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Random directed forest and the Brownian web

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Abstract

Consider the d dimensional lattice $\mathbb{Z}^d$ where each vertex is *open* or *closed* with probability p or $1-p$ respectively. An open vertex $\mathbf{u} := (\mathbf{u}(1), \mathbf{u}(2), \ldots, \mathbf{u}(d))$ is connected by an edge to another open vertex which has the minimum L_1 distance among all the open vertices with $\mathbf{x}(d) > \mathbf{u}(d)$. It is shown that this random graph is a tree almost surely for $d = 2$ and 3 and it is an infinite collection of disjoint trees for $d \geq 4$. In addition for $d = 2$, we show that when properly scaled, the family of its paths converge in distribution to the Brownian web.

Key words: Markov chain, Random walk, Directed spanning forest, Brownian web.

AMS 2000 Subject Classification: 60D05.

1 Introduction

Let $\mathcal{P}$ be the points of a Poisson point process on $\mathbb{R}^d$ of intensity 1. For each $\mathbf{x} \in \mathcal{P}$ let $h(\mathbf{x}) \in \mathcal{P}$ be the Poisson point in the half-space $\{\mathbf{u} : \mathbf{u}(d) > \mathbf{x}(d)\}$ which has the minimum Euclidean distance from $\mathbf{x}$, where $\mathbf{v}(j)$ denotes the j th co-ordinate of $\mathbf{v} \in \mathbb{R}^d$. The directed spanning forest (DSF) is the random graph with vertex set $\mathcal{P}$ and edge set $\{\langle \mathbf{x}, h(\mathbf{x}) \rangle : \mathbf{x} \in \mathcal{P}\}$. The study of the directed spanning forest (DSF) was initiated by Baccelli *et al.* [BB07]. Coupier *et al.* [CT11] proved that for $d = 2$ the DSF is a tree almost surely. Ferrari*et al.* [FLT04] also studied a directed random graph on a Poisson point process, however, the mechanism used to construct edges in that model incorporates more independence than is available in the DSF. They proved that their random graph is a connected tree in dimensions 2 and 3, and a forest in dimensions 4 and more.

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A similar construction, like the DSF arising from a Poisson point process, can be made from vertices of the integer lattice. Let $\{U_{\mathbf{v}} : \mathbf{v} \in \mathbb{Z}^d\}$ be a collection of i.i.d. Uniform $(0, 1)$ random variables. Fix $0 < p < 1$ and let $V := \{\mathbf{v} \in \mathbb{Z}^d : U_{\mathbf{v}} < p\}$ be the set of *open* vertices of $\mathbb{Z}^d$. Given $\mathbf{u} \in \mathbb{Z}^d$, let $\mathbf{v} \in V$ be such that

1. $\mathbf{u}(d) < \mathbf{v}(d)$,
2. there does not exist any $\mathbf{w} \in V$ with $\mathbf{w}(d) > \mathbf{u}(d)$ such that $||\mathbf{u} - \mathbf{w}||_1 < ||\mathbf{u} - \mathbf{v}||_1$, and
3. for all $\mathbf{w} \in V$ with $\mathbf{w}(d) > \mathbf{u}(d)$ and $||\mathbf{u} - \mathbf{w}||_1 = ||\mathbf{u} - \mathbf{v}||_1$ we have $U_{\mathbf{v}} \leq U_{\mathbf{w}}$.

Here and henceforth $||\mathbf{u}||_1$ denotes the L_1 norm of $\mathbf{u}$ on $\mathbb{R}^d$. Such a $\mathbf{v}$ is almost surely unique and clearly, is a function of $\mathbf{u}$ and $\mathbf{X} = \{U_{\mathbf{w}} : \mathbf{w} \in \mathbb{Z}^d, \mathbf{w}(d) > \mathbf{u}(d)\}$. We denote it by $h(\mathbf{u}, \mathbf{X})$. We will drop the second argument in h for the time being. Let $\langle \mathbf{u}, h(\mathbf{u}) \rangle$ be the edge joining $\mathbf{u}$ and $h(\mathbf{u})$ and let E denote the edge set given by,

$$E := \{\langle \mathbf{u}, h(\mathbf{u}) \rangle : \mathbf{u} \in V\}.$$

In this paper, we study the random graph $G := (V, E)$, which we will refer to as the *discrete DSF* henceforth.

$h(\mathbf{u})$

Figure 1: The construction of $h(\mathbf{u})$ from $\mathbf{u}$ on $\mathbb{Z}^2$. The shaded points are open, while the others are closed. Note that in order to get $h(\mathbf{u})$ from $\mathbf{u}$, we require information on the values of the Uniform random variables of the gray vertices.

Similar models of random graphs are known in the physics literature as drainage networks (see Scheidegger [S67]) and have been studied extensively (see Rodríguez-Iturbe *et al.* [RR97]). Mathematically, for similar discrete processes but with a condition for constructing edges which allows more independence, the dichotomy in dimensions of having a single connected tree vis-a-vis a forest has been studied (see Gangopadhyay *et al.* [GRS04], Athreya *et al.* [ARS08]).

Our first result shows that the tree/forest dichotomy in dimension holds in the discrete DSF. Thus, our paper may be viewed as an extension, albeit in the discrete setting, of the result of Coupier *et al.* [CT11] to any dimension. Our proof is different from that of [CT11]; while their argument is percolation theoretic and crucially depends on the planarity of $\mathbb{R}^2$, our argument exploits a Markovian structure inherent in the DSF which allows us to extend the result to any dimension.

Theorem 1.1 *For $d = 2$ and $d = 3$ the random graph G is connected almost surely and consists of a single tree while for $d \geq 4$, it is a disconnected forest with each connected component being an infinite tree almost surely.*

Our second result in this paper is the convergence of the random graph G for $d = 2$, under a suitable diffusive scaling, to the Brownian web. The standard Brownian web originated in the work of Arratia [A79], [A81] as the scaling limit of the voter model on $\mathbb{Z}$. It arises naturally as the diffusive scaling limit of the coalescing simple random walk paths starting from every point on the space-time lattice. We can thus think of the Brownian web as a collection of one-dimensional coalescing Brownian motions starting from every point in the space time plane $\mathbb{R}^2$. Detailed analysis of the Brownian web was carried out in Tóth *et al.* [TW98]. Later Fontes *et al.* [FINR04] introduced a framework in which the Brownian web is realized as a random variable taking values in a Polish space. We recall relevant details from Fontes *et al.* [FINR04].

Let $\mathbb{R}^2_c$ denote the completion of the space time plane $\mathbb{R}^2$ with respect to the metric

$$\rho((x_1,t_1),(x_2,t_2)) = |\tanh(t_1) - \tanh(t_2)| \vee \left|\frac{\tanh(x_1)}{1+|t_1|} - \frac{\tanh(x_2)}{1+|t_2|}\right|.$$

As a topological space $\mathbb{R}^2_c$ can be identified with the continuous image of $[-\infty,\infty]^2$ under a map that identifies the line $[-\infty,\infty] \times \{\infty\}$ with the point $(*,\infty)$, and the line $[-\infty,\infty] \times \{-\infty\}$ with the point $(*,-\infty)$. A path π in $\mathbb{R}^2_c$ with starting time $\sigma_\pi \in [-\infty,\infty]$ is a mapping $\pi : [\sigma_\pi,\infty] \to [-\infty,\infty]$ such that $\pi(\infty) = \pi(-\infty) = *$ and $t \to (\pi(t),t)$ is a continuous map from $[\sigma_\pi,\infty]$ to $(\mathbb{R}^2_c,\rho)$. We then define Π to be the space of all paths in $\mathbb{R}^2_c$ with all possible starting times in $[-\infty,\infty]$. The following metric, for $\pi_1,\pi_2 \in \Pi$

$$d_\Pi(\pi_1,\pi_2) = |\tanh(\sigma_{\pi_1}) - \tanh(\sigma_{\pi_2})| \vee \sup_{t \geq \sigma_{\pi_1} \wedge \sigma_{\pi_2}} \left|\frac{\tanh(\pi_1(t \vee \sigma_{\pi_1}))}{1+|t|} - \frac{\tanh(\pi_2(t \vee \sigma_{\pi_2}))}{1+|t|}\right|$$

makes Π a complete, separable metric space. Convergence in this metric can be described as locally uniform convergence of paths as well as convergence of starting times. Let $\mathcal{H}$ be the space of compact subsets of (Π, d_Π) equipped with the Hausdorff metric $d_{\mathcal{H}}$ given by,

$$d_{\mathcal{H}}(K_1, K_2) = \sup_{\pi_1 \in K_1} \inf_{\pi_2 \in K_2} d_\Pi(\pi_1, \pi_2) \vee \sup_{\pi_2 \in K_2} \inf_{\pi_1 \in K_1} d_\Pi(\pi_1, \pi_2).$$

The space $(\mathcal{H}, d_{\mathcal{H}})$ is a complete separable metric space. Let $B_{\mathcal{H}}$ be the Borel $\sigma-$algebra on the metric space $(\mathcal{H}, d_{\mathcal{H}})$. The Brownian web $\mathcal{W}$ is an $(\mathcal{H}, B_{\mathcal{H}})$ valued random variable.

Ferrari *et al.* [FFW05] have shown that, for $d = 2$, the random graph on the Poisson points introduced by [FLT04], converges to a Brownian web under a suitable diffusive scaling. Coletti *et al.* [CFD09] have a similar result for the discrete random graph studied in Gangopadhyay *et al.* [GRS04]. Baccelli *et al.* [BB07] have shown that scaled paths of the successive ancestors in the DSF converges weakly to the Brownian motion and also conjectured that the scaling limit of the DSF is the Brownian web.

For the random graph G we consider here, taking the edges $\{\langle h^{k-1}(\mathbf{u}), h^k(\mathbf{u})\rangle : k \geq 1)\}$ (with $h^0(\mathbf{u}) := \mathbf{u}$ and $h^k = h(h^{k-1})$) to be straight line segments we parametrize the path formed by these edges as the piecewise linear function $\pi^{\mathbf{u}} : [\mathbf{u}(2), \infty) \to \mathbb{R}$ such that $\pi^{\mathbf{u}}(h^k(\mathbf{u})(2)) = h^k(\mathbf{u})(1)$ for every $k \geq 0$ and $\pi^{\mathbf{u}}(t)$ is linear in the interval $[h^k(\mathbf{u})(2), h^{k+1}(\mathbf{u})(2)]$. Define $\mathcal{X} := \{\pi^{\mathbf{u}} : \mathbf{u} \in V\}$. For given $\gamma, \sigma > 0$, a path π with starting time σ_π and for each $n \geq 1$, the scaled path $\pi_n(\gamma, \sigma) : [\sigma_\pi/n^2\gamma, \infty] \to [-\infty, \infty]$ is given by $\pi_n(\gamma, \sigma)(t) = \pi(n^2\gamma t)/n\sigma$. Thus, the scaled path $\pi_n(\gamma, \sigma)$ has the starting time $\sigma_{\pi_n(\gamma,\sigma)} = \sigma_\pi/n^2\gamma$. For each $n \geq 1$, let $\mathcal{X}_n(\gamma, \sigma) = \{\pi_n^{\mathbf{u}}(\gamma, \sigma) : \mathbf{u} \in V\}$ be the collection of the scaled paths. The closure $\bar{\mathcal{X}}_n(\gamma, \sigma)$ of $\mathcal{X}_n(\gamma, \sigma)$ in (Π, d_Π) is a $(\mathcal{H}, \mathcal{B}_{\mathcal{H}})$ valued random variable. We have

Theorem 1.2 *There exist $\sigma := \sigma(p)$ and $\gamma := \gamma(p)$ such that as $n \to \infty$, $\bar{\mathcal{X}}_n(\gamma, \sigma)$ converges weakly to the standard Brownian Web $\mathcal{W}$ as $(\mathcal{H}, \mathcal{B}_{\mathcal{H}})$ valued random variables.*

For the proof of Theorem 1.1 we obtain a Markovian structure in our model and define suitable stopping times for this Markov process. From these stopping times the process regenerates which allows us to phrase the problem as a question of recurrence or transience of the Markov chain. This we do by obtaining a martingale for $d = 2$, using a Lyapunov function technique for $d = 3$ and a suitable coupling with a random walk with independent steps for $d = 4$.

The martingale obtained for $d = 2$ and the fact that the distributions of the stopping times have exponentially decaying tails are used to prove Theorem 1.2.

Finally, although our results are obtained for the random graph constructed by connecting edges between L_1 nearest open vertices, they should also hold for the model constructed with the L_2 metric.

The paper is structured as follows – in the next section we construct the paths of the graph G starting from k distinct vertices and obtain some properties of these paths. In Section 3, we derive the martingale (for $d = 2$) and also provide a method of approximation of the paths by independent processes, which is used later to prove Theorem 1.1 and Theorem 1.2. In Section 4 we prove Theorem 1.1 and in Section 5, we prove Theorem 1.2.

2 Construction of the process

We first detail a construction of the graph G which brings out a Markovian structure. Later we obtain a martingale for $d = 2$ which is used in the next two sections. Before proceeding further we fix some notation: for $\mathbf{u} \in \mathbb{Z}^d$ and $r > 0$, let $S(\mathbf{u}, r) := \{\mathbf{w} \in \mathbb{Z}^d : ||\mathbf{u} - \mathbf{w}||_1 \leq r\}$ be the closed L_1 ball of radius r and $\mathbb{H}(r) := \{\mathbf{w} \in \mathbb{Z}^d : \mathbf{w}(d) \leq r\}$ be the half-space.

From k ($k \geq 1$) vertices $\mathbf{u}^1, \ldots, \mathbf{u}^k \in \mathbb{Z}^d$, we obtain the vertices $\{h^n(\mathbf{u}^l),\ n \geq 0,\ 1 \leq l \leq k\}$ as a stochastic process. The vertices with the smallest d th coordinate are allowed to move, while the others stay put. Each of these vertices explores a region in the half space 'above' it to obtain the vertex to which it moves. During this exploration a vertex may encounter regions which have been already explored by other vertices earlier. While the information for the region explored earlier is known, the information about the freshly explored region is new and is obtained during the exploration process of the vertices which are moving at that time. The region which has been explored till the n th move of the entire process and which are needed for the $n+1$ th move is called the *history region* and denoted by $\Delta_n = \Delta_n(\mathbf{u}^1, \ldots, \mathbf{u}^k)$ and $\{(\mathbf{w}, U_{\mathbf{w}}) : \mathbf{w} \in \Delta_n\}$ constitutes the *history* $H_n = H_n(\mathbf{u}^1, \ldots, \mathbf{u}^k)$. Formally, let

(i) $g_0(\mathbf{u}^i) = \mathbf{u}^i$ for all $1 \leq i \leq k$ and $r_0 = \min\{g_0(\mathbf{u}^i)(d) : 1 \leq i \leq k\}$;

(ii) $W_0^{\text{move}} := \{g_0(\mathbf{w}) : \mathbf{w} \in \{\mathbf{u}^1, \ldots, \mathbf{u}^k\},\ g_0(\mathbf{w})(d) = r_0\}$ and $W_0^{\text{stay}} := \{g_0(\mathbf{u}^1), \ldots, g_0(\mathbf{u}^k)\} \setminus W_0^{\text{move}}$;

(iii) $\Delta_0 = \Delta_0(\mathbf{u}^1, \ldots, \mathbf{u}^k) := \{\mathbf{w} : \mathbf{w} \in W_0^{\text{stay}}\}$, and $H_0 = H_0(\mathbf{u}^1, \ldots, \mathbf{u}^k) := \{(\mathbf{w}, x) : \mathbf{w} \in \Delta_0,\ x = U_{\mathbf{w}}\}$.

Having obtained $g_n(\mathbf{u}^l)$, r_n, W_n^{move}, W_n^{stay}, Δ_n and H_n, for $1 \leq l \leq k$, we set

(i) $g_{n+1}(\mathbf{u}) := h(g_n(\mathbf{u}))$ for all $g_n(\mathbf{u}) \in W_n^{\text{move}}$ and $g_{n+1}(\mathbf{v}) := g_n(\mathbf{v})$ for all $g_n(\mathbf{v}) \in W_n^{\text{stay}}$, $r_{n+1} := \min\{g_{n+1}(\mathbf{u}^i)(d),\ 1 \leq i \leq k\}$;

(ii) $W_{n+1}^{\text{move}} := \{g_{n+1}(\mathbf{w}) : \mathbf{w} \in \{\mathbf{u}^1, \ldots, \mathbf{u}^k\},\ g_{n+1}(\mathbf{w})(d) = r_{n+1}\}$ and $W_{n+1}^{\text{stay}} := \{g_{n+1}(\mathbf{u}^1), \ldots, g_{n+1}(\mathbf{u}^k)\} \setminus W_{n+1}^{\text{move}}$;

(iii) $\Delta_{n+1} = \Delta_{n+1}(\mathbf{u}^1, \ldots, \mathbf{u}^k) := \big(\Delta_n \cup \cup_{\mathbf{u} \in W_n^{\text{move}}} S(\mathbf{u}, ||h(\mathbf{u}) - \mathbf{u}||_1)\big) \setminus \mathbb{H}(r_{n+1})$, and $H_{n+1} = H_{n+1}(\mathbf{u}^1, \ldots, \mathbf{u}^k) := \{(\mathbf{w}, x) : \mathbf{w} \in \Delta_{n+1},\ x = U_{\mathbf{w}}\}$.

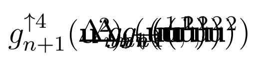

Figure 2: The vertices $g_{n+1}(\mathbf{u}^1), g_{n+1}(\mathbf{u}^2)$ and the history set $\Delta_{n+1}(\mathbf{u}^1, \mathbf{u}^2)$ when $W_n^{\text{move}} = \{g_n(\mathbf{u}^1), g_n(\mathbf{u}^2)\}$, $W_n^{\text{stay}} = \emptyset$. Note the vertices above $g_{n+1}(\mathbf{u}^1)$ and $g_{n+1}(\mathbf{u}^2)$ are unexplored.

Remark 2.1 *If* $\max\{\mathbf{u}^i(d) - \mathbf{u}^j(d) : \ 1 \leq i, j \leq k\} = m_0$, *then by the* $m_0 + 1$ *th move all the vertices would have moved from their initial positions* $\mathbf{u}^1, \dots, \mathbf{u}^k$.

For $n > m_0$, *the history region* Δ_n *formed at the* n *th step is a finite union of* d-*dimensional tetrahedrons, with each tetrahedron in* Δ_n *having a* $(d-1)$-*dimensional cube as a base on the hyperplane* $Q_{r_n+1} := \{\mathbf{x} \in \mathbb{Z}^d : \mathbf{x}(d) = r_n + 1\}$ *where* r_n *is as defined earlier. Clearly, by definition of* r_n, $W_n^{move} \subseteq Q_{r_n}$.

Furthermore, for each $n > m_0$, *all vertices in the set* $\Xi_n := \mathbb{Z}^d \setminus \Big(\Delta_n \cup \mathbb{H}(r_n)\Big)$ *are unexplored until the* $n+1$ *th step. Thus, each* $n > m_0$, $i = 1, \dots, k$, *and* $m \geq 1$, *the vertex* $g_n^{\uparrow m}(\mathbf{u}^i)$ *does never belong to* Δ_n *and* always unexplored, *where* $g_n^{\uparrow m}(\mathbf{u}^i)$ *is defined by*

$$g_n^{\uparrow m}(\mathbf{u}^i)(j) := \begin{cases} g_n(\mathbf{u}^i)(j) & \text{for } 1 \leq j \leq d-1, \\ m + g_n(\mathbf{u}^i)(d) & \text{for } j = d. \end{cases} \qquad \square$$

Returning back, we now obtain the Markov process implicit in our construction.

Figure 3: The vertices $g_{n+1}(\mathbf{u}^1), g_{n+1}(\mathbf{u}^2)$ and the history region $\Delta_{n+1}(\mathbf{u}^1, \mathbf{u}^2)$: $j \geq 0\}$ when $W_n^{\text{move}} = \{g_n(\mathbf{u}^2)\}$, $W_n^{\text{stay}} = \{g_n(\mathbf{u}^1)\}$.

Let us denote $\mathcal{S} := (\mathbb{Z}^d)^k \times \{(\mathbf{w}, x) : x \in [0,1],\ \mathbf{w} \in \Delta,\ \Delta \subseteq \mathbb{Z}^d,\ \Delta \text{ finite}\}$. Let $\mathbf{Y} = \{V_{\mathbf{w}} : \mathbf{w} \in \mathbb{Z}^d, \mathbf{w}(d) > 0\}$ be an independent collection of i.i.d. uniform $[0,1]$-valued random variables. For any $n \geq 1$, given $\big\{(g_n(\mathbf{u}^1), \ldots, g_n(\mathbf{u}^k), H_n(\mathbf{u}^1, \ldots, \mathbf{u}^k)) = (\mathbf{v}^1, \ldots, \mathbf{v}^k, H)\big\}$, we define the collection of random variables $\tilde{\mathbf{Y}} = \{\tilde{V}_{\mathbf{w}} : \mathbf{w} \in \mathbb{Z}^d, \mathbf{w}(d) > r_n\}$ as follows: for Δ the associated history region of the history H,

$$\tilde{V}_{\mathbf{w}} = \begin{cases} x & \text{if } \mathbf{w} \in \Delta, (\mathbf{w}, x) \in H \\ V_{\mathbf{w}'} & \text{if } \mathbf{w} \notin \Delta, \mathbf{w}(j) = \mathbf{w}'(j), \mathbf{w}(d) = \mathbf{w}'(d) + r_n. \end{cases}$$

The above definition implies that $\tilde{\mathbf{Y}}$ is a function of $\mathbf{Y}$ and H, say $\tilde{\mathbf{Y}} = f(\mathbf{Y}, H)$ where f is a function from $[0,1]^{\mathbb{Z}^d \setminus \mathbb{H}(0)}$ to $[0,1]^{\mathbb{Z}^d \setminus \mathbb{H}(r_n)}$. From the above definition and the fact that the vertices in $\Xi_n = \mathbb{Z}^d \setminus \Big(\Delta \cup \mathbb{H}(r_n)\Big)$ are unexplored, and hence can be replaced by another set of i.i.d. uniform random variables, for the family $\mathbf{X} = \{U_{\mathbf{w}} : \mathbf{w} \in \mathbb{Z}^d, \mathbf{w}(d) > r_n\}$, we have

$$\mathbf{X} \mid \big\{(g_n(\mathbf{u}^1), \ldots, g_n(\mathbf{u}^k), H_n(\mathbf{u}^1, \ldots, \mathbf{u}^k)) = (\mathbf{v}^1, \ldots, \mathbf{v}^k, H)\big\} \stackrel{d}{=} \tilde{\mathbf{Y}}.$$

From the definition of the process, we obtain that $g_{n+1}(\mathbf{u}^1), \ldots, g_{n+1}(\mathbf{u}^k)$ and H_{n+1} is a function of $g_n(\mathbf{u}^1), \ldots, g_n(\mathbf{u}^k), H_n$ and $\mathbf{X}$, i.e.,

$$(g_{n+1}(\mathbf{u}^1), \ldots, g_{n+1}(\mathbf{u}^k), H_{n+1}) = f_1((g_n(\mathbf{u}^1), \ldots, g_n(\mathbf{u}^k), H_n, \mathbf{X})$$

where f_1 is a function on $\mathcal{S} \times [0,1]^{\mathbb{Z}^d \setminus \mathbb{H}(r_n)}$. Therefore, from the above observation, given $g_n(\mathbf{u}^1) = \mathbf{v}^1, \dots, g_n(\mathbf{u}^k) = \mathbf{v}^k$ and $H_n(\mathbf{u}^1, \dots, \mathbf{u}^k) = H$, the conditional distribution of $g_{n+1}(\mathbf{u}^1), \dots, g_{n+1}(\mathbf{u}^k)$ and $H_{n+1}(\mathbf{u}^1, \dots, \mathbf{u}^k)$ is the same as that of $f_1((\mathbf{u}^1, \dots, \mathbf{u}^k, H), f(\mathbf{Y}, H))$. Hence, the process $\{(g_{n+1}(\mathbf{u}^1), \dots, g_{n+1}(\mathbf{u}^k), H_{n+1}) : n \geq 1\}$ admits a random mapping representation, which proves the Markov property (see, for example, Levin *et al.* [LPW]).

Proposition 2.1 *The process* $\{(g_n(\mathbf{u}^1), g_n(\mathbf{u}^2), \dots, g_n(\mathbf{u}^k), H_n),\ n \geq 0\}$ *is Markov with state space* $\mathcal{S} := (\mathbb{Z}^d)^k \times \{(\mathbf{w}, x) : x \in [0,1],\ \mathbf{w} \in S,\ S \subseteq \mathbb{Z}^d,\ S \text{ finite}\}$.

For the remainder of this section we fix $\mathbf{u}^1, \dots, \mathbf{u}^k$ with $\mathbf{u}^1(d) = \cdots = \mathbf{u}^k(d)$. Set $\tau_0 = \tau(\mathbf{u}^1, \dots, \mathbf{u}^k) := 0$ and, for $l \geq 1$, define

$$\tau_l = \tau_l(\mathbf{u}^1, \dots, \mathbf{u}^k) := \inf\{n > \tau_{l-1} : H_n = \emptyset\}. \tag{1}$$

We call this the *simultaneous regeneration of k joint paths.* We note here that τ_l denotes the number of steps (in the above construction) required for the joint process to regenerate i.e., to reach a state of empty history, for the l th time. This is not the same as the time (measured as the distance in the d th co-ordinate) for regeneration, which we will later denote by T_l. At each regeneration step τ_l, the paths must be at the same level in terms of their d th co-ordinate, i.e., $g_{\tau_l}(\mathbf{u}^1)(d) = \cdots = g_{\tau_l}(\mathbf{u}^k)(d)$ (see Figure 4).

$g_{\tau_1}(\mathbf{u}^1)$ $g_{\tau_1}(\mathbf{u}^2)$

Figure 4: At regeneration step $\tau_1(\mathbf{u}^1, \mathbf{u}^2)$ of the process $g_{\tau_1}(\mathbf{u}^1)(d) = g_{\tau_1}(\mathbf{u}^2)(d)$ and $\Delta_n = \emptyset$

Our first task is to show that the Markov process, defined in Proposition 2.1, regenerates almost surely. In fact, we prove a much stronger statement that the inter-regeneration times have exponentially decaying tail probabilities. More precisely, for $l \geq 1$, define $\sigma_l = \sigma_l(\mathbf{u}^1, \dots, \mathbf{u}^k) := \tau_l(\mathbf{u}^1, \dots, \mathbf{u}^k) - \tau_{l-1}(\mathbf{u}^1, \dots, \mathbf{u}^k)$.

Proposition 2.2 *For any $l \geq 1$ and $\mathbf{u}^1, \dots, \mathbf{u}^k \in \mathbb{Z}^d$ with $\mathbf{u}^1(d) = \cdots = \mathbf{u}^k(d)$, we have*

$$\mathbb{P}(\sigma_l \geq n) \leq C_1 \exp(-C_2 n) \tag{2}$$

for all $n \geq 1$, where C_1 and C_2 are positive constants, not depending on l, n or $\mathbf{u}^1, \dots, \mathbf{u}^k$.

To prove Proposition 2.2, we need an auxiliary lemma on Markov chains, whose proof is given in Appendix. Let $\{\theta_n : n \geq 1\}$ be a sequence of i.i.d. positive integer valued random variables with $\mathbb{P}(\theta_1 = 1) > 0$ and $\mathbb{P}(\theta_1 \geq n) \leq C_3 \exp(-C_4 n)$ for all $n \geq 1$ where C_3, C_4 are positive constants. Define a sequence of random variables as follows: $M_0 := 0$ and for $l \geq 0, M_{l+1} := \max\{M_l, \theta_{l+1}\} - 1$. Let $\tau^M := \inf\{l \geq 1 : M_l = 0\}$ be the first return time of M_l to 0.

Lemma 2.1 *For $n \geq 1$, we have*

$$\mathbb{P}(\tau^M \geq n) \leq C_5 \exp(-C_6 n)$$

where C_5 and C_6 are positive constants.

In order to prove Proposition 2.2, we define a random variable L_n which represents the *height* of the history region Δ_n, measured along the d th co-ordinate from the lowest vertex among $g_n(\mathbf{u}^1), \dots, g_n(\mathbf{u}^k)$ and construct a coupling with a Markov chain M_n which dominates the height random variable. Hence, the Markov chain's return time to 0 will dominate the return time of L_n to 0. The Markov chain can be constructed so that it uses an independent sequence of random variables when L_n has already returned to 0 but M_n is positive.

Proof of Proposition 2.2: We first observe that by the Markov property (Proposition 2.1) it is enough to show the result for $l = 1$. In order to study that, we define,

$$L_n := \begin{cases} \max\{\mathbf{w}(d) : \mathbf{w} \in \Delta_n\} - r_n & \text{if } \Delta_n \neq \emptyset \\ 0 & \text{if } \Delta_n = \emptyset \end{cases} \tag{3}$$

where $r_n = \min\{g_n(\mathbf{u}^i)(d) : i = 1, \dots, k\}$. We set,

$$\tau^L = \inf\{n \geq 1 : L_n = 0\}$$

and observe that $\tau_1 = \tau^L$.

Using the fact that $g_n^{\uparrow m}(\mathbf{u}^i) \notin \Delta_n$ for $m \geq 1$ and $1 \leq i \leq k$, where $g_n^{\uparrow m}(\mathbf{u}^i)$ is as defined in Remark 2.1, for any fixed $n \geq 0$, we define the collection of random variables

$$\Big\{ J_{n+1}(\mathbf{u}^i) := \inf\{m \geq 1 : g_n^{\uparrow m}(\mathbf{u}^i) \in V\} : \ 1 \leq i \leq k \Big\}, \tag{4}$$

which is an i.i.d. collection of geometric random variables with parameter p, i.e. each of the random variables takes the value m with probability $p(1-p)^{m-1}$ for $m = 1, 2, \ldots$. Also,

$$||g_n(\mathbf{w}) - g_{n+1}(\mathbf{w})||_1 \leq J_{n+1}(\mathbf{w}) \text{ for all } \mathbf{w} \text{ with } g_n(\mathbf{w}) \in W_n^{\text{move}}. \tag{5}$$

Let $\{G_n^{i,1} : 1 \leq i \leq k, n \geq 0\}$ be another family of i.i.d. geometric random variables with parameter p, independent of $\{U_{\mathbf{w}} : \mathbf{w} \in \mathbb{Z}^d\}$.

Now given $g_n(\mathbf{u}^1), \ldots, g_n(\mathbf{u}^k)$ and H_n, we define $\{M_n := M_n(\mathbf{u}^1, \ldots, \mathbf{u}^k), X_n := X_n(\mathbf{u}^1, \ldots, \mathbf{u}^k) : n \geq 0\}$ as follows:

$$\text{set } M_0 = 0 = X_0 \text{ and } M_{n+1} = \max\{M_n, J_{n+1}^1\} - 1 \text{ for } n \geq 0$$

where

$$J_{n+1}^1 := \begin{cases} \max\{J_{n+1}(\mathbf{u}) : g_n(\mathbf{u}) \in W_n^{\text{move}}\} & \text{if } \#W_n^{\text{move}} = k \text{ and } X_n = 0, \\ \max\{G_{n+1}^{i,1}, J_{n+1}(\mathbf{u}) : \\ \quad g_n(\mathbf{u}) \in W_n^{\text{move}}, i = 1, \ldots, k - k'\} & \text{if } \#W_n^{\text{move}} = k' < k \text{ and } X_n = 0, \\ \max\{G_{n+1}^{i,1} : 1 \leq i \leq k\} & \text{if } X_n = 1, \end{cases} \tag{6}$$

and

$$X_{n+1} := \begin{cases} 1 & \text{if } X_n = 0, L_{n+1} = 0 \\ X_n & \text{otherwise.} \end{cases} \tag{7}$$

From (6) it follows that $\{J_{n+1}^1 : n \geq 0\}$ is a family of i.i.d. copies of J where for any $m \geq 1$,

$$\mathbb{P}(J \leq m) = (1 - (1-p)^m)^k \tag{8}$$

and hence the sequence $\{J_n^1 : n \geq 1\}$ satisfies the conditions of Lemma 2.1.

Further, we claim that $0 \leq L_n \leq M_n$ for all $0 \leq n \leq \tau^L$. Indeed, this holds for $n = 0$, and assume that it holds for some $0 \leq n < \tau^L$. If $\Delta_{n+1} = \emptyset$ then we have $0 = L_{n+1} \leq M_{n+1}$. Otherwise if $\mathbf{w} \in \Delta_{n+1}$, then, from the definition of Δ_{n+1}, either $\mathbf{w} \in \Delta_n$ or $\mathbf{w} \in S(\mathbf{u}, ||\mathbf{u} - h(\mathbf{u})||_1)$ for some $\mathbf{u} \in W_n^{\text{move}}$. Therefore, from (5) and (6), $\mathbf{w}(d) \leq \max\{\max\{\mathbf{u}(d) : \mathbf{u} \in \Delta_n\}, \min\{g_n(\mathbf{u}^i)(d),\ 1 \leq i \leq k\} + ||\mathbf{u} - h(\mathbf{u})||_1 : \mathbf{u} \in W_n^{\text{move}}\} \leq \max\{L_n + r_n, r_n + J_{n+1}\} = \max\{L_n, J_{n+1}\} + r_n$. Also $r_{n+1} = \min\{g_{n+1}(\mathbf{u}^i)(d),\ 1 \leq i \leq k\} \geq \min\{g_n(\mathbf{u}^i)(d),\ 1 \leq i \leq k\} + 1 = r_n + 1$. Thus $L_{n+1} \leq \max\{L_n, J_{n+1}\} - 1 \leq \max\{M_n, J_{n+1}\} - 1 = M_{n+1}$.

Define,

$$\tau^M = \tau^M(\mathbf{u}^1, \ldots, \mathbf{u}^k) := \inf\{n \geq 1 : M_n = 0\}.$$

Note that the distribution of $\tau^M(\mathbf{u}^1,\dots,\mathbf{u}^k)$ does not depend on $\mathbf{u}^1,\dots,\mathbf{u}^k$. From the above observation that $0 \le L_n \le M_n$ for $0 \le n \le \tau_1$, we obtain that

$$\tau_1 = \tau^L \le \tau^M.$$

Using Lemma 2.1, we obtain Proposition 2.2. $\square$

Since $\tau_l < \infty$ almost surely, we obtain that

$$\{(g_{\tau_l}(\mathbf{u}^1),\dots,g_{\tau_l}(\mathbf{u}^k)) : l \ge 0\} \text{ is a Markov chain on } (\mathbb{Z}^d)^k.$$

Next we consider the width of the explored region between the $l-1$ and l th regenerations. For the process starting from $\mathbf{u}^1,\dots,\mathbf{u}^k$ with $\mathbf{u}^1(d) = \cdots = \mathbf{u}^k(d)$ define

$$W_l = W_l(\mathbf{u}^1,\dots,\mathbf{u}^k) := \sum_{n=\tau_{l-1}+1}^{\tau_l} \sum_{i=1}^{k} ||g_n(\mathbf{u}^i) - g_{n-1}(\mathbf{u}^i)||_1. \tag{9}$$

Further using $\{G_n^{i,l+1} :\ 1 \le i \le k, n \ge 0\}$ another family of i.i.d. geometric random variables with parameter p, independent of $\{U_{\mathbf{w}} : \mathbf{w} \in \mathbb{Z}^d\}$ and $\{G_n^{i,j} :\ 1 \le i \le k, 1 \le j \le l, n \ge 0\}$ we construct $\{M_n(g_{\tau_l}(\mathbf{u}^1),\dots,g_{\tau_l}(\mathbf{u}^k)), X_n(g_{\tau_l}(\mathbf{u}^1),\dots,g_{\tau_l}(\mathbf{u}^k)) : n \ge 0\}$ such that $\sigma_{l+1}(\mathbf{u}^1,\dots,\mathbf{u}^k) \le \tau^M(g_{\tau_l}(\mathbf{u}^1),\dots,g_{\tau_l}(\mathbf{u}^k))$ and $\tau^M(g_{\tau_l}(\mathbf{u}^1),\dots,g_{\tau_l}(\mathbf{u}^k))$ is an i.i.d. copy of $\tau^M(\mathbf{u}_1,\dots,\mathbf{u}_k)$. Also for $\tau_l \le n < \tau_{l+1}$ we have

$$\sum_{i=1}^{k} ||g_{n+1}(\mathbf{u}^i) - g_n(\mathbf{u}^i)||_1 \le \sum_{g_n(\mathbf{u}^i)\in W^n_{\text{move}}} J_{n+1}(\mathbf{u}^i) \le kJ^{l+1}_{(n-\tau_l)+1}$$

where the last sum is over distinct elements of W_n^{move} to avoid double counting and J_i^{l+1} is defined as in (6) using $\{G_n^{i,l+1} : 1 \le i \le k, n \ge 0\}$ instead of $\{G_n^{i,1} : 1 \le i \le k, n \ge 0\}$. Further it follows that $W_{l+1} \le \sum_{i=1}^{\tau^M(g_{\tau_l}(\mathbf{u}_1),\dots,g_{\tau_l}(\mathbf{u}_k))} kJ_i^{l+1}$ and $\sum_{i=1}^{\tau^M(g_{\tau_l}(\mathbf{u}_1),\dots,g_{\tau_l}(\mathbf{u}_k))} kJ_i^{l+1}$ is an i.i.d. copy of $W^M := \sum_{i=1}^{\tau^M(\mathbf{u}_1,\dots,\mathbf{u}_k)} kJ_i^1$.

The time for the l th regeneration (measured by the distance travelled by process in the d th co-ordinate) is defined as

$$T_l = T_l(\mathbf{u}^1,\dots,\mathbf{u}^k) := g_{\tau_l}(\mathbf{u}^1)(d) - \mathbf{u}^1(d) = g_{\tau_l}(\mathbf{u}^i)(d) - \mathbf{u}^i(d) \text{ for } 1 \le i \le k. \tag{10}$$

Clearly $T_l - T_{l-1} \le W_l$ and we have

Proposition 2.3 *For any $l \ge 1$ and $\mathbf{u}^1,\dots,\mathbf{u}^k$ with $\mathbf{u}^1(d) = \cdots = \mathbf{u}^k(d)$ we have*

$$\mathbb{P}(T_l - T_{l-1} \ge n) \le \mathbb{P}(W_l \ge n) \le \mathbb{P}(W^M \ge n) \le C_7 \exp(-C_8 n) \tag{11}$$

for all $n \ge 1$, where C_7 and C_8 are positive constants, not depending on l or $\mathbf{u}^1,\dots,\mathbf{u}^k$.

Proof: As in the proof of Lemma 2.1, it suffices to show that $\mathbb{E}\big(\exp(\alpha W^M)\big) < \infty$ for some $\alpha > 0$. Since $\{J_n^1 : n \geq 1\}$ are i.i.d. random variables, each with an exponentially decaying tail probability, there exists $\beta_0 > 0$ such that the moment generating function of $\Psi_J(\alpha) := E(\exp(\alpha J_1^1)) < \infty$ for all $\alpha < \beta_0$. Since the function $\Psi_J(\alpha)$ is continuous at 0 and $\Psi_J(0) = 1$, we may choose $\alpha_0 > 0$ so that $\Psi_J(2k\alpha)\exp(-C_2) < 1$ for all $\alpha \leq \alpha_0$ where C_2 is the constant as in (2). Now, we have, for $0 < \alpha \leq \alpha_0$,

$$\begin{aligned}
&\mathbb{E}\big[\exp(\alpha W^M)\big] \leq \mathbb{E}\Big[\exp(\alpha k \sum_{n=1}^{\tau^M} J_n^1)\Big] = \mathbb{E}\Big[\sum_{n=1}^{\infty} \mathbf{1}(\tau^M = n)\exp(\alpha k \sum_{i=1}^{n} J_i^1)\Big] \\
&= \sum_{n=1}^{\infty} \mathbb{E}\Big[\mathbf{1}(\tau^M = n)\exp(\alpha k \sum_{i=1}^{n} J_i^1)\Big] \leq \sum_{n=1}^{\infty} \big[\mathbb{P}(\tau^M = n)\big]^{1/2} \big[\mathbb{E}\big(\exp(2k\alpha \sum_{i=1}^{n} J_i^1)\big)\big]^{1/2} \\
&\leq \sum_{n=1}^{\infty} \sqrt{C_1}\exp(-nC_2/2)\big[\Psi_J(2k\alpha)\big]^{n/2} = \sqrt{C_1}\sum_{n=1}^{\infty}\big[\exp(-C_2)\Psi_J(2k\alpha)\big]^{n/2} < \infty;
\end{aligned}$$

here the first inequality follows from the Cauchy-Schwartz inequality. This completes the proof. $\square$

Remark 2.2 *For the process starting from just one vertex* $\mathbf{u}$*, from the translation invariance of the model, we have that* $\{\big(\sigma_l(\mathbf{u}), W_l(\mathbf{u}), (T_l(\mathbf{u}) - T_{l-1}(\mathbf{u}))\big) : l \geq 1\}$ *is an i.i.d. family of random vectors taking values in* $\{1, 2, 3, \dots\}^3$ *whose distribution does not depend on the choice of the starting vertex* $\mathbf{u}$*. Furthermore, each of the marginals of this random vector has exponentially decaying tail probability.*

Letting

$$\bar{\mathbf{u}} := (\mathbf{u}(1), \dots, \mathbf{u}(d-1)) \text{ for } \mathbf{u} = (\mathbf{u}(1), \dots, \mathbf{u}(d)) \tag{12}$$

denote the first $d-1$ *co-ordinates of* $\mathbf{u}$*, we have*

$$\{Y_{l+1}^{(\mathbf{u})} := \bar{g}_{\tau_{l+1}}(\mathbf{u}) - \bar{g}_{\tau_l}(\mathbf{u}) : l \geq 0\} \tag{13}$$

is a sequence of i.i.d. $\mathbb{Z}^{d-1}$ *valued random vectors, whose distribution does not depend on* $\mathbf{u}$*. For* $\mathbf{u} = \mathbf{0}$*, we denote* $Y_{l+1}^{(\mathbf{0})}$ *as* Y_{l+1}*. It should be noted, from the rotational symmetry in the first* $(d-1)$ *co-ordinates and reflection symmetry for each of the first* $(d-1)$ *co-ordinates of* $g_{\tau_l}(\mathbf{0})$*, that the distribution of the co-ordinates of* Y_{l+1} *are uncorrelated and marginally each of them is symmetric about* 0*. More precisely,* $\mathbb{P}(Y_l(i) = +m) = \mathbb{P}(Y_l(i) = -m)$ *for* $m \geq 1$ *where* $Y_l(i)$ *denotes the* i *th co-ordinate of* Y_l*. Further, for all* $i, j \in \{1, 2, \dots, d-1\}, i \neq j$,

$$\mathbb{E}\big[(Y_l(i))^2\big] = \sigma^2 \text{ and } \mathbb{E}\big[\big(Y_l(i)\big)^{m_1}\big(Y_l(j)\big)^{m_2}\big] = 0 \tag{14}$$

for some $\sigma^2 > 0$ and if at least one of m_1, m_2 is odd. Denoting the L_1 norm in $(d-1)$ dimensions by $||\cdot||_{1,d-1}$, we also observe that $||Y_l||_{1,d-1} \leq W_l(\mathbf{0})$, so that we also have

$$\mathbb{P}(||Y_l||_{1,d-1} > n) \leq C_7 \exp(-C_8 n) \tag{15}$$

for all $n \geq 1$, where C_7 and C_8 are as in (11). □

3 Martingale and independent processes

For the process starting from $\mathbf{u}^1, \ldots, \mathbf{u}^k$ with $\mathbf{u}^1(d) = \cdots = \mathbf{u}^k(d)$, the process $\{(g_{\mathcal{T}_l}(\mathbf{u}^1), \ldots, g_{\mathcal{T}_l}(\mathbf{u}^k)) : l \geq 0\}$ is a spatially invariant Markov chain on $(\mathbb{Z}^d)^k$. Hence, for any pair $1 \leq i \neq j \leq k$, the process $\{g_{\mathcal{T}_l}(\mathbf{u}^i) - g_{\mathcal{T}_l}(\mathbf{u}^j) : l \geq 0\}$ is also a Markov chain on $\mathbb{Z}^d$. However, as observed earlier, (see Figure 4), $g_{\mathcal{T}_l}(\mathbf{u}^i)(d) = g_{\mathcal{T}_l}(\mathbf{u}^j)(d)$ for every $l \geq 1$. Thus, using notation as in (12), for any pair, $1 \leq i \neq j \leq k$,

$$\{Z_l = Z_l(\mathbf{u}^i, \mathbf{u}^j) := \overline{g}_{\mathcal{T}_l}(\mathbf{u}^i) - \overline{g}_{\mathcal{T}_l}(\mathbf{u}^j) : l \geq 0\} \tag{16}$$

is a $\mathbb{Z}^{d-1}$ valued Markov chain.

In this section we first show that, for $d = 2$ and $k = 2$, the process Z_l is a martingale. Later, for a general d, we study how different are the two processes, one which starts from k ($k \geq 2$) distinct vertices far apart and the other being a collection of k independent processes starting from these k vertices.

3.1 Martingale

In this subsection we restrict ourselves to $d = 2$. Consider the process starting from two vertices $\mathbf{u}, \mathbf{v} \in \mathbb{Z}^2$ with $\mathbf{u}(2) = \mathbf{v}(2)$. We first observe that, for $l \geq 0$, the regeneration time $T_l = T_l(\mathbf{u}, \mathbf{v})$ is a stopping time with respect to the filtration $\{\mathcal{F}_t : t \geq 0\}$ where $\mathcal{F}_t := \sigma\{U_{\mathbf{w}} : \mathbf{w}(2) \leq \mathbf{u}(2) + t\}$. By our construction, $g_{\mathcal{T}_l}(\mathbf{u})$ is $\mathcal{F}_{T_l}$ measurable. Therefore, $\overline{g}_{\mathcal{T}_l}(\mathbf{u})$, given by the projection from $\mathbb{Z}^d \to \mathbb{Z}^{d-1}$, is also $\mathcal{F}_{T_l}$ measurable.

Proposition 3.1 *For $d = 2$ and $\mathbf{u}, \mathbf{v} \in \mathbb{Z}^2$ with $\mathbf{u}(2) = \mathbf{v}(2)$, the process $\{\overline{g}_{\mathcal{T}_l}(\mathbf{u}) : l \geq 0\}$ is a martingale with respect to the filtration $\{\mathcal{F}_{T_l} : l \geq 0\}$.*

The above proposition also holds for $\overline{g}_{\mathcal{T}_l}(\mathbf{v})$, so we obtain

Corollary 3.1 *For $d = 2$ and any $\mathbf{u}, \mathbf{v} \in \mathbb{Z}^2$ with $\mathbf{u}(2) = \mathbf{v}(2)$, the process $\{Z_l = Z_l(\mathbf{u}, \mathbf{v}) : l \geq 0\}$ is a martingale with respect to the filtration $\{\mathcal{F}_{T_l} : l \geq 0\}$.*

Proof of Proposition 3.1: We construct the process $(g_n(\mathbf{u}), g_n(\mathbf{v}), H_n(\mathbf{u}, \mathbf{v}))$ starting form $\mathbf{u}, \mathbf{v}$ with $\mathbf{u}(2) = \mathbf{v}(2)$, and the process $(g_n(\mathbf{u}), H_n(\mathbf{u}))$ starting from $\mathbf{u}$ with the same set of uniform random variables $\{U_{\mathbf{w}} : \mathbf{w} \in \mathbb{Z}^2\}$. Observe that every joint regeneration of the paths from a pair of vertices $\mathbf{u}, \mathbf{v}$ is also a regeneration of the single path from $\mathbf{u}$, i.e., for every $l \geq 0$, we have

$$T_l(\mathbf{u}, \mathbf{v}) = T_{N_l}(\mathbf{u})$$

for some sequence $N_l = N_l(\mathbf{u}, \mathbf{v})$. Therefore, we have,

$$\overline{g}_{\tau_l(\mathbf{u},\mathbf{v})}(\mathbf{u}) = \overline{g}_{\tau_{N_l}(\mathbf{u})}(\mathbf{u}) = \sum_{i=1}^{N_l} Y_i^{(\mathbf{u})}$$

where $\{Y_i^{(\mathbf{u})} := \overline{g}_{\tau_i(\mathbf{u})}(\mathbf{u}) - \overline{g}_{\tau_{i-1}(\mathbf{u})}(\mathbf{u}) : i \geq 1\}$ is as in Remark 2.2. Since $N_l \leq T_l(\mathbf{u}, \mathbf{v})$, and each of $T_i(\mathbf{u}, \mathbf{v}) - T_{i-1}(\mathbf{u}, \mathbf{v})$ and $Y_i^{(\mathbf{u})}$ has an exponentially decaying tail probability (see Proposition 2.3 and equation (15)), for every $l \geq 0$, we have that $\mathbb{E}(|\overline{g}_{\tau_l(\mathbf{u},\mathbf{v})}(\mathbf{u})|) < \infty$.

Further we need to show that

$$\mathbb{E}\big[\overline{g}_{\tau_{l+1}(\mathbf{u},\mathbf{v})}(\mathbf{u}) - \overline{g}_{\tau_l(\mathbf{u},\mathbf{v})}(\mathbf{u}) | \mathcal{F}_{T_l(\mathbf{u},\mathbf{v})}\big] = \mathbb{E}\big[\sum_{i=N_l+1}^{N_{l+1}} Y_i^{(\mathbf{u})} | \mathcal{F}_{T_{N_l}(\mathbf{u})}\big] = 0 \text{ a.s.} \qquad (17)$$

Denoting $\mathcal{G}_i := \mathcal{F}_{T_i(\mathbf{u})}$, we have that $Y_{i+1}^{(\mathbf{u})}$ is independent of $\mathcal{G}_i$. We also observe that N_l is $\{\mathcal{G}_i : i \geq 0\}$ adapted for each $l \geq 0$, i.e., $\{N_l \leq m\} \in \mathcal{G}_m$. Therefore, for any $A \in \mathcal{F}_{T_{N_l}(\mathbf{u})} = \mathcal{G}_{N_l}$, we have

$$\begin{aligned}
&\mathbb{E}\big[\mathbf{1}(A) \sum_{i=N_l+1}^{N_{l+1}} Y_i^{(\mathbf{u})}\big] = \mathbb{E}\big[\mathbf{1}(A) \sum_{n_l=1}^{\infty} \sum_{m=1}^{\infty} \mathbf{1}(N_l = n_l) \mathbf{1}(N_{l+1} = n_l + m) \sum_{i=1}^{m} Y_{n_l+i}^{(\mathbf{u})}\big] \\
&= \mathbb{E}\big[\mathbf{1}(A) \sum_{n_l=1}^{\infty} \sum_{m=1}^{\infty} \mathbf{1}(N_l = n_l) \mathbf{1}(N_{l+1} \geq n_l + m) Y_{n_l+m}^{(\mathbf{u})}\big] \\
&= \sum_{n_l=1}^{\infty} \sum_{m=1}^{\infty} \mathbb{E}\big[\mathbf{1}(A) \mathbf{1}(N_l = n_l)[1 - \mathbf{1}(N_{l+1} \leq n_l + m - 1)] Y_{n_l+m}^{(\mathbf{u})}\big] \\
&= \sum_{n_l=1}^{\infty} \sum_{m=1}^{\infty} \mathbb{E}\Big[\mathbb{E}\big[\mathbf{1}(A) \mathbf{1}(N_l = n_l)[1 - \mathbf{1}(N_{l+1} \leq n_l + m - 1)] Y_{n_l+m}^{(\mathbf{u})} \mid \mathcal{G}_{n_l+m-1}\big]\Big] \\
&= \sum_{n_l=1}^{\infty} \sum_{m=1}^{\infty} \mathbb{E}\Big[\mathbf{1}(A) \mathbf{1}(N_l = n_l)[1 - \mathbf{1}(N_{l+1} \leq n_l + m - 1)] \mathbb{E}\big[Y_{n_l+m}^{(\mathbf{u})} \mid \mathcal{G}_{n_l+m-1}\big]\Big] \\
&= \sum_{n_l=1}^{\infty} \sum_{m=1}^{\infty} \mathbb{E}\Big[\mathbf{1}(A) \mathbf{1}(N_l = n_l)[1 - \mathbf{1}(N_{l+1} \leq n_l + m - 1)] \mathbb{E}\big[Y_{n_l+m}^{(\mathbf{u})}\big]\Big] = 0.
\end{aligned}$$

In the above we have used (14) and the fact that, since A is $\mathcal{G}_{N_l}$ measurable, $A \cap \{N_l = n_l\} \in \mathcal{G}_{n_l} \subseteq \mathcal{G}_{n_l+m-1}$ for all $m \geq 1$. Also, $\{N_{l+1} \leq n_l + m - 1\} \in \mathcal{G}_{n_l+m-1}$ and $Y_{n_l+m}^{(\mathbf{u})}$ is independent of $\mathcal{G}_{n_l+m-1}$. $\square$

3.2 Independent processes

In this subsection, we describe *simultaneous regenerations of k independent paths.* This will be used to approximate the paths at simultaneous regenerations of joint paths when the starting points are far apart. We start with a result (Lemma 3.1) about renewal processes, which is proved in the Appendix.

Let $\{\xi_n^{(i)} : n \geq 1\}$, $i = 1, \ldots, k$, be k independent sequences of i.i.d. inter-arrival times (positive integer valued random variables) with $\mathbb{P}(\xi_n^{(i)} = j) = f_j^{(i)}$ for $i = 1, \ldots, k$. We assume that $\sum_{j=n}^{\infty} f_j^{(i)} \leq C_9 \exp(-C_{10} n)$ for all $n \geq 1$ and some universal positive constants C_9 and C_{10} and $f_1^{(i)} > 0$ for $i = 1, \ldots, k$. Let $S_0^{(i)} := 0$ and $S_n^{(i)} := \sum_{j=1}^{n} \xi_j^{(i)}$ for all $n \geq 1$. For any $n \geq 1$ and $i = 1, \ldots, k$, define the residual life of the i th component at time n by

$$R_n^{(i)} := \inf\{S_k^{(i)} : S_k^{(i)} \geq n\} - n. \tag{18}$$

We consider the joint residual process $(R_n^{(i)} : i = 1, \ldots, k)$ and define

$$\tau^R := \inf\{n \geq 1 : R_n^{(i)} = 0 \text{ for } i = 1, \ldots, k\}.$$

Lemma 3.1 *For any $n \geq 1$, we have*

$$\mathbb{P}(\tau^R \geq n) \leq C_{11}^{(k)} \exp(-C_{12}^{(k)} n)$$

where $C_{11}^{(k)}$ and $C_{12}^{(k)}$ are positive constants, depending on k and the distribution of $\xi_n^{(i)}$'s only.

Now we consider k independent constructions of the marginal paths starting from the vertices $\mathbf{u}^1, \ldots, \mathbf{u}^k$ respectively, with $\mathbf{u}^1(d) = \cdots = \mathbf{u}^k(d)$. More precisely, we start with k i.i.d. collections $\{U_{\mathbf{w}}^i : \mathbf{w} \in \mathbb{Z}^d\}$, $i = 1, \ldots, k$, of uniform random variables. For each $i = 1, \ldots, k$, we construct the path process, starting from $\mathbf{u}^i$ only, as in Section 2, using only the collection $\{U_{\mathbf{w}}^i : \mathbf{w} \in \mathbb{Z}^d\}$ and we denote this process by $\{g_n^{(\mathrm{Ind})}(\mathbf{u}^i) : n \geq 0\}$. Therefore, for each i, we have a collection of regeneration times, which we denote by $\{T_l^{(\mathrm{Ind})}(\mathbf{u}^i) : l \geq 0\}$ (see equation (10) for definition). Since the collection $\{T_l^{(\mathrm{Ind})}(\mathbf{u}^i) : l \geq 0\}$ uses only the random variables $\{U_{\mathbf{w}}^i : \mathbf{w} \in \mathbb{Z}^d\}$, these families are independent. Furthermore, as mentioned in Remark 2.2, for a single starting point, the distribution of the collection $\{T_l^{(\mathrm{Ind})}(\mathbf{u}^i) : l \geq 0\}$, does not depend on the starting point $\mathbf{u}^i$ and is an independent copy of the family of random variable $\{T_l(\mathbf{0}) : l \geq 0\}$.

Define, $R_n^{(i)} = \inf\{T_l^{(\mathrm{Ind})}(\mathbf{u}^i) : T_l^{(\mathrm{Ind})}(\mathbf{u}^i) \geq n\} - n$; note here that $\{T_{l+1}^{(\mathrm{Ind})}(\mathbf{u}^i) - T_l^{(\mathrm{Ind})}(\mathbf{u}^i) : l \geq 0\}$ is an i.i.d. sequence of random variables which plays the role of $\{\xi_l^{(i)} : l \geq 0\}$ of the previous lemma. Set, $T_0^{(\mathrm{Ind})}(\mathbf{u}^1, \ldots, \mathbf{u}^k) = 0$ and, for $l \geq 0$,

$$T_{l+1}^{(\mathrm{Ind})}(\mathbf{u}^1, \ldots, \mathbf{u}^k) := \inf\{n > T_l^{(\mathrm{Ind})}(\mathbf{u}^1, \ldots, \mathbf{u}^k) : R_n^{(i)} = 0 \text{ for } i = 1, \ldots, k\}. \tag{19}$$

We call $T_l^{(\mathrm{Ind})}(\mathbf{u}^1,\dots,\mathbf{u}^k)$, the time for the l th *simultaneous regeneration time of k independent paths.*

Applying Lemma 3.1 and observing that each $T_l^{(\mathrm{Ind})}(\mathbf{u}^1,\dots,\mathbf{u}^k)$ represents the occurrence of a renewal event, we obtain the following proposition.

Proposition 3.2 *The family $\{T_{l+1}^{(\mathrm{Ind})}(\mathbf{u}^1,\dots,\mathbf{u}^k) - T_l^{(\mathrm{Ind})}(\mathbf{u}^1,\dots,\mathbf{u}^k) : l \geq 0\}$ is an i.i.d. sequence of random variables taking values in $\{1,2,3,\dots\}$ and, for all $n \geq 1$,*

$$\mathbb{P}(T_1^{(\mathrm{Ind})}(\mathbf{u}^1,\dots,\mathbf{u}^k) \geq n) \leq C_{13}^{(k)} \exp(-C_{14}^{(k)} n) \tag{20}$$

where $C_{13}^{(k)}$ and $C_{14}^{(k)}$ are positive constants, depending on k and the distribution of $T_1(\mathbf{0})$ only.

Now let $\{g_n^{(\mathrm{Ind})}(\mathbf{u}^i) : n \geq 0\}$, $1 \leq i \leq k$, be the k independent versions of the processes starting from $\mathbf{u}^1,\dots,\mathbf{u}^k$ respectively, as described above. Also, let $\{g_{\tau_l(\mathbf{u}^i)}^{(\mathrm{Ind})}(\mathbf{u}^i) : l \geq 0\}$ be the i th process evaluated at its regeneration steps $\tau_l(\mathbf{u}^i)$, $l \geq 0$. Let $T_l^{(\mathrm{Ind})}(\mathbf{u}^1,\dots,\mathbf{u}^k)$ be the l th simultaneous regeneration time as defined in (19). Clearly the simultaneous regeneration time $T_l^{(\mathrm{Ind})}(\mathbf{u}^1,\dots,\mathbf{u}^k)$ is also a regeneration time for each of the $i = 1,\dots,k$ processes. Suppose $N_l(i)$ ($i = 1,\dots,k$, $l \geq 0$) is such that the l th joint regeneration coincides with the $N_l(i)$ th regeneration of the i th process, i.e.

$$T_l^{(\mathrm{Ind})}(\mathbf{u}^1,\dots,\mathbf{u}^k) = T_{N_l(i)}^{(\mathrm{Ind})}(\mathbf{u}^i),\ i = 1,\dots,k$$

where $T_{N_l(i)}^{(\mathrm{Ind})}(\mathbf{u}^i)$ is the $N_l(i)$ th regeneration time of the i th process starting only from $\mathbf{u}^i$.

As in (9) consider the width of the explored region between the $l-1$ and l th regenerations of the i th process. We define

$$W_l^{(\mathrm{Ind})}(\mathbf{u}^1,\dots,\mathbf{u}^k) := \sum_{i=1}^{k} \sum_{t=N_{l-1}(i)+1}^{N_l(i)} W_t^{(\mathrm{Ind})}(\mathbf{u}^i) \tag{21}$$

where $\{W_l^{(\mathrm{Ind})}(\mathbf{u}^i) : l \geq 1\}$ is the explored width process, associated with the process $\{g_l^{(\mathrm{Ind})}(\mathbf{u}^i) : l \geq 0\}$ for $i = 1,\dots,k$.

Since $N_l(i) \leq T_l^{(\mathrm{Ind})}(\mathbf{u}^1,\dots,\mathbf{u}^k)$ for every $i = 1,\dots,k$ and $T_l^{(\mathrm{Ind})}(\mathbf{u}^1,\dots,\mathbf{u}^k) - T_{l-1}^{(\mathrm{Ind})}(\mathbf{u}^1,\dots,\mathbf{u}^k)$ satisfies (20), so calculations similar to that in the proof of Proposition 2.3 yields

$$\mathbb{P}(W_l^{(\mathrm{Ind})}(\mathbf{u}^1,\dots,\mathbf{u}^k) \geq n) \leq C_{15}^{(k)} \exp(-C_{16}^{(k)} n) \tag{22}$$

where $C_{15}^{(k)}$ and $C_{16}^{(k)}$ are positive constants, depending on k and the distribution of $W_1(\mathbf{0})$ only.

From (13), we have that

$$\overline{g}^{(\mathrm{Ind})}_{\tau_{N_l(i)}(\mathbf{u}^i)}(\mathbf{u}^i) = \overline{\mathbf{u}^i} + \sum_{t=1}^{N_l(i)} Y_t^{(\mathbf{u}^i)}.$$

At each simultaneous regeneration time $T_l^{(\mathrm{Ind})}(\mathbf{u}^1, \dots, \mathbf{u}^k)$, the d th co-ordinates of each of the processes $g_n^{(\mathrm{Ind})}(\mathbf{u}^i)$ coincide and equal $T_l^{(\mathrm{Ind})}(\mathbf{u}^1, \dots, \mathbf{u}^k)$. We consider the first $d-1$ co-ordinates of these processes and for $i = 1, \dots, k$, denote

$$\psi_l^{(i)} := \overline{g}^{(\mathrm{Ind})}_{\tau_{N_l(i)}(\mathbf{u}^i)}(\mathbf{u}^i) - \overline{g}^{(\mathrm{Ind})}_{\tau_{N_{l-1}(i)}(\mathbf{u}^i)}(\mathbf{u}^i) = \sum_{t=N_{l-1}(i)+1}^{N_l(i)} Y_t^{(\mathbf{u}^i)} \tag{23}$$

The process $\psi_l^{(i)}$ represents the increment in the first $(d-1)$ co-ordinates of the path starting from $\mathbf{u}^i$ between the $(l-1)$ th and the l th simultaneous regeneration times of k independent paths.

Since $T_l^{(\mathrm{Ind})}(\mathbf{u}^1, \dots, \mathbf{u}^k)$ represents the occurrence of a renewal event and all random variables $\{Y_t^{(\mathbf{u}^i)} : t \geq 1, i = 1, \dots, k\}$ are independent, we have that

> $\{(\psi_l^{(1)}, \dots, \psi_l^{(k)}) : l \geq 1\}$ is an i.i.d. collection of random variables taking values in $\mathbb{Z}^{(d-1)k}$. As earlier, we also observe that $\sum_{i=1}^k ||\psi_l^{(i)}||_{1,d-1} \leq W_l^{(\mathrm{Ind})}(\mathbf{u}^1, \dots, \mathbf{u}^k)$, so that we have
>
> $$\mathbb{P}(\sum_{i=1}^k ||\psi_l^{(i)}||_{1,d-1} > n) \leq C_{15}^{(k)} \exp(-C_{16}^{(k)} n)$$
>
> for all $n \geq 1$, where $C_{15}^{(k)}$ and $C_{16}^{(k)}$ are as above.

Using the inherent symmetries of the marginals of $\psi_l^{(i)}$ (with $\psi_l^{(i)}(j)$ being the j th co-ordinate of $\psi_l^{(i)}$) and carrying out calculations, similar to that in the proof of Proposition 3.1, we obtain the following.

(a) $\mathbb{P}\big(\psi_l^{(i)}(j) = r\big) = \mathbb{P}\big(\psi_l^{(i)}(j) = -r\big)$ for all $r \geq 1$ for $i = 1, \dots, k$ and $j = 1, \dots, d-1$. Furthermore, $\mathbb{P}\big(\psi_l^{(i)}(j) = r\big)$ is independent of $i = 1, \dots, k$ and $j = 1, \dots, d-1$ for $r \in \mathbb{Z}$.

(b) $\mathbb{E}\Big[\big(\psi_l^{(i_1)}(j_1)\big)^{m_1}\big(\psi_l^{(i_2)}(j_2)\big)^{m_2}\Big] = 0$ if at least one of m_1, m_2 is odd, for all $1 \leq i_1 \neq i_2 \leq k$ and $1 \leq j_1, j_2 \leq d-1$.

(c) $\mathbb{E}\Big[\big(\psi_l^{(i_1)}(j_1)\big)^{m_1}\big(\psi_l^{(i_2)}(j_2)\big)^{m_2}\Big]$ is independent of $i_1, i_2, i_1 \neq i_2, j_1, j_2$, and depends only on m_1, m_2.

3.3 Coupling of joint process and independent process

In this subsection, we describe a coupling of the joint paths starting from $\mathbf{u}^1, \dots, \mathbf{u}^k$, for $k \geq 2$, with $\mathbf{u}^1(d) = \cdots = \mathbf{u}^k(d)$ and k independent paths starting from $\mathbf{u}^1, \dots, \mathbf{u}^k$ respectively. Without loss of generality, we may assume $\mathbf{u}^1(d) = 0$. Let $d_{\min} := \min\{||\mathbf{u}^i - \mathbf{u}^j||_1 : 1 \leq i \neq j \leq k\}$. As in last subsection, we start with k independent collections of i.i.d. uniform random variables $\{U^i_{\mathbf{w}} : \mathbf{w} \in \mathbb{Z}^d, \mathbf{w}(d) > 0, i = 1, \dots, k\}$. For each of $i = 1, \dots, k$, we construct the i th independent process $\{g^{(\mathrm{Ind})}_n(\mathbf{u}^i) : n \geq 0\}$, starting at $\mathbf{u}^i$, using only the uniform random variables $\{U^i_{\mathbf{w}} : \mathbf{w} \in \mathbb{Z}^d, \mathbf{w}(d) > 0\}$.

Fix $r < d_{\min}/2$ and another independent collection of uniform random variables $\{U^{k+1}_{\mathbf{w}} : \mathbf{w} \in \mathbb{Z}^d, \mathbf{w}(d) > 0\}$. We define a new collection of uniform random variables $\{\tilde{U}_{\mathbf{w}} : \mathbf{w} \in \mathbb{Z}^d, \mathbf{w}(d) > 0\}$ by

$$\tilde{U}_{\mathbf{w}} := \begin{cases} U^i_{\mathbf{w}} & \text{if } ||\overline{\mathbf{w}} - \overline{\mathbf{u}}^i||_{1,d-1} < r \text{ for some } i = 1, \dots, k \\ U^{k+1}_{\mathbf{w}} & \text{otherwise.} \end{cases}$$

Using this collection of uniform random variables, we construct the joint process (as in Section 2) from the points $\mathbf{u}^1, \dots, \mathbf{u}^k$ until its first simultaneous regeneration time $T_1(\mathbf{u}^1, \dots, \mathbf{u}^k)$ of joint paths from $\mathbf{u}^1, \dots, \mathbf{u}^k$.

Now, as defined in (19), let $T^{(\mathrm{Ind})}_1(\mathbf{u}^1, \dots, \mathbf{u}^k)$ be the first simultaneous regeneration time of the k independent processes and $N_1(i)$ the number of individual regenerations of the i th process until the first simultaneous regeneration. With the width of the explored region of the k independent processes as defined in (21), we consider the event where the total width of the explored region of each of the i th processes until the first joint regeneration time is less than r.

Figure 5: The shaded regions represent part of the cylinders (up to $T^{(\mathrm{Ind})}_1(\mathbf{u}^1, \mathbf{u}^2)$) of width r around $\mathbf{u}^1$ and $\mathbf{u}^2$. In the left cylinder we use the collection $\{U^1_{\mathbf{w}}\}$, on the right cylinder we use the collection $\{U^2_{\mathbf{w}}\}$ and in the remaining region, we use $\{U^3_{\mathbf{w}}\}$.

More precisely, we consider the event

$$A^{\text{Good}}(r) := \Big\{ W_1^{(\text{Ind})}(\mathbf{u}^1, \ldots, \mathbf{u}^k) \le r \Big\}.$$

On the event $A^{\text{Good}}(r)$, the joint path process $(g_n(\mathbf{u}^1), \ldots, g_n(\mathbf{u}^k))$ started simultaneously from $(\mathbf{u}^1, \ldots, \mathbf{u}^k)$, using the collection $\{\tilde{U}_{\mathbf{w}} : \mathbf{w} \in \mathbb{Z}^d, \mathbf{w}(d) > \mathbf{u}^1(d)\}$, until the first simultaneous regeneration time $T_1(\mathbf{u}^1, \ldots, \mathbf{u}^k)$ of joint paths, coincides with the collection of independent paths $\{g_n^{(\text{Ind})}(\mathbf{u}^i) : n \ge 0, i = 1, \ldots, k\}$, with the i th independent path, starting from $\mathbf{u}^i$, constructed using only the collection $\{U^i_{\mathbf{w}} : \mathbf{w} \in \mathbb{Z}^d, \mathbf{w}(d) > \mathbf{u}^i(d)\}$, until the first simultaneous regeneration of the independent paths $T_1^{(\text{Ind})}(\mathbf{u}^1, \ldots, \mathbf{u}^k)$. It should be kept in mind that the paths may be indexed differently than those for the independent paths, however as geometric paths they are identical. Therefore, we must have

$$T_1^{(\text{Ind})}(\mathbf{u}^1, \ldots, \mathbf{u}^k) = T_1(\mathbf{u}^1, \ldots, \mathbf{u}^k)$$

and hence we have,

$$\begin{aligned} &\mathbb{P}\big[\big(\overline{g}_{T_1(\mathbf{u}^1,\ldots,\mathbf{u}^k)}(\mathbf{u}^1), \ldots, \overline{g}_{T_1(\mathbf{u}^1,\ldots,\mathbf{u}^k)}(\mathbf{u}^k)\big) = \big(\overline{\mathbf{u}^1} + \psi_1^{(1)}, \ldots, \overline{\mathbf{u}^k} + \psi_1^{(k)}\big)\big] \\ &\quad \ge \mathbb{P}(A^{\text{Good}}(r)) \ge 1 - C_{15}^{(k)} \exp(-C_{16}^{(k)} r). \end{aligned} \tag{24}$$

Finally, using the Markov property, we can use this coupling for each subsequent joint regeneration step. The new value of $d_{\min}$ for the l th regeneration has to be computed from the position of the processes at the $l-1$ th joint regeneration and the value of r has to be chosen accordingly.

4 Trees and Forest

In this section we prove Theorem 1.1. For $d = 2, 3$, we need to prove that for any $\mathbf{u}, \mathbf{v} \in V$, the paths $\pi^{\mathbf{u}}$ and $\pi^{\mathbf{v}}$ coincide eventually, i.e., $\pi^{\mathbf{u}}(t) = \pi^{\mathbf{v}}(t)$ for all $t \ge t_0$ for some $t_0 < \infty$.

First, we claim that it is enough to prove that

$$\pi^{\mathbf{u}} \text{ and } \pi^{\mathbf{v}} \text{ coincide eventually for } \mathbf{u}, \mathbf{v} \in V \text{ with } \mathbf{u}(d) = \mathbf{v}(d). \tag{25}$$

Indeed, for $\mathbf{u}, \mathbf{v} \in V$ with $\mathbf{u}(d) < \mathbf{v}(d)$ we have, from (25),

$$\mathbb{P}\big[\bigcap_{\mathbf{w} \in V, \mathbf{u}(d) = \mathbf{w}(d)} \{\text{the paths } \pi^{\mathbf{u}} \text{ and } \pi^{\mathbf{w}} \text{ coincide eventually}\}\big] = 1;$$

$$\mathbb{P}\big[\bigcap_{\mathbf{w}' \in V, \mathbf{w}'(d) = \mathbf{v}(d)} \{\text{the paths } \pi^{\mathbf{v}} \text{ and } \pi^{\mathbf{w}'} \text{ coincide eventually}\}\big] = 1$$

Further, $\mathbb{P}\big[\text{there exist } \mathbf{w}, \mathbf{w}' \in V \text{ with } \mathbf{w}(d) = \mathbf{u}(d), \mathbf{w}'(d) = \mathbf{v}(d), h(\mathbf{w}) = \mathbf{w}'\big] = 1$. Since, the intersection of these three events has probability 1, $\pi^{\mathbf{u}}$ and $\pi^{\mathbf{v}}$ meet.

Now, to prove that for any two vertices $\mathbf{u}^0$ and $\mathbf{v}^0$ with $\mathbf{u}^0(d) = \mathbf{v}^0(d)$, the paths coincide eventually, we show that $\mathbb{P}(Z_l(\mathbf{u}^0, \mathbf{v}^0) = 0 \text{ for some } l \geq 0) = 1$ where Z_l is as in (16). Recall, at the beginning of Section 3, we had observed that $\{Z_l(\mathbf{u}^0, \mathbf{v}^0) : l \geq 0\}$ is a Markov chain taking values in $\mathbb{Z}^{d-1}$ with $\mathbf{0} \in \mathbb{Z}^{d-1}$ being its only absorbing state.

4.1 d=2

For the proof of Theorem 1.1 in the case $d = 2$ we consider the process constructed from the two vertices $\mathbf{u}^0$ and $\mathbf{v}^0$ with $\mathbf{u}^0(d) = \mathbf{v}^0(d)$. Without loss of generality, we may assume that $\mathbf{u}^0(1) > \mathbf{v}^0(1)$. Since the paths $\{g_n(\mathbf{u}^0) : n \geq 0\}$ and $\{g_n(\mathbf{v}^0) : n \geq 0\}$ do not cross[1] each other, from Corollary 3.1 we have that $\{Z_l(\mathbf{u}^0, \mathbf{v}^0) = g_{\tau_l(\mathbf{u}^0,\mathbf{v}^0)}(\mathbf{u}^0)(1) - g_{\tau_l(\mathbf{u}^0,\mathbf{v}^0)}(\mathbf{v}^0)(1) : l \geq 0\}$ is a non-negative martingale. By the martingale convergence theorem, there exists a random variable Z_∞ such that $Z_l(\mathbf{u}^0, \mathbf{v}^0) \to Z_\infty$ a.s. as $l \to \infty$. Also, 0 being the only absorbing state of the Markov chain $\{Z_l(\mathbf{u}^0, \mathbf{v}^0) : l \geq 0\}$ we have $Z_\infty = 0$ a.s. and hence $Z_l(\mathbf{u}^0, \mathbf{v}^0) = 0$ for some l a.s. This completes the proof of Theorem 1.1 for $d = 2$. $\square$

4.2 $d = 3$

We show that the Lyapunov method used in Gangopadhyay *et al.* [GRS04] is applicable here. We start with the process constructed from the vertices $\mathbf{u}^0, \mathbf{v}^0 \in \mathbb{Z}^3$ with $\mathbf{u}^0(3) = \mathbf{v}^0(3)$ and consider the process $Z_l = Z_l(\mathbf{u}^0, \mathbf{v}^0)$ where Z_l is as defined in (16). Also, changing the transition probability of Z_l from the state $\mathbf{0} = (0, 0)$, so the state $\mathbf{0}$ is no longer absorbing, we make the Markov chain $\{Z_l(\mathbf{u}^0, \mathbf{v}^0) : l \geq 0\}$ irreducible. With a slight abuse of notation, we continue to denote the modified chain by $\{Z_l(\mathbf{u}^0, \mathbf{v}^0) : l \geq 0\}$.

As in [GRS04], to show that the modified chain is recurrent, we show that the estimates (3), (4) and (5) of [GRS04] hold in this case too, i.e., we need appropriate bounds of $\mathbb{E}\Big[\big(||Z_{l+1}(\mathbf{u}^0, \mathbf{v}^0)||_2^2 - ||Z_l(\mathbf{u}^0, \mathbf{v}^0)||_2^2\big)^m \mid Z_l(\mathbf{u}^0, \mathbf{v}^0) = \mathrm{x}\Big]$ for $\mathrm{x} \in \mathbb{Z}^2$ and $m = 1, 2, 3$.

---

[1]Suppose $\mathbf{u} := (1, 0)$ and $\mathbf{v} := (0, 0)$ and $h(\mathbf{v}) = (i, j)$. If $i > 0$ then the region $\{(x, y) : y > 0 \text{ and } ||(x, y) - \mathbf{u}||_1 \leq i + j - 1\} \subset \{(x, y) : y > 0 \text{ and } ||(x, y) - \mathbf{v}||_1 \leq i + j\}$ and since the interior of neither of these regions contain open vertices, so $h(\mathbf{u}) = (i, j)$. If $i \leq 0$, then for the edge $\langle \mathbf{u}, h(\mathbf{u})\rangle$ to cross the edge $\langle \mathbf{v}, h(\mathbf{v})\rangle$ it must be that $||h(\mathbf{u}) - \mathbf{u}||_1 > i + j + 1$; which is not possible because $||h(\mathbf{v}) - \mathbf{u}||_1 = i + j + 1$. Now for vertices $\mathbf{u}$ and $\mathbf{v}$ which are not neighbours consider all the vertices in between. Since the edges from any pair of neighbouring vertices do not cross each other, neither do the edges from $\mathbf{u}$ and $\mathbf{v}$.

Since our model is translation invariant spatially and $Z_l(\mathbf{u}^0, \mathbf{v}^0)$ is a time homogeneous Markov chain, we may take $\mathbf{v}^0 = \mathbf{0} = (0,0,0)$ and $\mathbf{u}^0 = (\mathrm{x}, 0)$ and $l = 0$.

Now, we use the coupling described in last section, with $k = 2$ and $r = d_{\min}/3 = (|\mathrm{x}(1)| + |\mathrm{x}(2)|)/3$. First we observe that $||Z_1(\mathbf{u}^0, \mathbf{v}^0) - \mathrm{x}||_2 \leq ||Z_1(\mathbf{u}^0, \mathbf{v}^0) - \mathrm{x}||_1 \leq W_1(\mathbf{u}^0, \mathbf{v}^0)$ and $||\psi_1^{(1)} - \psi_1^{(2)}||_2 \leq ||\psi_1^{(1)} - \psi_1^{(2)}||_1 \leq W_1^{(\mathrm{Ind})}(\mathbf{u}^0, \mathbf{v}^0)$, where $W_1(\mathbf{u}^0, \mathbf{v}^0)$ and $W_1^{(\mathrm{Ind})}(\mathbf{u}^0, \mathbf{v}^0)$ are as defined in (9) and (21) respectively. Also, on the event $(A^{\mathrm{Good}}(r))^c$, we have $W_1(\mathbf{u}^0, \mathbf{v}^0) > d_{\min}/3 = (|\mathrm{x}(1)|+|\mathrm{x}(2)|)/3$ and $W_1^{(\mathrm{Ind})}(\mathbf{u}^0, \mathbf{v}^0) > d_{\min}/3 = (|\mathrm{x}(1)| + |\mathrm{x}(2)|)/3$. Thus, with $\psi_l^{(i)}$ as in (23), from the definition of $A^{\mathrm{Good}}(r)$ and the equation (24), we have

$$\begin{aligned}
&\Big|\mathbb{E}\Big[\big(||Z_1(\mathbf{u}^0, \mathbf{v}^0)||_2^2 - ||\mathrm{x}||_2^2\big)^m \mid Z_0(\mathbf{u}^0, \mathbf{v}^0) = \mathrm{x}\Big] \\
&\quad - \mathbb{E}\Big[\big(||(\overline{\mathbf{u}^0} + \psi_1^{(1)}) - (\overline{\mathbf{v}^0} + \psi_1^{(2)})||_2^2 - ||\mathrm{x}||_2^2\big)^m\Big]\Big| \\
&= \Big|\mathbb{E}\Big[\big(||Z_1(\mathbf{u}^0, \mathbf{v}^0)||_2^2 - ||\mathrm{x}||_2^2\big)^m \mathbf{1}\big((A^{\mathrm{Good}}(r))^c\big) \mid Z_0(\mathbf{u}^0, \mathbf{v}^0) = \mathrm{x}\Big] \\
&\quad - \mathbb{E}\Big[\big(||(\overline{\mathbf{u}^0} + \psi_1^{(1)}) - (\overline{\mathbf{v}^0} + \psi_1^{(2)})||_2^2 - ||\mathrm{x}||_2^2\big)^m \mathbf{1}\big((A^{\mathrm{Good}}(r))^c\big)\Big]\Big| \\
&\leq \mathbb{E}\big[2^m\big(||Z_1(\mathbf{u}^0, \mathbf{v}^0)||_2^{2m} + ||\mathrm{x}||_2^{2m}\big)\mathbf{1}\big((A^{\mathrm{Good}}(r))^c\big)\big] \\
&\qquad + \mathbb{E}\big[2^m\big(||(\psi_1^{(1)} - \psi_1^{(2)}) + \mathrm{x}||_2^{2m} + ||\mathrm{x}||_2^{2m}\big)\mathbf{1}\big((A^{\mathrm{Good}}(r))^c\big)\big] \\
&\leq 2^m\mathbb{E}\Big[\Big(||\mathrm{x}||_2^{2m} + 2^{2m}\big[||Z_1(\mathbf{u}^0, \mathbf{v}^0) - \mathrm{x}||_2^{2m} + ||\mathrm{x}||_2^{2m}\big]\Big)\mathbf{1}\big((A^{\mathrm{Good}}(r))^c\big)\Big] \\
&\quad + 2^m\mathbb{E}\Big[\Big(||\mathrm{x}||_2^{2m} + 2^{2m}\big[||\psi_1^{(1)} - \psi_1^{(2)}||_2^{2m} + ||\mathrm{x}||_2^{2m}\big]\Big)\mathbf{1}\big((A^{\mathrm{Good}}(r))^c\big)\Big] \\
&\leq 2^{4m}\Big[||\mathrm{x}||_2^{2m}\mathbb{P}\big((A^{\mathrm{Good}}(r))^c\big) + \mathbb{E}\big[\big(W_1(\mathbf{u}^0, \mathbf{v}^0)\big)^{2m}\mathbf{1}(W_1(\mathbf{u}^0, \mathbf{v}^0) > d_{\min}/3)\big] \\
&\qquad + \mathbb{E}\big[\big(W_1^{(\mathrm{Ind})}(\mathbf{u}^0, \mathbf{v}^0)\big)^{2m}\mathbf{1}(W_1^{(\mathrm{Ind})}(\mathbf{u}^0, \mathbf{v}^0) > d_{\min}/3)\big]\Big] \\
&\leq C_{17}\exp(-C_{18}||\mathrm{x}||_2)
\end{aligned}$$

for a proper choice of $C_{17}, C_{18} > 0$.

Now, the estimates (3), (4) and (5) of [GRS04] follow from direct computations of the moments of the marginals $\psi_1^{(1)}$ and $\psi_1^{(2)}$. For example, when $m = 1$, with $\psi_1^{(i)}(j)$ denoting the the j th co-ordinate of $\psi_1^{(i)}$ for $i = 1, 2$ and $j = 1, 2$, using the observations made about the marginals in Section 3, we have $\mathbb{E}\Big[||(\overline{\mathbf{u}^0} + \psi_1^{(1)}) - (\overline{\mathbf{v}^0} + \psi_1^{(2)})||_2^2 - ||\mathrm{x}||_2^2\Big] = \mathbb{E}\Big[\big(\mathrm{x}(1) + \psi_1^{(1)}(1) - \psi_1^{(2)}(1)\big)^2 + \big(\mathrm{x}(2) + \psi_1^{(1)}(2) - \psi_1^{(2)}(2)\big)^2 - \big(\mathrm{x}(1)\big)^2 - \big(\mathrm{x}(2)\big)^2\Big] = 4\mathbb{E}\big[\big(\psi_1^{(1)}(1)\big)^2\big]$ which yields (3). Similar calculations yield (4) and (5). This completes the proof for $d = 3$. □

4.3 $d \geq 4$

We present the proof for $d = 4$; the argument being similar for $d > 4$. We first show that $\mathbb{P}(G \text{ is disconnected}) > 0$, which by ergodicity of the model implies that $\mathbb{P}(G \text{ is disconnected}) = 1$.

We start with two open vertices $\mathbf{u}^0$ and $\mathbf{v}^0$ in $\mathbb{Z}^4$ with $\mathbf{u}^0(4) = \mathbf{v}^0(4) = 0$ and show that for the Markov chain $\{Z_l(\mathbf{u}^0, \mathbf{v}^0); l \geq 0\}$ there is a positive probability that it does not get absorbed at $(0, 0, 0)$. We follow the same technique as in [GRS04] to achieve the above, *viz.*, we run the chain for n^4 time units, starting from $\mathbf{u}^0$ and $\mathbf{v}^0$ sufficiently far apart (of the order n). Then, with a very high probability the Markov chain has travelled further away (of the order of n^2), and using the Markov property, we start from these new vertices and continue this process.

More precisely, for $\epsilon > 0$, define the event

$$A_{n,\epsilon}(\mathbf{u}^0, \mathbf{v}^0) := \Big\{ Z_{n^4}(\mathbf{u}^0, \mathbf{v}^0) \in D_{n^{2(1+\epsilon)}} \setminus D_{n^{2(1-\epsilon)}} \Big\},$$

where $D_r := \{\mathbf{x} \in \mathbb{Z}^3 : ||\mathbf{x}||_1 \leq r\}$. We show

Proposition 4.1 *For $0 < \epsilon < \frac{1}{3}$, there exist constants $C_{19}, \beta > 0$ and $n_0 \geq 1$ such that, for all $n \geq n_0$,*

$$\inf_{\overline{\mathbf{v}^0} \in \overline{\mathbf{u}^0} + D_{n^{1+\epsilon}} \setminus D_{n^{1-\epsilon}}} \mathbb{P}\big(A_{n,\epsilon}(\mathbf{u}^0, \mathbf{v}^0)\big) \geq 1 - C_{19} n^{-\beta}.$$

Following the steps of [GRS04], it is enough to prove Proposition 4.1. Again we use the coupling described earlier for $k = 2$. Consider the event

$$\begin{aligned} A^{(\mathrm{Ind})}_{n,\epsilon}(\mathbf{u}^0, \mathbf{v}^0) := \Big\{ & \overline{\mathbf{v}^0} + \sum_{l=1}^{n^4} \psi^{(2)}_l \in \overline{\mathbf{u}^0} + \sum_{l=1}^{n^4} \psi^{(1)}_l + D_{n^{2(1+\epsilon)}} \setminus D_{n^{2(1-\epsilon)}}, \\ & \overline{\mathbf{v}^0} + \sum_{l=1}^{j} \psi^{(2)}_l \notin \overline{\mathbf{u}^0} + \sum_{l=1}^{j} \psi^{(1)}_l + D_{K \log n} \text{ for all } j = 1, \ldots, n^4 \Big\}, \end{aligned}$$

where K is a suitably chosen large constant. This event corresponds to the event (19) defined in [GRS04] for which it was shown that there exists n_0 such that

$$\inf_{\overline{\mathbf{v}^0} \in \overline{\mathbf{u}^0} + D_{n^{1+\epsilon}} \setminus D_{n^{1-\epsilon}}} \mathbb{P}\big(A^{(\mathrm{Ind})}_{n,\epsilon}(\mathbf{u}^0, \mathbf{v}^0)\big) \geq 1 - C_{20} n^{-\alpha},$$

for some constant $C_{20}, \alpha > 0$ and for all $n \geq n_0$.

Now, we employ the coupling described earlier in Subsection 3.3 with $k = 2$. This time we will continue this coupling step by step for n^4 simultaneous regeneration steps of independent paths. At each step we choose $r = K\log n/3$ and say that the coupling is successful at step j if the event $A^{\text{Good}}(r)$ occurs. We do the coupling at step $j+1$ if the coupling is successful at step j. Note, if the coupling is successful at every step $j = 1,\dots,n^4$, we have, for $j = 1,2,\dots,n^4$,

$$\overline{\mathbf{u}^0} + \sum_{l=1}^{j} \psi_l^{(1)} = \overline{g}_{\tau_j(\mathbf{u}^0,\mathbf{v}^0)}(\mathbf{u}^0) \text{ and } \overline{\mathbf{v}^0} + \sum_{l=1}^{j} \psi_l^{(2)} = \overline{g}_{\tau_j(\mathbf{u}^0,\mathbf{v}^0)}(\mathbf{v}^0).$$

Therefore, we get

$$\mathbb{P}\big(A_{n,\epsilon}(\mathbf{u}^0,\mathbf{v}^0)\big) \geq \mathbb{P}\big(A^{(\text{Ind})}_{n,\epsilon}(\mathbf{u}^0,\mathbf{v}^0) \cap \{\text{Coupling is successful for } j = 1,2,\dots,n^4\}\big).$$

Using the Markov property and the estimate of the coupling being successful, given in (24), we obtain, for all sufficiently large n,

$$\mathbb{P}\big(A_{n,\epsilon}(\mathbf{u}^0,\mathbf{v}^0)\big) \geq 1 - C_{20}n^{-\alpha} - C^{(2)}_{15} n^4 \exp(-C^{(2)}_{16} K \log n/3) \geq 1 - C_{19} n^{-\beta}$$

for suitable choice of $\beta > 0$ and C_{19}. This proves the proposition and completes the proof for this case. $\square$

5 Brownian Web

In this section we prove Theorem 1.2. We begin by recalling that the Brownian web takes values in the metric space $\mathcal{H}$ equipped with the Hausdorff metric $d_{\mathcal{H}}$ where $\mathcal{H}$ is the space of compact subsets of the path space (Π, d_Π) (see the discussion in the paragraphs after the statement of Theorem 1.1 in Section 1). As introduced earlier, for any $n \geq 1$, the collection of scaled paths $\mathcal{X}_n(\gamma,\sigma)$ is obtained from G with normalization constants γ,σ and we had remarked that the closure of $\mathcal{X}_n(\gamma,\sigma)$ in (Π, d_Π) denoted by $\overline{\mathcal{X}}_n(\gamma,\sigma)$ is a $(\mathcal{H}, \mathcal{B}_{\mathcal{H}})$ valued random variable.

We require some more notations. For a compact set of paths $K \in \mathcal{H}$ and for $t \in \mathbb{R}$ let $K^t := \{\pi \in K : \sigma_\pi \leq t\}$ be the set of paths which start 'below' t. For $t > 0$ and $t_0, a, b \in \mathbb{R}$ with $a < b$, we define two counting random variables as follows

$$\begin{aligned}
\eta_K(t_0,t;a,b) &:= \#\{\pi(t_0+t) : \pi \in K^{t_0} \text{ and } \pi(t_0) \in [a,b]\} \text{ and}\\
\hat{\eta}_K(t_0,t;a,b) &:= \#\{\pi(t_0+t) : \pi \in K^{t_0} \text{ and } \pi(t_0+t) \in [a,b]\}.
\end{aligned}$$

Theorem 2.2 in Fontes *et al.* [FINR04] provided a criteria for a sequence of $(\mathcal{H}, \mathcal{B}_{\mathcal{H}})$ valued random variables with non-crossing paths to converge weakly to the Brownian web. In the following we denote, the standard independent Brownian motions starting from $\mathbf{x}^1,\dots,\mathbf{x}^k$ respectively, by $(B^{\mathbf{x}^1},\dots,B^{\mathbf{x}^k})$ and standard coalescing Brownian motions starting from $\mathbf{x}^1,\dots,\mathbf{x}^k$ respectively, by $(W^{\mathbf{x}^1},\dots,W^{\mathbf{x}^k})$.

Theorem 5.1 *[FINR04] Suppose $\chi_1, \chi_2, \ldots$ are $(\mathcal{H}, B_{\mathcal{H}})$ valued random variables with non-crossing paths. Assume that the following conditions hold.*

(I_1) *Let $\mathcal{D}$ be a deterministic countable dense subset of $\mathbb{R}^2$. Suppose that, for any $k \geq 1$, and for any $\mathbf{x}^1, \ldots, \mathbf{x}^k \in \mathcal{D}$, there exists $\pi_n^1, \ldots, \pi_n^k \in \chi_n$ such that, as $n \to \infty$,*

$$(\pi_n^1, \ldots, \pi_n^k) \Rightarrow (W^{\mathbf{x}^1}, \ldots, W^{\mathbf{x}^k}).$$

(B_1) *For all $t > 0, \limsup_{n\to\infty} \sup_{(a,t_0)\in\mathbb{R}^2} \mathbb{P}(\eta_{\chi_n}(t_0, t; a, a+\epsilon) \geq 2) \to 0$ as $\epsilon \to 0^+$*

(B_2) *For all $t > 0, \frac{1}{\epsilon} \limsup_{n\to\infty} \sup_{(a,t_0)\in\mathbb{R}^2} \mathbb{P}(\eta_{\chi_n}(t_0, t; a, a+\epsilon) \geq 3) \to 0$ as $\epsilon \to 0^+$.*

Then χ_n converges in distribution to the standard Brownian web $\mathcal{W}$.

The convergence in (I_1) occurs in the space Π^k. Note that the convergence in Π implies that the starting points converge as points in $\mathbb{R}^2$ and the paths converge uniformly on the compact sets of time.

In Theorem 1.4 and Lemma 6.1 of Newman *et al.* [NRS05], it was further proved that the condition (B_2) can be replaced by (E_1') where

(E_1') if $\mathcal{Z}^{t_0}$ is any subsequential limit of $\chi_n^{t_0}$ for any $t_0 \in \mathbb{R}$, then for all $t, a, b \in \mathbb{R}$ with $t > 0$ and $a < b$, $\mathbb{E}[\hat{\eta}_{\mathcal{Z}^{t_0}}(t_0, t; a, b)] \leq \mathbb{E}[\hat{\eta}_{\mathcal{W}}(t_0, t; a, b)] = \frac{b-a}{\sqrt{t\pi}}$.

It is worthwhile mentioning here that for a sequence of $(\mathcal{H}, \mathcal{B}_{\mathcal{H}})$ valued random variables χ_n with non-crossing paths, property (I_1) implies tightness (see Proposition B.2 in the Appendix of [FINR04]) and hence such a subsequential limit $\mathcal{Z}^{t_0}$ exists. Our model $\overline{\mathcal{X}}_1$ consists of non-crossing paths only. Thus, to prove Theorem 1.2 we need to show that for some $\gamma(p) > 0$ and $\sigma(p) > 0$ the sequence $\overline{\mathcal{X}}_n(\gamma, \sigma)$ satisfies the conditions (I_1), (B_1) and (E_1') and hence converges to the standard Brownian web.

5.1 Proof of condition (I_1)

We follow the argument of Ferrari *et al.* [FFW05]. It should be noted here that dependency structure of our model is quite different from that in [FFW05] and we require significant modifications of their argument. First, to show (I_1) we need to control the tail of the distribution of the coalescing time of two paths starting at the same instant of time but at a unit distance apart.

Lemma 5.1 *Fix $\mathbf{u} = (1, 0)$, $\mathbf{v} = (0, 0) \in \mathbb{Z}^2$, let $\nu = \inf\{l : g_{\tau_l(\mathbf{u},\mathbf{v})}(\mathbf{u}) = g_{\tau_l(\mathbf{u},\mathbf{v})}(\mathbf{v})\}$, where $\tau_l(\mathbf{u}, \mathbf{v})$ is the l th regeneration step as defined in (1). For the ν*

th regeneration time $T_\nu(\mathbf{u},\mathbf{v})$ as defined in (10), there exist positive constants C_{21} and C_{22}, such that, we have

$$\mathbb{P}(\nu > t) \leq \frac{C_{21}}{\sqrt{t}} \quad \text{and} \quad \mathbb{P}(T_\nu(\mathbf{u},\mathbf{v}) > t) \leq \frac{C_{22}}{\sqrt{t}}.$$

Corollary 5.1 *For $\mathbf{u} = (a,0)$, $\mathbf{v} = (0,0) \in \mathbb{Z}^2$, let $\nu(a) = \inf\{l : g_{T_l(\mathbf{u},\mathbf{v})}(\mathbf{u}) = g_{T_l(\mathbf{u},\mathbf{v})}(\mathbf{v})\}$, where $a > 0$ is any positive integer. Then*

$$\mathbb{P}(\nu(a) > t) \leq \frac{C_{21}a}{\sqrt{t}} \quad \text{and} \quad \mathbb{P}(T_{\nu(a)}(\mathbf{u},\mathbf{v}) > t) \leq \frac{C_{22}a}{\sqrt{t}}.$$

The corollary follows from Lemma 5.1, using the fact that the the coalescence time $\nu(a)$ is given by the maximum of the coalescence times of the martingales starting from the pairs $(i,0)$ and $(i-1,0)$ for $i = 1,\ldots,a$.

Proof of Lemma 5.1: Consider the nonnegative martingale, $\{Z_j(\mathbf{u},\mathbf{v}), \mathcal{F}_{T_j(\mathbf{u},\mathbf{v})}\}$ defined in (16). We use the method in Theorem 4 of Coletti *et al.* [CFD09] to achieve the bound on $T_\nu(\mathbf{u},\mathbf{v})$. In order to use the result of [CFD09], it is enough to prove that

$$\sup\{\mathbb{P}(Z_{j+1}(\mathbf{u},\mathbf{v}) = m | Z_j(\mathbf{u},\mathbf{v}) = m) : m \geq 1\} \leq \theta \tag{26}$$

for some $\theta \in (0,1)$.

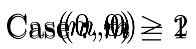

Figure 6: One possible realization of the event $\{Z_{j+1} = m+1 \mid Z_j = m\}$. The bold vertices are open and all other vertices depicted are closed.

To show (26), we observe that for $m \geq 2$, $\mathbb{P}(Z_{j+1}(\mathbf{u},\mathbf{v}) = m+1 | Z_j(\mathbf{u},\mathbf{v}) = m) \geq (1-p)^6 p^3$ and $\mathbb{P}(Z_{j+1}(\mathbf{u},\mathbf{v}) = 2 | Z_j(\mathbf{u},\mathbf{v}) = 1) \geq (1-p)^4 p^3$ (see Figure 6). Therefore, we have

$$\begin{aligned}
&\mathbb{P}(Z_{j+1}(\mathbf{u},\mathbf{v}) = m | Z_j(\mathbf{u},\mathbf{v}) = m) \\
&\leq 1 - \mathbb{P}(Z_1(\mathbf{u},(m,0)) = m+1 | Z_0(\mathbf{u},(m,0)) = m) \\
&\leq 1 - \min\{(1-p)^6 p^3, (1-p)^4 p^3\} = 1 - (1-p)^6 p^3.
\end{aligned}$$

This establishes

$$\mathbb{P}(\nu > t) \leq \frac{C_{21}}{\sqrt{t}}. \tag{27}$$

To complete the result, we choose $C_{23} = \mathbb{E}(W^M)/2$ where W^M is as in Proposition 2.3. Note that, it is also the case that $T_l(\mathbf{u},\mathbf{v}) \leq \sum_{i=1}^{l} W^M(i)$, for any $l \geq 1$, where $\{W^M(i) : i \geq 1\}$ is an i.i.d. sequence, each having the same distribution as that of W^M (see discussion before the statement of Proposition 2.3). Using (27),

$$\begin{aligned}
&\mathbb{P}(T_\nu(\mathbf{u},\mathbf{v}) > t) \leq \mathbb{P}(T_\nu(\mathbf{u},\mathbf{v}) > t, \nu \leq C_{23}t) + \mathbb{P}(\nu > C_{23}t) \\
&\leq \mathbb{P}(T_{\lfloor C_{23}t \rfloor}(\mathbf{u},\mathbf{v}) > t) + \frac{C_{21}}{\sqrt{C_{23}t}} \\
&\leq \mathbb{P}\Big(\sum_{i=1}^{\lfloor C_{23}t \rfloor} W^M(i) - \mathbb{E}(W^M(i)) > t - \mathbb{E}(W^M)\lfloor C_{23}t \rfloor\Big) + \frac{C_{21}}{\sqrt{C_{23}t}} \\
&\leq \frac{\mathbb{V}\mathrm{ar}\big(\sum_{i=1}^{\lfloor C_{23}t \rfloor} W^M(i)\big)}{(t - \mathbb{E}(W^M)\lfloor C_{23}t \rfloor)^2} + \frac{C_{21}}{\sqrt{C_{23}t}} \leq \frac{C_{22}}{\sqrt{t}}
\end{aligned}$$

for a suitable choice of constant C_{22}. This completes the proof. $\square$

Let $\mathbf{x}^1, \ldots, \mathbf{x}^k \in \mathbb{R}^2$ be k fixed points, ordered in such a way that either $\mathbf{x}^i(2) < \mathbf{x}^{i+1}(2)$ or $\mathbf{x}^i(2) = \mathbf{x}^{i+1}(2), \mathbf{x}^i(1) < \mathbf{x}^{i+1}(1)$ for all $i = 1, \ldots, k-1$. Let $\mathbf{x}_n^i = \mathbf{x}_n^i(\gamma,\sigma)$ be such that for any $n \geq 1$, $\mathbf{x}_n^i \in \mathbb{Z}^2$ and $\mathbf{x}_n^i(1)/n\sigma \to \mathbf{x}^i(1)$ and $\mathbf{x}_n^i(2) = \lfloor n^2\gamma \mathbf{x}^i(2) \rfloor$ so that $(\mathbf{x}_n^i(1)/n\sigma, \mathbf{x}_n^i(2)/n^2\gamma) \to \mathbf{x}^i$ as $n \to \infty$, for $i = 1, \ldots, k$. Note that we have kept the choice of $\mathbf{x}_n^i(1)$ in our hand and we can suitably adapt our choice later. There are several choices which will work. For example, we can choose $\mathbf{x}_n^i(1)$ so that it corresponds to first open point to the left of $(\lfloor n\sigma\mathbf{x}^i(1)\rfloor, \mathbf{x}_n^i(2))$, i.e., $\mathbf{x}_n^i(1) = \max\{\lfloor n\sigma\mathbf{x}^i(1)\rfloor + j : j \leq 0, (\lfloor n\sigma\mathbf{x}^i(1)\rfloor + j, \mathbf{x}_n^i(2)) \in V\}$. Another choice that we may consider is any open point on the interval, $\big[\lfloor n\sigma\mathbf{x}^i(1)\rfloor - \lfloor n^\alpha \rfloor, \lfloor n\sigma\mathbf{x}^i(1)\rfloor + \lfloor n^\alpha \rfloor\big] \times \{\mathbf{x}_n^i(2)\}$ for some $0 < \alpha < 1$. We now consider the joint process starting from $\mathbf{x}_n^1, \ldots, \mathbf{x}_n^k$ and define, for $i = 1, \ldots, k$, the n th order diffusively scaled version of these paths as

$$\pi_n^i(t) = \pi_n^i(\gamma,\sigma)(t) := \frac{1}{n\sigma}\pi^{\mathbf{x}_n^i}(\gamma n^2 t) \quad \text{for } t \geq \mathbf{x}_n^i(2)/n^2\gamma. \tag{28}$$

To obtain (I_1), it is sufficient to show

Proposition 5.1 *There exist $\gamma := \gamma(p)$ and $\sigma := \sigma(p)$ such that, as $n \to \infty$,*

$$\{\pi_n^1(\gamma,\sigma), \ldots, \pi_n^k(\gamma,\sigma)\} \text{ converges weakly to } \{W^{\mathbf{x}^1}, \ldots, W^{\mathbf{x}^k}\}.$$

We prove the Proposition 5.1 through a series of other propositions. The strategy we adopt to prove this proposition is, to show that until the time when two paths come '*close to each other*', they can be approximated by independent paths, and after that time they coalesce very quickly, which is negligible in the scaling.

Towards this end, we start with k independent collections of i.i.d. uniform random variables and we construct the path $\pi_n^{i,(\mathrm{Ind})}$, starting from $\mathbf{x}_n^i$ using the i th collection only for each $i = 1, \ldots, k$. Therefore, we have k independent processes, $(\pi_n^{1,(\mathrm{Ind})}, \ldots, \pi_n^{k,(\mathrm{Ind})})$ where the i th path, $\pi_n^{i,(\mathrm{Ind})}$, is an independent copy of the π_n^i, for $i = 1, \ldots, k$ with π_n^i as defined in (28).

We first show that

Proposition 5.2 *There exist $\gamma := \gamma(p)$ and $\sigma := \sigma(p)$ such that, as $n \to \infty$,*

$$(\pi_n^{1,(\mathrm{Ind})}, \ldots, \pi_n^{k,(\mathrm{Ind})}) \Rightarrow (B^{\mathbf{x}^1}, \ldots, B^{\mathbf{x}^k}).$$

Proof: Since the paths $\pi_n^{i,(\mathrm{Ind})}$ are pairwise independent, it is enough to show that the marginals converge. So, it suffices to prove that $\pi_n^{1,(\mathrm{Ind})}$ converges weakly to the standard Brownian motion $B^{\mathbf{x}^1}$, starting at $\mathbf{x}^1$. Furthermore, by translation invariance of our model, we may write

$$\{g_m(\mathbf{x}_n^1) : m \geq 0\} \stackrel{d}{=} \mathbf{x}_n^1 + \{g_m(\mathbf{0}) : m \geq 0\}$$

where $g_m(\mathbf{u})$ is the position of the path after the m th step, starting from $\mathbf{u}$. Since, $\mathbf{x}_n^1 \to \mathbf{x}^1$, it is enough to show that the path starting from $\mathbf{0}$, scaled diffusively, will converge to the standard Brownian motion, starting from $\mathbf{0}$. In other words, it is enough to consider $\mathbf{x}^1 = 0$.

Let τ_j and T_j denote the j th regeneration step and time respectively for the process starting from $\mathbf{0}$ (see (1) and (10)). Let $Y_j = Y_j^{(\mathbf{0})} = \overline{g}_{\tau_j}(\mathbf{0}) - \overline{g}_{\tau_{j-1}}(\mathbf{0})$ (see (13)). Now, we define a piecewise linear path $\tilde{\pi}^{1,(\mathrm{Ind})}$ and its diffusively scaled version $\tilde{\pi}_n^{1,(\mathrm{Ind})}$ as

$$\begin{aligned}
\tilde{\pi}^{1,(\mathrm{Ind})} :=& g_{\tau_n}(\mathbf{0}) + \frac{t - T_n}{T_{n+1} - T_n}(g_{\tau_{n+1}}(\mathbf{0}) - g_{\tau_n}(\mathbf{0})) \quad \text{for } T_n \leq t < T_{n+1} \\
\tilde{\pi}_n^{1,(\mathrm{Ind})}(t) =& \tilde{\pi}_n^{1,(\mathrm{Ind})}(\gamma, \sigma)(t) := \frac{1}{n\sigma} \tilde{\pi}^{1,(\mathrm{Ind})}(\gamma n^2 t) \quad \text{for } t \geq 0.
\end{aligned}$$

Next we define another stochastic process, S on $[0, \infty)$, as follows:

$$S(t) = T_j + (t - j)(T_{j+1} - T_j) \text{ for } j \leq t < j + 1, j \geq 0.$$

Clearly, $S(t)$ is a strictly increasing process. Hence, $t \to S(t)$ admits an inverse $S^{-1}(t)$ which is also strictly increasing. The process $S(t)$ denotes the time change required to track the path $\tilde{\pi}^{1,(\mathrm{Ind})}$. More precisely, we have, for $t \geq 0$,

$$\tilde{\pi}_n^1(t) = X_n\big(S^{-1}(n^2\gamma t)/n^2\big).$$

where the process $X_n = X_n(\gamma, \sigma)$ on $[0, \infty)$ is defined as follows: $X_n(0) = 0$ and for $t > 0$,

$$X_n(t) := \frac{1}{n\sigma}\Big[(n^2t - \lfloor n^2t \rfloor)Y_{\lfloor n^2t \rfloor + 1} + \sum_{i=1}^{\lfloor n^2t \rfloor} Y_i\Big].$$

From Remark in 2.2, Y_i's are symmetric and i.i.d., so that $\mathbb{E}(Y_1) = 0$. Thus, from Donsker's invariance principle, it follows that, for $\sigma = \sigma_0 := \sqrt{\mathrm{Var}(Y_1)}$, the process X_n converges weakly to the standard Brownian motion starting from 0.

Let $N(t)$ be the number of the renewals for the process $S(t)$ up to time t, i.e., $N(t) = \lfloor S^{-1}(t) \rfloor$ so that, $N(t) \leq S^{-1}(t) \leq N(t)+1$. Hence by the renewal theorem $S^{-1}(n^2\gamma t)/n^2 \to g(t) := \frac{\gamma t}{\mathbb{E}(T_1)}$, $t \geq 0$ almost surely (see Theorem 4.4.1 of Durrett [D10]). Taking $\gamma = \gamma_0 := \mathbb{E}[T_1]$, we conclude that

$$\tilde{\pi}_n^{1,(\mathrm{Ind})} \Rightarrow B^{\mathbf{x}^1}.$$

Finally to conclude the result, it is enough to show that, for any $s > 0$ and $\epsilon > 0$

$$\begin{aligned} &\mathbb{P}\big[\sup\{|\tilde{\pi}_n^{1,(\mathrm{Ind})}(t) - \pi_n^{1,(\mathrm{Ind})}(t)| : t \in [0, s]\} > \epsilon\big] \\ &= \mathbb{P}\big[\sup\{|\tilde{\pi}^{1,(\mathrm{Ind})}(t) - \pi^{1,(\mathrm{Ind})}(t)| : t \in [0, sn^2]\} > \epsilon n\big] \to 0 \end{aligned}$$

as $n \to \infty$. Since $N(sn^2) \leq \lfloor sn^2 \rfloor$, we have

$$\begin{aligned} &\Big\{\sup\{|\tilde{\pi}^{1,(\mathrm{Ind})}(t) - \pi^{1,(\mathrm{Ind})}(t)| : t \in [0, sn^2]\} > \epsilon n\Big\} \\ &\subseteq \bigcup_{j=1}^{\lfloor sn^2 \rfloor} \Big\{\sup\{|\tilde{\pi}^{1,(\mathrm{Ind})}(t) - \pi^{1,(\mathrm{Ind})}(t)| : t \in [T_j, T_{j+1}]\} > \epsilon n\Big\}. \end{aligned}$$

Now, on the interval $[T_j, T_{j+1}]$, both the paths $\pi^{1,(\mathrm{Ind})}$ and $\tilde{\pi}^{1,(\mathrm{Ind})}$ agree at the end points. Furthermore, both the paths are piecewise linear. From the definition of $W_j(\mathbf{0})$ (see equation (9)), for any $t \in [T_j, T_{j+1}]$, $|\pi^{1,(\mathrm{Ind})}(t) - \pi^{1,(\mathrm{Ind})}(T_j)| \leq W_j$ and $|\tilde{\pi}^{1,(\mathrm{Ind})}(t) - \tilde{\pi}^{1,(\mathrm{Ind})}(T_j)| \leq W_j$. Thus, we have

$$\Big\{\sup\{|\tilde{\pi}^{1,(\mathrm{Ind})}(t) - \pi^{1,(\mathrm{Ind})}(t)| : t \in [T_j, T_{j+1}]\} > \epsilon n\Big\} \subseteq \{2W_j(\mathbf{0}) > \epsilon n\},$$

and

$$\begin{aligned} &\mathbb{P}\big[\sup\{|\tilde{\pi}^{1,(\mathrm{Ind})}(t) - \pi^{1,(\mathrm{Ind})}(t)| : t \in [0, sn^2]\} > \epsilon n\big] \\ &\leq \mathbb{P}\big[2\max\{W_j(\mathbf{0}) : j = 1, \ldots, \lfloor sn^2 \rfloor\} > \epsilon n\big] \leq \lfloor sn^2 \rfloor \mathbb{P}(2W_1(\mathbf{0}) > \epsilon n) \to 0 \end{aligned}$$

as $n \to \infty$. This proves Proposition 5.2. $\square$

Henceforth, we assume that we are working with $\gamma = \gamma_0$ and $\sigma = \sigma_0$ and for the ease of writing we drop (γ, σ) from our notation unless required.

First we consider the case where second co-ordinate of all $\mathbf{x}^i$'s equal 0, i.e., $\mathbf{x}^1(2) = \cdots = \mathbf{x}^k(2) = 0$. Hence, we must have $\mathbf{x}_n^1(2) = \cdots = \mathbf{x}_n^k(2) = 0$. Note that 0 is not important, it is just easy for notation.

We think of the paths as coming from a metric space $(C[0,\infty), \tilde{d})$ with $\tilde{d}$ given by

$$\tilde{d}(\pi_1, \pi_2) := \sum_{j=0}^{\infty} 2^{-j} \min\big\{1, \sup\{|\pi_1(t) - \pi_2(t)| : t \in [j, j+1]\}\big\}$$

for $\pi_1, \pi_2 \in C[0,\infty)$. Consider the product metric space $(C[0,\infty)^k, \tilde{d}^k)$, where $\tilde{d}^k$ is a metric on $C[0,\infty)^k$ such that the topology generated by it coincides with the corresponding product topology. In particular, we consider the metric

$$\tilde{d}^k\Big((\pi_{1,1}, \ldots, \pi_{k,1}), (\pi_{1,1}, \ldots, \pi_{k,2})\Big) := \max\big\{\tilde{d}(\pi_{i,1}, \pi_{i,2}) : i = 1, \ldots, k\big\}$$

for $\pi_{i,1}, \pi_{i,2} \in C[0,\infty),\ i = 1, \ldots, k$.

First, we prove a sandwich lemma, which we require later for proving our result.

Lemma 5.2 (Sandwich Lemma) *Let $\{\pi_n : n \geq 1\}$ be any family of diffusively scaled paths such that $\sigma_{\pi_n} \leq 0$ for each $n \geq 1$. Further, let $\mathbf{u}_n = (u_n, 0)$ and $\mathbf{v}_n = (v_n, 0)$ be two open points with $u_n < 0 < v_n$ and $(v_n - u_n)/n \to 0$. Further, we assume that $u_n \leq \pi_n(0) n\sigma_0 \leq v_n$, then*

$$(\pi_n^{\mathbf{u}_n}, \pi_n|_{[0,\infty)}, \pi_n^{\mathbf{v}_n}) \Rightarrow (B^{\mathbf{0}}, B^{\mathbf{0}}, B^{\mathbf{0}}) \tag{29}$$

where $\pi_n^{\mathbf{u}_n}$ and $\pi_n^{\mathbf{v}_n}$ are the diffusively scaled paths starting from $\mathbf{u}_n$ and $\mathbf{v}_n$ respectively, $\pi_n|_{[0,\infty)}$ is the restriction of the path π_n on $[0,\infty)$, and $B^{\mathbf{0}}$ is the standard Brownian motion starting at $\mathbf{0}$.

Proof of Sandwich Lemma: In view of the Proposition 5.2, we have, for any $\mathbf{x} \in \mathbb{R}^2$, $\pi^{\mathbf{x}_n}(\gamma_0 n^2 t)/n\sigma_0 \Rightarrow B^{\mathbf{x}}$ such that $\mathbf{x}_n(1)/\sigma_0 n \to \mathbf{x}(1)$ and $\mathbf{x}_n(2) = \lfloor \mathbf{x}(2)\gamma_0 n^2 \rfloor$. Taking $\mathbf{x} = \mathbf{0}$, we observe that $\pi_n^{\mathbf{u}_n} \Rightarrow B^{\mathbf{0}}$.

Since the paths in our model are non-crossing and $u_n \leq \pi_n(0)n\sigma_0 \leq v_n$, we also have $\pi_n^{\mathbf{u}_n}(t) \leq \pi_n|_{[0,\infty)}(t) \leq \pi_n^{\mathbf{v}_n}(t)$ for all $t \geq 0$. Therefore, to prove our result, it is enough to show that $\pi_n^{\mathbf{v}_n}$ converges to the same Brownian motion $B^{\mathbf{0}}$. Thus, to complete the proof, we need to show that for any $\epsilon > 0$, $\mathbb{P}\big[\tilde{d}\big(\pi_n^{\mathbf{v}_n}, \pi_n^{\mathbf{u}_n}\big) \geq \epsilon\big] \to 0$ for which it is sufficient to show

$$\mathbb{P}\Big[\sup_{t \geq 0}\big|\pi_n^{\mathbf{v}_n}(t) - \pi_n^{\mathbf{u}_n}(t)\big| \geq \epsilon\Big] = \mathbb{P}\Big[\sup_{t \geq 0}\big|\pi^{\mathbf{v}_n}(t) - \pi^{\mathbf{u}_n}(t)\big| \geq \epsilon\sigma_0 n\Big] \to 0.$$

Recall that the difference between the two paths starting at $\mathbf{v}_n$ and $\mathbf{u}_n$ observed at the simultaneous regeneration steps of the joint path forms a nonnegative martingale, denoted by $\{Z_l = (Z_l(\mathbf{v}_n, \mathbf{u}_n)) : l \geq 0\}$, as described in (16). Fix any

$0 < \alpha < 1$ and let $\nu_n := \inf\{l \geq 0 : Z_l = 0\}$. On the intersection of the events $\{\nu_n < n^2\}$ and $\{W_j = W_j(\mathbf{v}_n, \mathbf{u}_n) < n^\alpha$ for all $j = 1, \ldots, n^2\}$, we have that $\Big\{\sup_{t\geq 0}|\pi^{\mathbf{v}_n}(t) - \pi^{\mathbf{u}_n}(t)| \geq \epsilon\sigma_0 n\Big\} \subseteq \big\{\max\{Z_l : l = 1, \ldots, n^2\} \geq \epsilon\sigma_0 n - 2n^\alpha\big\} \subseteq \big\{\max\{Z_l : l = 1, \ldots, n^2\} \geq \epsilon\sigma_0 n/2\big\}$ for all large n. Thus, we have,

$$\begin{aligned}
&\mathbb{P}\Big[\sup_{t\geq 0}\big|\pi^{\mathbf{v}_n}(t) - \pi^{\mathbf{u}_n}(t)\big| \geq \epsilon\sigma_0 n\Big] \\
&\leq \mathbb{P}\Big[\max\{Z_l : l = 1, \ldots, n^2\} \geq \epsilon\sigma_0 n/2\Big] + \mathbb{P}(\nu_n \geq n^2) + \mathbb{P}\big(\cup_{j=1}^{n^2}\{W_j > n^\alpha\}\big).
\end{aligned}$$

From Corollary 5.1, we have $\mathbb{P}(\nu_n \geq n^2) \leq C_{21}(v_n - u_n)/n \to 0$ as $n \to \infty$. The third term converges to 0 by Proposition 2.3. For the first term, using by Doob's maximal inequality, we have $\mathbb{P}\big[\max\{Z_l : l = 1, \ldots, n^2\} \geq \epsilon\sigma_0 n/2\big] \leq 2\mathbb{E}(|Z_{n^2}|)/\epsilon\sigma_0 n = 2\mathbb{E}(Z_{n^2})/\epsilon\sigma_0 n = 2\mathbb{E}(Z_0)/n\sigma_0\epsilon = 2(v_n - u_n)/\epsilon\sigma_0 n \to 0$ as $n \to \infty$. This completes the proof. $\square$

Now, we concentrate on the coalescence of paths starting far apart. Towards that, we define a subset A of $C[0,\infty)^k$ as follows:

$$\begin{aligned}
A = \Big\{&(\pi_1, \ldots, \pi_k) \in C[0,\infty)^k : \pi_i\text{'s satisfy the following conditions} \\
&a)\quad \pi_1(0) < \pi_2(0) < \cdots < \pi_k(0); \\
&b)\quad t^{(i,j)} := \inf\{t \geq 0 : \pi_i(t) = \pi_j(t)\} < \infty \text{ for all } i, j = 1, \ldots, k, i < j; \\
&c)\quad t^{(i_1,j_1)} \neq t^{(i_2,j_2)} \text{ for all } i_1, j_1, i_2, j_2 = 1, \ldots, k, i_1 < j_1, i_2 < j_2 \\
&\qquad \text{and } (i_1, j_1) \neq (i_2, j_2); \\
&d)\quad \text{for any } \delta > 0 \text{ and } i, j = 1, \ldots, k, i < j, \text{ there exist } t \in (t^{(i,j)} - \delta, t^{(i,j)}) \\
&\qquad \text{and } s \in (t^{(i,j)}, t^{(i,j)} + \delta) \text{ such that } (\pi_i(t) - \pi_j(t))(\pi_i(s) - \pi_j(s)) < 0\Big\}.
\end{aligned}$$

Note that A consists of all k-tuples of continuous paths ordered by their starting points where each pair of paths intersect at distinct time points and two paths cross each other instantaneously after they intersect. Clearly, from the indexing of $\mathbf{x}^i$'s and the path property of standard independent Brownian motions, we have

$$\mathbb{P}\big[(B^{\mathbf{x}^1}, \ldots, B^{\mathbf{x}^k}) \in A\big] = 1. \tag{30}$$

We define a '*coalescence map*' $f : C[0,\infty)^k \to C[0,\infty)^k$ as follows:

$$f(\pi_1, \ldots, \pi_k) := \begin{cases} (\overline{\pi}_1, \ldots, \overline{\pi}_k) & \text{for } (\pi_1, \ldots, \pi_k) \in A \\ (\pi_1, \ldots, \pi_k) & \text{otherwise} \end{cases}$$

where $\overline{\pi}_1 \equiv \pi_1$ and for $1 < l \leq k$

$$\overline{\pi}_l(t) := \begin{cases} \pi_l(t) & \text{for } t \leq s^{(l)} \\ \overline{\pi}_{l-1}(t) & \text{for } t > s^{(l)} \end{cases}$$

where $s^{(l)} := \inf\{t \geq 0 : \pi_l(t) = \overline{\pi}_{l-1}(t)\}$. For $\mathbf{x}^1, \ldots, \mathbf{x}^k \in \mathbb{R}^2$ with $\mathbf{x}^1(2) = \cdots = \mathbf{x}^k(2) = 0$ and $\mathbf{x}^1(1) < \ldots < \mathbf{x}^k(1)$, from the strong Markov property, it follows that $f(B^{\mathbf{x}^1}, \ldots, B^{\mathbf{x}^k})$ has the same distribution as that of k standard coalescing Brownian motions starting from $\mathbf{x}^1, \ldots, \mathbf{x}^k$, (see Arratia [A79]), i.e.,

$$f(B^{\mathbf{x}^1}, \ldots, B^{\mathbf{x}^k}) \stackrel{d}{=} (W^{\mathbf{x}^1}, \ldots, W^{\mathbf{x}^k}). \tag{31}$$

Next, we define a sequence of subsets of $C[0,\infty)^k$ where the pair of functions *come close to each other*. We fix $\alpha \in (0,1)$ for the rest of this section. For $n \geq 1$, define

$$\begin{aligned} A_n^\alpha = \Big\{ & (\pi_1, \ldots, \pi_k) \in C[0,\infty)^k : \pi_i\text{'s satisfy the following conditions} \\ & a) \quad \pi_1(0) < \pi_2(0) < \cdots < \pi_k(0); \\ & b) \quad t_n^{(i,j)} := \inf\{t \geq 0 : |\pi_i(t) - \pi_j(t)| \leq 3n^{\alpha-1}\} < \infty \\ & \qquad\quad \text{for all } i, j = 1, \ldots, k, i < j; \\ & c) \quad |t_n^{(i_1,j_1)} - t_n^{(i_2,j_2)}| > \frac{1}{n^2} \text{ for all } i_1, j_1, i_2, j_2 = 1, \ldots, k, \\ & \qquad\quad i_1 < j_1, i_2 < j_2 \text{ and } (i_1, j_1) \neq (i_2, j_2) \Big\}. \end{aligned} \tag{32}$$

Next we define the 'α-*coalescence map*' $f_n^{(\alpha)} : C[0,\infty)^k \to C[0,\infty)^k$ as follows:

$$f_n^{(\alpha)}(\pi_1, \ldots, \pi_k) := \begin{cases} (\overline{\pi}_1, \ldots, \overline{\pi}_k) & \text{for } (\pi_1, \ldots, \pi_k) \in A_n^\alpha \\ (\pi_1, \ldots, \pi_k) & \text{otherwise} \end{cases}$$

where $\overline{\pi}_1 \equiv \pi_1$ and, for $1 < l \leq k$, we define it inductively, by setting

$$\overline{\pi}_l(t) := \begin{cases} \pi_l(t) & \text{for } t \leq s_n^{(l)} \\ \pi_l(s_n^{(l)}) + n^2(t - s_n^{(l)})\big[\overline{\pi}_{l-1}(s_n^{(l)} + \frac{1}{n^2}) - \pi_l(s_n^{(l)})\big] & \text{for } s_n^{(l)} < t \leq s_n^{(l)} + \frac{1}{n^2} \\ \overline{\pi}_{l-1}(t) & \text{for } t > s_n^{(l)} + \frac{1}{n^2} \end{cases}$$

where $s_n^{(l)} = \inf\{t \geq 0 : \pi_l(t) - \overline{\pi}_{l-1}(t) \leq 3n^{\alpha-1}\}$. The α-coalescence map tracks π_l until $s_n^{(l)}$, then linearly interpolates to $\overline{\pi}_{l-1}(s_n^{(l)} + \frac{1}{n^2})$ in the time interval $[s_n^{(l)}, s_n^{(l)} + \frac{1}{n^2}]$ and then tracks the function $\overline{\pi}_{l-1}$ after that time point, for every $l \geq 2$.

Before proceeding, we state the following deterministic lemma (which is a slightly stronger version of Lemma 19 of Colletti *et al.* [CFD09]). The proof of this lemma has been relegated to the appendix and it will be used in the proof of Proposition 5.3.

Lemma 5.3 *Let* $(\pi_1, \ldots, \pi_k) \in A$ *and* $\{(\pi_{1,n}, \ldots, \pi_{k,n}) : n \geq 1\} \subseteq C[0,\infty)$ *be such that* $\tilde{d}(\pi_{i,n}, \pi_i) \to 0$ *as* $n \to \infty$. *Then, for n large enough, we have* $(\pi_{1,n}, \ldots, \pi_{k,n}) \in$

A_n^α *and* $\lim_{n\to\infty} s_n^{(l)} = s^{(l)}$ *for all* $l = 2,\ldots,k$*, where* $\{s^{(l)}, s_n^{(l)} : \ l = 2,\ldots,k\}$ *are as defined above. Further,*

$$\tilde{d}^k\Big(f_n^{(\alpha)}(\pi_{1,n},\ldots,\pi_{k,n}), f(\pi_1,\ldots,\pi_k)\Big) \to 0 \text{ as } n\to\infty. \tag{33}$$

Now, we consider the case when $\mathbf{x}^1,\ldots,\mathbf{x}^k \in \mathbb{R}^2$ have the same second coordinates as 0, i.e., $\mathbf{x}^1(2) = \cdots = \mathbf{x}^k(2) = 0$. We prove the following

Proposition 5.3 *We have*

(a) $f_n^{(\alpha)}(\pi_n^{1,(\mathrm{Ind})},\ldots,\pi_n^{k,(\mathrm{Ind})}) \Rightarrow (W^{\mathbf{x}^1},\ldots,W^{\mathbf{x}^k})$;

(b) $f_n^{(\alpha)}(\pi_n^1,\ldots,\pi_n^k) \Rightarrow (W^{\mathbf{x}^1},\ldots,W^{\mathbf{x}^k})$;

(c) $\mathbb{P}\big[\tilde{d}^k\big(f_n^{(\alpha)}(\pi_n^1,\ldots,\pi_n^k),(\pi_n^1,\ldots,\pi_n^k)\big) \geq \epsilon\big] \to 0$ *for any* $\epsilon > 0$ *as* $n\to\infty$.

Proof of Proposition 5.3: Lemma 5.3 and the observation in equation (30), allow us to use the extended continuous mapping theorem (see Theorem 4.27 in Kallenberg [K02]) to conclude that $f_n^{(\alpha)}(\pi_n^{1,(\mathrm{Ind})},\ldots,\pi_n^{k,(\mathrm{Ind})}) \Rightarrow f(B^{\mathbf{x}^1},\ldots,B^{\mathbf{x}^k})$. We conclude the result of (a) using equation (31).

To prove (b), we show that $f_n^{(\alpha)}(\pi_n^1,\ldots,\pi_n^k)$ and $f_n^{(\alpha)}(\pi_n^{1,(\mathrm{Ind})},\ldots,\pi_n^{k,(\mathrm{Ind})})$ have the same limit. Towards that end, it is enough to show that for any $s > 0$ and for any bounded uniformly continuous function $H : C[0,s]^k \to \mathbb{R}$, we have

$$\lim_{n\to\infty}\Big|\mathbb{E}\big[H\big(f_n^{(\alpha)}(\pi_n^{1,(\mathrm{Ind})},\ldots,\pi_n^{k,(\mathrm{Ind})})\big|_{[0,s]}\big)\big] - \mathbb{E}\big[H\big(f_n^{(\alpha)}(\pi_n^1,\ldots,\pi_n^k)\big|_{[0,s]}\big)\big]\Big| = 0.$$

where $(\pi_1,\ldots,\pi_k)|_{[0,s]}$ denotes the restriction of $(\pi_1,\ldots,\pi_k)$ over time $[0,s]$. To achieve this, we describe a coupling procedure such that the individual paths of the joint process $(\pi_n^1,\ldots,\pi_n^k)$ starting sufficiently far apart, agree with the corresponding individual paths of the independent processes $(\pi_n^{1,(\mathrm{Ind})},\ldots,\pi_n^{k,(\mathrm{Ind})})$.

We employ the same coupling idea used earlier. Starting from $k+1$ independent families of i.i.d. $U[0,1]$ random variables $\{U_{\mathbf{w}}^i : \mathbf{w}\in\mathbb{Z}^2,\ 1\le i\le k+1\}$ and vertices $\mathbf{x}^1,\ldots,\mathbf{x}^k$ with $\mathbf{x}_n^1(2) = \cdots = \mathbf{x}_n^k(2) = 0$ and $\mathbf{x}_n^1(1) < \cdots < \mathbf{x}_n^k(1)$ we construct the individual process $\{g_m^{(\mathrm{Ind})}(\mathbf{x}_i), m\ge 0\}$ using the random variables $\{U_{\mathbf{w}}^i : \mathbf{w}\in\mathbb{Z}^2\}$ for each $i = 1,\ldots,k$. Let $\pi^{i,(\mathrm{Ind})} = \pi^{\mathbf{x}_n^i,(\mathrm{Ind})}$ be the piecewise linear path obtained for the i th process and $\pi_n^{i,(\mathrm{Ind})}$ be the n th order diffusively scaled version.

For the joint process we first select the set of vertices such that their horizontal distances are at least $3n^\alpha$ apart, where $\alpha\in(0,1)$ is as chosen in the definition of $f_n^{(\alpha)}$. We call these vertices *special vertices*. More precisely, let $\mathbf{x}_n^{i_1} = \mathbf{x}_n^1$ and having defined $\mathbf{x}_n^{i_1},\ldots,\mathbf{x}_n^{i_{m-1}}$ we define $\mathbf{x}_n^{i_m}$ with

$$i_m := \inf\{j > i_{m-1} : \mathbf{x}_n^j(1) - \mathbf{x}_n^{i_{m-1}}(1) > 3n^\alpha\}$$

if such an i_m exists, otherwise we stop the selection. Let $I_1 := \{\mathbf{x}_n^{i_1}, \ldots, \mathbf{x}_n^{i_l}\}$ be the special vertices thus selected. We define a new collection of uniform random variables $\{\tilde{U}^1_{\mathbf{w}} : \mathbf{w} \in \mathbb{Z}^2, \mathbf{w}(2) \geq \mathbf{x}^n_{i_1}(2) = 0\}$ by

$$\tilde{U}^1_{\mathbf{w}} := \begin{cases} U^i_{\mathbf{w}} & \text{if } |\mathbf{w}(1) - \mathbf{x}^i_n(1)| < n^\alpha \text{ for } \mathbf{x}^i_n \in I_1 \\ U^{k+1}_{\mathbf{w}} & \text{otherwise} \end{cases}$$

and construct the joint process $\big\{(g_m(\mathbf{x}^1_n), \ldots, g_m(\mathbf{x}^k_n)),\ m \geq 0\big\}$, as in Section 2, using the collection $\{\tilde{U}^1_{\mathbf{w}} : \mathbf{w} \in \mathbb{Z}^2, \mathbf{w}(2) \geq 0\}$ till the first simultaneous regeneration time $T_1(\mathbf{x}^1_n, \ldots, \mathbf{x}^k_n)$ of the joint process.

Having constructed the joint process till the $j-1$ th simultaneous regeneration time $T_{j-1}(\mathbf{x}^1_n, \ldots, \mathbf{x}^k_n)$ of the joint processes with the I_{j-1} being the special vertices, we first choose the next set I_j of special vertices as follows $\mathbf{x}^{j_1}_n = \mathbf{x}^{i_1}_n \in I_j$. For $\mathbf{x}^{j_{m-1}}_n \in I_j$, we choose $\mathbf{x}^{j_m}_n \in I_j$ with

$$j_m := \inf\{j > j_{m-1} : \mathbf{x}^j_n \in I_{j-1} \text{ and } g_{\tau_{j-1}}(\mathbf{x}^j_n)(1) - g_{\tau_{j-1}}(\mathbf{x}^{j_{m-1}}_n)(1) > 3n^\alpha\}$$

if such an j_m exists, otherwise we stop the selection. The process from time T_{j-1} onwards is constructed with the collection

$$\tilde{U}^j_{\mathbf{w}} := \begin{cases} U^i_{\mathbf{w}} & \text{if } |\mathbf{w}(1) - g_{\tau_{j-1}}(\mathbf{x}^i_n)(1)| < n^\alpha \text{ for } \mathbf{x}^i_n \in I_j \\ U^{k+1}_{\mathbf{w}} & \text{otherwise} \end{cases}$$

until the time of the next simultaneous regeneration of the joint process.

Let $(\pi^{\mathbf{x}^1_n}, \ldots, \pi^{\mathbf{x}^k_n})$ be the paths obtained from this joint process starting from $(\mathbf{x}^1_n, \ldots, \mathbf{x}^k_n)$ and with $(\pi^1_n, \ldots, \pi^k_n)$ being the diffusively scaled versions. To compare $f^{(\alpha)}_n(\pi^1_n, \ldots, \pi^k_n)$ with $f^{(\alpha)}_n(\pi^{1,(\text{Ind})}_n, \ldots, \pi^{k,(\text{Ind})}_n)$ we define the events

$$\begin{aligned} E_n &:= \{(\pi^{1,(\text{Ind})}_n, \ldots, \pi^{k,(\text{Ind})}_n) \in A^\alpha_n\} \\ B_{n,s} &:= \{W^{(\text{Ind})}_j(k) < n^\alpha : \text{ for all } j \geq 1 \text{ with } T^{(\text{Ind})}_j(k) \leq sn^2\}, \end{aligned}$$

where A^α_n, $W^{(\text{Ind})}_j(k)$ and $T^{(\text{Ind})}_j(k)$ are as defined in (32), (21) and (19) respectively.

Note that, from Proposition 5.2 and using Skorohod's representation theorem, we can assume that $(\pi^{1,(\text{Ind})}_n, \ldots, \pi^{k,(\text{Ind})}_n)$ converges to $(B^{\mathbf{x}^1}, \ldots, B^{\mathbf{x}^k})$ as elements in $(C[0,\infty)^k, \tilde{d}^k)$, almost surely. Observe that on the event $B_{n,s} \cap E_n$, the coupled processes satisfy $f^{(\alpha)}_n(\pi^1_n, \ldots, \pi^k_n)|_{[0,s]} = f^{(\alpha)}_n(\pi^{1,(\text{Ind})}_n, \ldots, \pi^{k,(\text{Ind})}_n)|_{[0,s]}$ for n large. This observation yields

$$\begin{aligned} &\Big|\mathbb{E}\big[H\big(f^{(\alpha)}_n(\pi^{1,(\text{Ind})}_n, \ldots, \pi^{k,(\text{Ind})}_n)\big|_{[0,s]}\big)\big] - \mathbb{E}\big[H\big(f^{(\alpha)}_n(\pi^1_n, \ldots, \pi^k_n)\big|_{[0,s]}\big)\big]\Big| \\ &\leq \Big|\mathbb{E}\Big[\big[H\big(f^{(\alpha)}_n(\pi^{1,(\text{Ind})}_n, \ldots, \pi^{k,(\text{Ind})}_n)\big) - H\big(f^{(\alpha)}_n(\pi^1_n, \ldots, \pi^k_n)\big)\big]\mathbf{1}_{B^c_{n,s} \cup E^c_n}\Big]\Big| \\ &\leq 2||H||_\infty \mathbb{P}(B^c_{n,s} \cup E^c_n). \end{aligned}$$

Finally, from Lemma 5.3, $\mathbb{P}(E_n^c) \to 0$. Using a similar calculation, used in the last part of Proposition 5.2, $\mathbb{P}(B_{n,s}^c) \to 0$ as $n \to \infty$, which completes the proof of (b).

We write the proof of (c) for $k = 2$; the argument for general k being similar. By definition of $f_n^{(\alpha)}$, we have $\overline{\pi}_n^1(t) = \pi_n^1(t)$ for all $t \geq 0$ and $\overline{\pi}_n^1(t) = \pi_n^2(t)$ for $t \leq s_n^{(2)}$, where $s_n^{(2)} = \inf\{t \geq 0 : \pi_n^2(t) - \overline{\pi}_n^1(t) \leq 3n^{\alpha-1}\} = \inf\{t \geq 0 : \pi_n^2(t) - \pi_n^1(t) \leq 3n^{\alpha-1}\}$. From the definition of the metric $\tilde{d}^k$, it suffices to show that, for any $\epsilon > 0$,

$$\mathbb{P}\Big[\sup_{t \geq s_n^{(2)}} \big|\pi_n^1(t) - \pi_n^2(t)\big| \geq \epsilon\Big] = \mathbb{P}\Big[\sup_{t \geq s_n^{(2)}\gamma_0 n^2} \big|\pi^1(t) - \pi^2(t)\big| \geq \epsilon\sigma_0 n\Big] \to 0$$

as $n \to \infty$.

Let $\kappa_n := \inf\{j : T_j(\mathbf{x}_n^1, \mathbf{x}_n^2) \geq s_n^{(2)}\gamma_0 n^2\}$ where $T_j(\mathbf{x}_n^1, \mathbf{x}_n^2)$ is the j th simultaneous regeneration time of the joint paths starting from $(\mathbf{x}_n^1, \mathbf{x}_n^2)$ and define the event $D = \{W_{\kappa_n}(\mathbf{x}_n^1, \mathbf{x}_n^2) < n^\alpha\}$. We have

$$\mathbb{P}\Big[\sup_{t \geq s_n^{(2)}\gamma_0 n^2} \big|\pi^1(t) - \pi^2(t)\big| \geq \epsilon\sigma_0 n\Big] \leq \mathbb{P}(D^c) + \mathbb{P}\Big[\Big\{\sup_{t \geq s_n^{(2)}\gamma_0 n^2} \big|\pi^1(t) - \pi^2(t)\big| \geq \epsilon\sigma_0 n\Big\} \cap D\Big].$$

The first term $\mathbb{P}(D^c) = \mathbb{P}\{W_{\kappa_n}(\mathbf{x}_n^1, \mathbf{x}_n^2) \geq n^\alpha\} \to 0$ from Proposition 2.3. For the second term, we note that, on the event D, the width of the explored region is at most n^α, hence the difference of the paths, at the previous simultaneous regeneration time before $s_n^{(2)}\gamma_0 n^2$, is at most $5n^\alpha$, i.e., $|\pi^1(T_{\kappa_n - 1}(\mathbf{x}_n^1, \mathbf{x}_n^2)) - \pi^2(T_{\kappa_n - 1}(\mathbf{x}_n^1, \mathbf{x}_n^2))| \leq 5n^\alpha$. Also, for $t \in [T_{\kappa_n - 1}(\mathbf{x}_n^1, \mathbf{x}_n^2), T_{\kappa_n}(\mathbf{x}_n^1, \mathbf{x}_n^2)]$, we have $|\pi^1(t) - \pi^2(t)| \leq 7n^\alpha < \epsilon\sigma_0 n$ for all large n. Hence, using the Markov property, non-crossing property of paths and the translation invariance of our model, it follows that, for all large n,

$$\begin{aligned}
&\mathbb{P}\Big[\Big\{\sup_{t \geq s_n^{(2)}\gamma_0 n^2} \big|\pi^1(t) - \pi^2(t)\big| \geq \epsilon\sigma_0 n\Big\} \cap D\Big] \leq \mathbb{P}\Big[\sup_{t \geq T_{\kappa_n}(\mathbf{x}_n^1, \mathbf{x}_n^2)} \big|\pi^1(t) - \pi^2(t)\big| \geq \epsilon\sigma_0 n\Big] \\
&\leq \mathbb{P}\Big[\sup_{t \geq 0} \big|\pi^{(7n^\alpha, 0)}(t) - \pi^{(0,0)}(t)\big| \geq \epsilon\sigma_0 n\Big] \to 0
\end{aligned}$$

as $n \to \infty$, using the same calculations as in the Sandwich Lemma. □

Proof of Proposition 5.1: First we consider the case when $\mathbf{x}^1(2) = \cdots = \mathbf{x}^k(2)$. With the choices of $\mathbf{x}_n^1, \ldots, \mathbf{x}_n^k$, made in equation (28) and using the translation invariance of the model, we note that the joint distribution of the paths starting from $(\mathbf{x}_n^1, \ldots, \mathbf{x}_n^k)$ is the same as that of the paths starting from $((\mathbf{x}_n^1(1), 0), \ldots, (\mathbf{x}_n^k(1), 0))$ and then translated by $(0, \mathbf{x}_n^1(2))$. By Proposition 5.3, the diffusively scaled paths starting from $((\mathbf{x}_n^1(1), 0), \ldots, (\mathbf{x}_n^k(1), 0))$ converge weakly to coalescing Brownian motions starting from $((\mathbf{x}^1(1), 0), \ldots, (\mathbf{x}^k(1), 0))$. Since $(0, \mathbf{x}_n^1(2)) \to (0, \mathbf{x}^1(2))$ as $n \to \infty$, we have the result for this case.

For the general case, using similar arguments as above, it is enough to consider $\mathbf{x}^1, \ldots, \mathbf{x}^k$ starting at different levels with $\mathbf{x}^1(2) \leq \cdots \leq \mathbf{x}^k(2) = 0$. We proceed by induction using arguments similar to that of Colletti *et al* [CFD09]. First, the result holds for $k = 1$, from Proposition 5.2. For $k \geq 2$, we assume that the result holds for $k-1$, and we start with any k points $\mathbf{x}^1, \dots, \mathbf{x}^k \in \mathbb{R}^2$ with $\mathbf{x}^1(2) \leq \dots \leq \mathbf{x}^k(2) = 0$ and let $i_0 := \inf\{i : \mathbf{x}^i(2) = 0\}$.

We construct the process from these k vertices, noting that the vertices at the highest level do not move until the process from the vertices below catches up with the vertices at the highest level. We stop the process as soon as it needs information of vertices on or above the x-axis. As per the construction of the process, at any step m, the points at the lowest level, *viz.,* W_m^{move}, are allowed to move and they have the same second co-ordinate. If all of them can find the next vertex such that the explored region lies below the x-axis we run the process; otherwise we stop at that step. More precisely, define

$$l_n := \max\big\{m : \text{ for each } \mathbf{u} \in W_m^{\text{move}}, \big(S(\mathbf{u}, -\mathbf{u}(2) - 1) \setminus \{y \leq \mathbf{u}(2)\}\big) \cap V \neq \emptyset\big\},$$

where $S(\mathbf{u}, r) = \{\mathbf{w} \in \mathbb{Z}^2 : ||\mathbf{w} - \mathbf{u}||_1 \leq r\}$.

At this step l_n, all vertices lie below the x-axis and no points on or above x-axis have been explored. For the vertices below the x-axis we consider their projection on the x-axis. We think of the process is being restarted from these projected vertices (even though they many not be open) together with the paths from the vertices which were originally on the x-axis. Since the number of vertices below the x-axis from which we started the process is $k-1$ or less, the induction hypothesis gives us that the scaling limit of these paths until they reach the x-axis is a system of coalescing Brownian motions. Using the previous case of this proof and the sandwich lemma, we show that the scaling limit of the paths from the projected vertices on the x-axis and the original vertices is the same system coalescing Brownian motions. We also show that at the l_n th step the location of the vertices of the process below the x-axis is close to the x-axis so that on scaling this difference vanishes.

First we observe that, if the maximum gap of the second co-ordinates at step l_n from the x-axis is at least $m+1$, then for some $\mathbf{u} \in W_{l_n}^{\text{move}}$, all the m vertices lying above $\mathbf{u}$ (and below the x-axis) must be closed. Thus,

$$\begin{aligned}
&\mathbb{P}\Big[\max\big\{-g_{l_n}(\mathbf{x}_n^i)(2) : 1 \leq i \leq i_0 - 1\big\} \geq m + 1\Big] \\
&\leq \mathbb{P}\Big[\text{for some } \mathbf{x}_n^i \text{ with } g_{l_n}(\mathbf{x}_n^i) \in W_{l_n}^{\text{move}}, g_{l_n}^{\uparrow j}(\mathbf{x}_n^i) \notin V \text{ for } 1 \leq j \leq m\Big] \\
&\leq (i_0 - 1)(1 - p)^m.
\end{aligned}$$

Fix $0 < \nu < \alpha$. Define the event

$$F_n := \big\{\max\{-g_{l_n}(\mathbf{x}_n^i)(2) : 1 \leq i < i_0\} \geq n^\nu\big\}. \tag{34}$$

Applying Borel Cantelli lemma, we get that $\mathbb{P}(\limsup_n F_n) = 0$ and hence

$$g_{l_n}(\mathbf{x}_n^i)(2)/n^2\gamma_0 \to \mathbf{x}^{i_0}(2) = 0 \text{ as } n \to \infty \text{ for all } 1 \le i < i_0 \text{ almost surely.} \tag{35}$$

By the induction hypothesis, we have that for paths starting from first $i_0 - 1$ vertices,

$$(\pi_n^1, \dots, \pi_n^{i_0-1}) \Rightarrow (W^{\mathbf{x}^1}, \dots, W^{\mathbf{x}^{i_0-1}}).$$

Using Skorohod representation theorem, we can assume that the convergence is almost sure as elements in Π^{i_0-1}, i.e.,

$$(\pi_n^1, \dots, \pi_n^{i_0-1}) \to (W^{\mathbf{x}^1}, \dots, W^{\mathbf{x}^{i_0-1}}) \text{ as } n \to \infty \text{ in } \Pi^{i_0-1} \tag{36}$$

almost surely. From (35) and (36) we have,

$$\begin{aligned} &\big(\{\pi_n^i(t) : \sigma_{\pi_n^i} \le t \le \frac{g_{l_n}(\mathbf{x}_n^i)(2)}{n^2\gamma_0}\}, 1 \le i < i_0\big) \\ \to\ &\big(\{W^{\mathbf{x}^i}(t) : \mathbf{x}^i(2) \le t \le 0\}, 1 \le i < i_0\big) \quad \text{almost surely.} \end{aligned} \tag{37}$$

Here we need to remark that although the process starts with k vertices, till the l_n th step only the first $i_0 - 1$ vertices move and their movement till the l_n th step is as if the process started with only these $i_0 - 1$ vertices. Thus the induction hypothesis may be applied.

Let $(\mathcal{P}, \rho_{\mathcal{P}})$ be the space of compact subsets of $(\mathbb{R}_c^2, \rho)$ with $\rho_{\mathcal{P}}$ the induced Hausdorff metric, i.e., for $A_1, A_2 \in \mathcal{P}$,

$$\rho_{\mathcal{P}} := \sup\{\sup_{\mathbf{x}^1 \in A_1} \inf_{\mathbf{x}^2 \in A_2} \rho(\mathbf{x}^1, \mathbf{x}^2), \sup_{\mathbf{x}^2 \in A_2} \inf_{\mathbf{x}^1 \in A_1} \rho(\mathbf{x}^1, \mathbf{x}^2)\}.$$

Denoting by $\tilde{\mathbf{x}}_n^i$, the projection of the vertex $g_{l_n}(\mathbf{x}_n^i)$ on the x-axis and

$$Q_n := \bigcup_{1 \le i < i_0} \{(\frac{\tilde{\mathbf{x}}_n^i(1)}{n\sigma_0}, 0)\} \cup \bigcup_{i_0 \le i \le k} \{\mathbf{x}^i\} \text{ and } Q := \bigcup_{1 \le i < i_0} \{(W^{\mathbf{x}^i}(0), 0)\} \cup \bigcup_{i_0 \le i \le k} \{\mathbf{x}^i\},$$

we have, from (37), $Q_n \to Q$ as $n \to \infty$ almost surely in $(\mathcal{P}, \rho_{\mathcal{P}})$. Conditioned on Q_n, let $\mathcal{X}_n^{Q_n}$ be the n th order diffusively scaled version of the paths starting from the random point set $\{(n\sigma_0\mathbf{x}(1), n^2\gamma_0\mathbf{x}(2)) : \mathbf{x} \in Q_n\} = \{\tilde{\mathbf{x}}_n^i : 1 \le i < i_0\} \cup \{\mathbf{x}_n^i : i_0 \le i \le k\}$. Using the first part where $\mathbf{x}^1(2) = \cdots = \mathbf{x}^k(2)$, and Lemma 6.5 of Newman *et al.* [NRS05] we conclude that $\mathcal{X}_n^{Q_n}$ converges in distribution to $\mathcal{W}^Q$, i.e., coalescing Brownian motions starting from a random point set distributed as Q.

Finally we need to show that the paths starting from $\{g_{l_n}(\mathbf{x}_n^1), \dots, g_{l_n}(\mathbf{x}_n^{i_0-1})\} \cup \{\mathbf{x}_n^{i_0}, \dots, \mathbf{x}_n^k\}$, and the paths starting from the projected vertices $\{\tilde{\mathbf{x}}_n^1, \dots, \tilde{\mathbf{x}}_n^{i_0-1}\} \cup$

$\{\mathbf{x}_n^{i_0}, \ldots, \mathbf{x}_n^k\}$ converge to the same limit. Using (35), we conclude that the starting points of the paths converge almost surely. For the local convergence of paths from time 0 onwards, we will use the Sandwich Lemma.

We fix $\alpha < \beta < 1$ and for each $i = 1, \ldots, i_0 - 1$, two open points $\mathbf{u}_n^i = (u_n^i, 0)$ and $\mathbf{v}_n^i = (v_n^i, 0)$ such that $\tilde{\mathbf{x}}_n^i(1) - n^\beta < u_n^i < \tilde{\mathbf{x}}_n^i(1) - n^\alpha < \tilde{\mathbf{x}}_n^i(1) + n^\alpha < v_n^i < \tilde{\mathbf{x}}_n^i(1) + n^\beta$. Note that the probability of existence of such points converges to 1 as $n \to \infty$ (since the points on the x-axis are yet to be explored). From the Sandwich Lemma, any path which crosses the x-axis, in between u_n^i and v_n^i will converge to the same Brownian motion. It is therefore enough to prove that, for $i = 1, \ldots, i_0 - 1$,

$$\mathbb{P}\big\{\pi^{\mathbf{x}_n^i}(0) \notin [u_n^i, v_n^i]\big\} = \mathbb{P}\big\{\pi^{g_{l_n}(\mathbf{x}_n^i)}(0) \notin [u_n^i, v_n^i]\big\} \to 0$$

as $n \to \infty$. On the complement of the event F_n, defined in (34), the path from $g_{l_n}(\mathbf{x}_n^i)$ needs at most n^ν steps to ensure that thereafter all the vertices of the path are on or above the x-axis. Therefore, the maximum displacement of the first co-ordinate of the path starting from $g_{l_n}(\mathbf{x}_n^i)$ when it crosses the x-axis is bounded by $\sum_{j=1}^{n^\nu} ||h^j(g_{l_n}(\mathbf{x}_n^i)) - h^{j-1}(g_{l_n}(\mathbf{x}_n^i))||_1$. Define the event

$$D_n^{(i)} := \Big\{\sum_{j=1}^{n^\nu} ||h^j(g_{l_n}(\mathbf{x}_n^i)) - h^{j-1}(g_{l_n}(\mathbf{x}_n^i))||_1 < n^\alpha\Big\}.$$

If $(D_n^{(i)})^c$ occurs, then we must have $||h^j(g_{l_n}(\mathbf{x}_n^i)) - h^{j-1}(g_{l_n}(\mathbf{x}_n^i))||_1 \geq n^{\alpha-\nu}$ for some $j = 1, \ldots, n^\nu$. Thus, using the bound in (2.3), we conclude that $\mathbb{P}\big((D_n^{(i)})^c\big) \leq n^\nu C_7 \exp(-C_8 n^{\alpha-\nu}) \to 0$ as $n \to \infty$. Finally, on the event $D_n^{(i)} \cap F_n^c$, we have $\pi^{g_{l_n}(\mathbf{x}_n^i)}(0) \in [u_n^i, v_n^i]$. Thus,

$$\mathbb{P}\big\{\pi^{g_{l_n}(\mathbf{x}_n^i)}(0) \notin [u_n^i, v_n^i]\big\} \leq \mathbb{P}(F_n) + \mathbb{P}\big((D_n(i))^c\big) \to 0$$

as $n \to \infty$. This completes the proof. $\square$

5.2 Proof of (B_1) and (E_1')

The proof of condition (B_1) is standard and follows by the same argument as in Coletti *et al.* [CFD09].

In order to prove (E_1'), recall that $\mathcal{X}_n^{t_0}$ is the collection of n th order diffusively scaled paths obtained from our random graph G which start before time t_0 and $\overline{\mathcal{X}}_n^{t_0}$ is the closure of $\mathcal{X}_n^{t_0}$ in (Π, d_Π). Note that condition (I_1) guarantees that $\overline{\mathcal{X}}_n$ is tight. Being a subset of $\overline{\mathcal{X}}_n$, for any t_0, the sequence $\overline{\mathcal{X}}_n^{t_0}$ is also tight and hence such subsequential limit(s) exist.

For $K \in \mathcal{H}$, let $K(s) := \{(\pi(s), s) : \pi \in K, \sigma_\pi \leq s\} \subset \mathbb{R}^2$. Also define $\mathcal{Z}^{t_0;(t_0+\epsilon)+} := \{\pi : \sigma_\pi = t_0 + \epsilon$ and there exists $\pi' \in \mathcal{Z}^{t_0}$ such that $\pi(u) = \pi'(u)$ for all $u \geq t_0 + \epsilon\}$.

The equivalent of Lemma 6.2 and Lemma 6.3 of [NRS05] for our model is

Lemma 5.4 *$\mathcal{Z}^{t_0}(t_0+\epsilon)$ is a.s. locally finite for any $\epsilon > 0$, and $\mathcal{Z}^{t_0;(t_0+\epsilon)+}$ is distributed as $\mathcal{W}^{\mathcal{Z}^{t_0}(t_0+\epsilon)}$, i.e., coalescing Brownian motions starting from the random point set $\mathcal{Z}^{t_0}(t_0+\epsilon) \subset \mathbb{R}^2$.*

Proof: Using Lemma 5.1 and Proposition 5.1, the same proof as that of Lemma 6.2 of [NRS05] proves that $\mathcal{Z}^{t_0}(t_0+\epsilon)$ is a.s. locally finite for our model.

Finally we need to prove $\mathcal{Z}^{t_0;(t_0+\epsilon)+}$ is distributed as coalescing Brownian motions starting from the locally finite point set $\mathcal{Z}^{t_0}(t_0+\epsilon)$. Because of translation invariance of our model we choose $t_0 = 0$. Using Skorohod's representation theorem, we may assume that we are working on a probability space such that $\mathcal{X}_n^0 \to \mathcal{Z}^0$ as $n \to \infty$ in $(\mathcal{H}, d_{\mathcal{H}})$ almost surely. Hence we have

$$\rho_{\mathcal{P}}(\mathcal{X}_n^0(\epsilon), \mathcal{Z}^0(\epsilon)) \to 0 \text{ as } n \to \infty \text{ almost surely.} \tag{38}$$

The rest of the proof follows a similar line as in last part of Proposition 5.1 and we provide a sketch. We again create a set of open points, Q_n, on the line $\lfloor \gamma_0 \epsilon n^2 \rfloor$ such that $\rho_{\mathcal{P}}(\tilde{Q}_n, \mathcal{X}_n^0(\epsilon)) \to 0$ as $n \to \infty$ almost surely, where $\tilde{Q}_n$ is the scaled version of points of Q_n. Again, for any interval $[a, b]$, this can be done in a similar way as in Proposition 5.1, so that no information about points on the line $\{y = \lfloor \gamma_0 \epsilon n^2 \rfloor\}$ or above is required. Using the Proposition 5.1 and the fact that $\rho_{\mathcal{P}}(\mathcal{X}_n^0(\epsilon), \mathcal{Z}^0(\epsilon)) \to 0$ and $\rho_{\mathcal{P}}(\tilde{Q}_n, \mathcal{X}_n^0(\epsilon)) \to 0$, we now conclude that the scaled paths starting from $\tilde{Q}_n$ converge to a coalescing system of Brownian motions starting from $\mathcal{Z}^0(\epsilon)$. Finally, using the sandwich lemma, we conclude that $\mathcal{X}_n^0|_{[\epsilon,\infty)}$ converges to the same coalescing system of Brownian motions starting from $\mathcal{Z}^0(\epsilon)$ where $\mathcal{X}_n^0|_{[\epsilon,\infty)}$ is the restriction of paths in $\mathcal{X}_n^0$ on $[\epsilon, \infty)$. $\square$

Now, to prove (E_1), for any ϵ such that $0 < \epsilon < t$, we have,

$$\begin{aligned}\mathbb{E}[\hat{\eta}_{\mathcal{Z}^{t_0}}(t_0, t; a, b)] &= \mathbb{E}\big[\hat{\eta}_{\mathcal{Z}^{t_0;(t_0+\epsilon)+}}(t_0+\epsilon, t-\epsilon; a, b)\big] \\ &\leq \mathbb{E}[\hat{\eta}_{\mathcal{W}}(t_0+\epsilon, t-\epsilon; a, b)] = \frac{b-a}{\sqrt{\pi(t-\epsilon)}}.\end{aligned}$$

Letting $\epsilon \to 0$, we conclude (E_1').

6 Appendix

Proof of Lemma 2.1: It suffices to prove that, for some $\alpha > 0$, we have $\mathbb{E}(\exp(\alpha\tau_M)) < \infty$. Since M_{n+1} is a function of M_n and an independent sequence of random variables, $\{M_n : n \geq 0\}$ is a Markov chain. Furthermore, it is irreducible and recurrent.

Note first that, using $\mathbf{1}$ to denote the indicator function,

$$M_{n+1} - M_n = -\mathbf{1}(\theta_{n+1} \leq M_n) + \big[\theta_{n+1} - M_n - 1\big]\mathbf{1}(\theta_{n+1} > M_n).$$

Choose m_0 large such that $\mathbb{E}\big[(\theta_1 - m_0 - 1)\mathbf{1}(\theta_1 > m_0)\big] < \mathbb{P}(\theta_1 \le m_0)$. This is possible because $\mathbb{E}\big[(\theta_1 - m - 1)\mathbf{1}(\theta_1 > m)\big] - \mathbb{P}(\theta_1 \le m) \to -1$ as $m \to \infty$. By the choice of m_0, we have that $\mathbb{E}[M_{n+1} - M_n \mid M_n = m_0] < 0$. Since the random variable θ_1 has an exponentially decaying tail, we can choose $\alpha > 0$ sufficiently small and $r > 1$ so that $\mathbb{E}\big[\exp\big(\alpha(M_{n+1} - M_n)\big) \mid M_n = m_0\big] < 1/r$.

Furthermore, we observe that given $M_n = m > m_0$, the distribution of $M_{n+1} - M_n$ is the same as that of $-\mathbf{1}(\theta_{n+1} \le m) + \big[\theta_{n+1} - m - 1\big]\mathbf{1}(\theta_{n+1} > m)$; the latter random variable being dominated by $-\mathbf{1}(\theta_{n+1} \le m_0) + \big[\theta_{n+1} - m_0 - 1\big]\mathbf{1}(\theta_{n+1} > m_0)$. Therefore, we conclude that, for all $m > m_0$,

$$\mathbb{E}\big[\exp\big(\alpha(M_{n+1} - M_n)\big) \mid M_n = m\big] < 1/r.$$

Hence, taking $E = \{0, 1, \ldots, l_0 - 1\}$ and $f : \{0, 1, 2, \ldots, \} \to \mathbb{R}$ by $f(i) = \exp(\alpha i)$ and using Proposition 5.5, Chapter 1 of Asmussen [A03], we obtain the result. $\square$

Proof of Lemma 3.1: Define $L_n := \max\{R_n^{(i)} : i = 1, \ldots, k\}$ and set $\tau_L := \inf\{n \ge 1 : L_n = 0\}$. Then, we have, $\tau_R = \tau_L$. Again, we define a new Markov chain which dominates L_n and satisfies the conditions of Lemma 2.1, from which we will conclude the result.

We start with k families of independent copies of the inter-arrival times, say $\{\eta_n^{(i)} : n \ge 1\}$ with $\eta_1^{(i)} \overset{d}{=} \xi_1^{(i)}$ for $i = 1, \ldots, k$. Now, keeping the same notation as in the proof of Proposition 2.2, we set $W_n^{\text{move}} := \{i : R_n^{(i)} = 0, \text{ for } i = 1, \ldots, k\}$ and $W_n^{\text{stay}} := \{1, \ldots, k\} \setminus W_n^{\text{move}}$. Now, for $i \in W_n^{\text{move}}$ we have $S_{l_i(n)}^{(i)} = n$ for some $l_i(n) \ge 0$, and for $i \in W_n^{\text{stay}}$ we have $S_l^{(i)} \ne n$ for every $l \ge 0$. Define

$$J_{n+1} = \max\big\{\max\{\xi_{l_i(n)+1}^{(i)} : i \in W_n^{\text{move}}\}, \max\{\eta_{n+1}^{(i)} : i \in W_n^{\text{stay}}\}\big\}$$

and

$$M_0 := 0 \text{ and } M_{n+1} := \max\{M_n, J_{n+1}\} - 1 \text{ for } n \ge 0.$$

We now claim $M_n \ge L_n$ for all $n \ge 0$. Clearly, $M_0 = L_0 = 0$. Assume that the result holds for n. For $i \in W_n^{\text{stay}}$ (i.e., $R_n^{(i)} \ge 1$) we have $R_{n+1}^{(i)} = R_n^{(i)} - 1$. While, for $i \in W_n^{\text{move}}$ (i.e., $S_{l_i(n)}^{(i)} = n$ for some $l_i(n) \ge 0$) we have

$$\begin{aligned} R_{n+1}^{(i)} &= \inf\{S_k^{(i)} : S_k^{(i)} \ge n + 1\} - n - 1 = S_{l_i(n)+1}^{(i)} - n - 1 \\ &= S_{l_i(n)}^{(i)} + \xi_{l_i(n)+1}^{(i)} - n - 1 = \xi_{l_i(n)+1}^{(i)} - 1. \end{aligned}$$

Thus we have

$$\begin{aligned}
L_{n+1} &= \max\{R_{n+1}^{(i)} : i = 1, 2, \ldots, k\} \\
&= \max\big\{\max\{R_{n+1}^{(i)} : i \in W_n^{\text{move}}\}, \max\{R_{n+1}^{(i)} : i \in W_n^{\text{stay}}\}\big\} \\
&= \max\big\{\max\{\xi_{l_i(n)+1}^{(i)} - 1 : i \in W_n^{\text{move}}\}, \max\{R_n^{(i)} - 1 : i \in W_n^{\text{stay}}\}\big\} \\
&\leq \max\big\{\max\{\xi_{l_i(n)+1}^{(i)} : i \in W_n^{\text{move}}\}, \max\{R_n^{(i)} : i = 1, \ldots, k\}, \\
&\qquad\quad \max\{\eta_{n+1}^{(i)} : i \in W_n^{\text{stay}}\}\big\} - 1 \\
&= M_{n+1}.
\end{aligned}$$

The independence of the families of random variables, $\{\xi_n^{(i)}\}$ and $\{\eta_n^{(i)}\}$ and the fact that $\xi_1^{(i)} \stackrel{d}{=} \eta_1^{(i)}$, for $i = 1, \ldots, k$, implies that we can write $M_{n+1} = \max\{M_n, \theta_{n+1}\} - 1$ where $\{\theta_n : n \geq 1\}$ is a sequence of i.i.d. random variables with $\theta_1 \stackrel{d}{=} \max\{\xi_1^{(i)} : i = 1, \ldots, k\}$. The assumptions imposed on $\xi_n^{(i)}$'s imply that the Markov chain satisfies the conditions of Lemma 2.1 and the result follows from that. $\square$

Proof of Lemma 5.3: We prove this lemma for $k = 2$, the proof for general k being similar. Fix $0 < \epsilon < \min\{s^{(2)}, 1\}$. To prove $\lim_{n\to\infty} s_n^{(2)} = s^{(2)}$ we show that $\liminf_n s_n^{(2)} \geq s^{(2)} - \epsilon$ and $\limsup_n s_n^{(2)} \leq s^{(2)} + \epsilon$.

Let $\nu_1 := \inf_{t\in[0,s^{(2)}-\epsilon]}(\pi_2(t) - \pi_1(t)) > 0$ and choose n_1 large enough so that for all $n \geq n_1$ we have $\max\big\{\max\{\sup_{t\in[0,s^{(2)}+1]} |\pi_{in}(t) - \pi_i(t)| : i = 1, 2\}, n^{\alpha-1}\big\} < \nu_1/4$. Hence, for $t \leq s^{(2)} - \epsilon$, we have $\pi_{2n}(t) - \pi_{1n}(t) \geq \pi_2(t) - \pi_1(t) - |\pi_{2n}(t) - \pi_1(t)| - |\pi_{2n}(t) - \pi_2(t)| > \nu_1/2 > n^{\alpha-1}$ so that $s_n^{(2)} \geq s^{(2)} - \epsilon$ for all $n \geq n_1$ and hence $\liminf_n s_n^{(2)} \geq s^{(2)} - \epsilon$.

For the upper bound, fix $s \in [s^{(2)}, s^{(2)} + \epsilon]$, such that $\pi_1(s) - \pi_2(s) > 0$. Set $\nu_2 := \min\{\pi_1(s) - \pi_2(s), \inf_{[0,s^{(2)}/2]} \pi_2(t) - \pi_1(t)\}$. Now choose n_2 large enough so that for all $n \geq n_2$ we have $\sup_{t\in[0,s_1+1]} |\pi_{in} - \pi_i| < \frac{\nu_2}{4}$ for $i = 1, 2$. Thus, for all $n \geq n_2$, we have $\pi_{2n}(0) - \pi_{1n}(0) \geq \pi_2(0) - \pi_2(0) - |\pi_{1n}(0) - \pi_1(0)| - |\pi_{2n}(0) - \pi_2(0)| > \nu_2/2 > 0$, so that $\pi_{2n}(0) > \pi_{1n}(0)$. Also, $\pi_{1n}(s) - \pi_{2n}(s) \geq \pi_1(s) - \pi_2(s) - |\pi_{1n}(s) - \pi_1(s)| - |\pi_{2n}(s) - \pi_2(s)| > \nu_2/2 > 0$. Thus, π_{1n} and π_{2n} cross each other before time $s^{(2)} + \epsilon$ and hence $s_n^{(2)} \leq s^{(2)} + \epsilon$. This completes the proof of first part of the Lemma.

To prove the second part of the lemma, observe that it suffices to show that $\sup_{t\in[0,s^{(2)}+1]} |\overline{\pi}_{2n}(t) - \overline{\pi}_2(t)| \to 0$ as $n \to \infty$. Fix $\eta > 0$ and choose $\beta > 0$ such that $\sup_{t,s\in[s^{(2)}-\beta,s^{(2)}+\beta]} \max\{|\pi_1(t) - \pi_1(s)|, |\pi_2(t) - \pi_2(s)|, |(\pi_2 - \pi_1)(t) - (\pi_2 - \pi_1)(s)|\} < \eta$. Take n_0 such that for all $n \geq n_0$ we have (a) $\frac{1}{n^2} < \beta$, (b) $s_n^{(2)}, s_n^{(2)} + \frac{1}{n^2} \in (s^{(2)} - \beta, s^{(2)} + \beta)$ and (c) $\sup_{t\in[0,s^{(2)}+1]} \max\{|\pi_{1n}(t) - \pi_1(t)|, |\pi_{2n}(t) - \pi_2(t)|\} < \eta$.

Further

$$\sup_{t\in[0,s^{(2)}+1]} |\overline{\pi}_{2n}(t)-\overline{\pi}_2(t)| \leq \sup_{t\in[0,s^{(2)}-\beta]} |\pi_{2n}(t)-\pi_2(t)|$$
$$+ \sup_{t\in[s^{(2)}-\beta,s^{(2)}+\beta]} |\overline{\pi}_{2n}(t)-\overline{\pi}_2(t)| + \sup_{t\in[s^{(2)}+\beta,s^{(2)}+1]} |\pi_{1n}(t)-\pi_1(t)|.$$

Note that for $n \geq n_0$ we also have

$$\sup_{t\in[s^{(2)}-\beta,s^{(2)}+\beta]} \max\{|\overline{\pi}_{2n}(t)-\overline{\pi}_1(s)|, |\overline{\pi}_{1n}(t)-\overline{\pi}_2(s)|, |\overline{\pi}_{1n}(t)-\overline{\pi}_{2n}(s)|\} < 3\eta. \quad (39)$$

Since we obtain $\overline{\pi}_{2n}$ by linearly joining $\pi_{2n}(s_n^{(2)})$ and $\pi_{1n}(s_n^{(2)}+\frac{1}{n^2})$, for $n \geq n_0$ from (39) we have $\sup_{t\in[s^{(2)}-\beta,s^{(2)}+\beta]} |\overline{\pi}_{2n}(t)-\overline{\pi}_2(s)| < 3\eta$. Combining, we conclude that $\sup_{t\in[0,s^{(2)}+1]} |\overline{\pi}_{2n}(t)-\overline{\pi}_2(t)| \leq 3\eta$ for all $n \geq n_0$. $\square$

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