 0, 0, \ldots, 0, i_{n+2}, \ldots, i_{2n}).$$

Then there is a path in $\mathcal{G}$ of length $\sum_{l=n+2}^{2n} i_l + |i_1 - \sum_{l=n+2}^{2n} i_l|$ joining γ and γ' such that all vertices on this path are of the form $(r, 0, 0, , \ldots, 0, s_{n+2}, s_{n+3}, \ldots, s_{2n})$.

Proof: Let i_1' be $|i_1 - \sum_{l=n+2}^{2n} i_l|$. From (9.7), it is clear that if $\delta(i) = 0$ for $2 \le i \le 2n$, then there is an edge joining δ and $\delta + \epsilon_1$. Thus $\gamma \overset{P_1}{\leftrightarrow} (i_1', 0, 0, \ldots, 0)$ where $P_1 = (\gamma, \gamma + \epsilon_1, \gamma + 2\epsilon_1, \ldots, \gamma + i_1'\epsilon_1)$. Also, from (9.6), it follows that if $\delta(i) = 0$ for $2 \le i \le k-1$, then there is an edge joining δ and $\delta + \epsilon_1 + \epsilon_k$. Thus,

$$\gamma \overset{P_1}{\leftrightarrow} (i_1', 0, 0, \ldots, 0) \overset{P_2}{\leftrightarrow} (i_1', 0, 0, \ldots, i_{2n}) \overset{P_3}{\leftrightarrow} \cdots \overset{P_{n+1}}{\leftrightarrow} \gamma' = (i_1', 0, 0, \ldots, 0, i_{n+2}, \ldots, i_{2n})$$

where $P_l = (\gamma_l, \gamma_l + \epsilon_{2n+2-l}, \gamma_l + 2\epsilon_{2n+2-l}, \cdots + \gamma_l + i_{2n+2-l}\epsilon_{2n+2-l})$ and $\gamma_l = (i_1', 0, \cdots, 0, i_{2n+3-l}, \cdots, i_{2n})$. Length of this path can easily be shown to be equal to $\sum_{l=n+2}^{2n} i_l + |i_1 - \sum_{l=n+2}^{2n} i_l|$. □

Lemma 9.3. *The growth graph $\mathcal{G}$ is conneted. More specifically, Let*

$$\gamma = (0, 0, \dots, 0), \quad \gamma' = (\gamma(1), \gamma(2) \dots, \gamma(2n)).$$

Then there is a path in $\mathcal{G}$ of length $(\sum_{l=3}^{2n-1} \gamma(l)) + \gamma(n+1) +$ *max* $(\gamma(2), \gamma(2n)) + |\gamma(1)$ $- (\sum_{l=3}^{2n-1} \gamma(l)) + \gamma(n+1) +$ *max* $(\gamma(2), \gamma(2n))$ *joining γ and γ'.*

Proof: Define,

$$\begin{aligned} i_1 &= \gamma(1) - (\sum_{l=3}^{2n} \gamma(l)) + \gamma(n+1) + \max\ (\gamma(2), \gamma(2n)). \\ i_k &= \min\ (\gamma(2n+2-k) - \gamma(k), 0) + \min\ (\gamma(k-1), \gamma(2n-k+3)), \\ &\qquad\qquad\qquad\qquad\qquad \text{for, } n+2 \le k \le 2n. \end{aligned}$$

From previous lemma, we have, $\gamma \overset{P}{\leftrightarrow} (i_1, 0, 0, \dots, 0, i_{n+2}, \dots, i_{2n})$. By putting $k = n$ in (9.4), it is clear that if $\delta(i) = 0$ for $2 \le i \le n$, then there is an edge joining δ and $\delta + \epsilon_1 + \epsilon_{n+1} - \epsilon_{n+2}$. Using this, we get,

$$\begin{aligned} &(i_1, 0, 0, \cdots, 0, i_{n+2}, \dots, i_{2n}) \\ &\overset{P}{\leftrightarrow} (i_1 + \gamma(n+1), 0, 0, \dots, 0, \gamma(n+1), i_{n+2} - \gamma(n+1), i_{n+3}, \dots, i_{2n}). \end{aligned}$$

Case $1 : \gamma(n+2) > \gamma(n)$**.** Here, we have,

$$i_{n+2} - \gamma(n+1) = \gamma(n+2) - \gamma(n).$$

Now from (9.3), it follows that if $\delta(i) = 0$ for $2 \le i \le n-1$, then there is an edge joining δ and $\delta + \epsilon_1 + \epsilon_n + \epsilon_{n+2} - \epsilon_{n+3}$. Applying this, we get,

$$\begin{aligned} &(i_1 + \gamma(n+1), 0, , \dots, 0, \gamma(n+1), i_{n+2} - \gamma(n+1), i_{n+3}, \dots, i_{2n}) \\ &\overset{P}{\leftrightarrow} (i_1 + \gamma(n+1) + \gamma(n), 0, \dots, 0, \gamma(n), \gamma(n+1), \gamma(n+2), i_{n+3} - \gamma(n), i_{n+4}, \cdots, i_{2n}). \end{aligned}$$

Case $2 : \gamma(n+2) < \gamma(n)$**.** In this case, we have,

$$i_{n+2} - \gamma(n+1) = 0.$$

From (9.5), it is clear that if $\delta(i) = 0$ for $2 \le i \le n-1$ and $\delta(n+2) = 0$, there is an edge joining δ and $\delta + \epsilon_1 + \epsilon_n$. Hence,

$$\begin{aligned} &(i_1 + \gamma(n+1), 0, \dots, 0, \gamma(n+1), 0, i_{n+3}, \dots, i_{2n}) \\ &\overset{P}{\leftrightarrow} (i_1 + \gamma(n+1) + \gamma(n) - \gamma(n+2), 0, \dots, 0, \gamma(n) - \gamma(n+2), \gamma_{n+1}, i_{n+2}, i_{n+3}, \dots, i_{2n}). \end{aligned}$$

Now, from (9.3), it follows that if $\delta(i) = 0$ for $2 \le i \le n-1$, then there is an edge between δ and $\delta + \epsilon_1 + \epsilon_n + \epsilon_{n+2} - \epsilon_{n+3}$. Hence,

$$\begin{aligned} &(i_1 + \gamma(n+1) + \gamma(n) - \gamma(n+2), 0, 0, \dots, 0, \gamma(n) - \gamma(n+2), \gamma(n+1), i_{n+2}, i_{n+3}, \dots, i_{2n}) \\ &\overset{P}{\leftrightarrow} (i_1 + \gamma(n+1) + \gamma(n), 0, \dots, 0, \gamma(n), \gamma(n+1), \gamma(n+2), i_{n+3} - \gamma(n+2), i_{n+4}, \dots, i_{2n}). \end{aligned}$$

Proceeding in similar way, we get the required path. Also, length of this path can easily be shown to be equal to $(\sum_{l=3}^{2n-1}\gamma(l))+\gamma(n+1)+\ \max\ (\gamma(2),\gamma(2n))+|\gamma(1)-(\sum_{l=3}^{2n-1}\gamma(l))$
$+\gamma(n+1)+\ \max\ (\gamma(2),\gamma(2n))|$. $\square$

Theorem 9.4. *Let D_0 be the operator $e_\gamma\mapsto d(\gamma)e_\gamma$ on $\mathcal{H}$ where the $d(\gamma)$'s are given by*

$$d(\gamma)=\begin{cases}\gamma(1) & \text{if } \gamma(1)\geq(\sum_{l=3}^{2n-1}\gamma(l))+\gamma(n+1)\\ & \qquad +\ \max\ (\gamma(2),\gamma(2n)),\\ -2(\sum_{l=3}^{2n-1}\gamma(l))-\gamma(n+1)-\ \max\ (\gamma(2),\gamma(2n))+\gamma(1) & \\ & \text{if } \gamma(1)\leq(\sum_{l=3}^{2n-1}\gamma(l))+\gamma(n+1)\\ & \qquad +\ \max\ (\gamma(2),\gamma(2n)).\end{cases}$$

Then $(C(H_q^{2n}),\mathcal{H},D_0)$ is a nontrivial $2n$-summable spectral triple equivariant under torus action.

Proof: Clearly, D_0 is a selfadjoint operator with compact resolvent. That it has bounded commutators with the $\pi(z_j)$'s follow by direct verification. Let

$$u=1_{\{1\}}(z_{2n}^*z_{2n})(z_{2n}-1)+1.$$

It is easy to see that u is a unitary. We will now compute the pairing between D_0 and $\pi(u)$. First observe that the action of $\pi(u)$ on $\mathcal{H}$ is given by

$$\pi(u)e_\gamma=\begin{cases}e_{\gamma+\epsilon_1} & \text{if } \gamma(i)=0 \text{ for } 1<i\leq 2n,\\ e_\gamma & \textit{otherwise}.\end{cases}$$

Write $P=\frac{1}{2}(I+\operatorname{sign}D_0)$. Then P is the projection onto the closed linear span of $\{e_\gamma:\gamma(1)\geq(\sum_{l=3}^{2n-1}\gamma(l))+\gamma(n+1)+\ \max\ (\gamma(2),\gamma(2n))\}$. It follows that the index of PuP is -1.

Summability follows from the expression of D_0. $\square$

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