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On Stochastic Comparisons of Residual Life Time at Random Time

Isha Dewan and Baha-Eldin Khaledi

Indian Statistical Institute, Delhi Centre
7, SJSS Marg, New Delhi–110 016, India

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Isha Dewan
Indian Statistical Institute
New Delhi, India

[1] [2] Baha-Eldin Khaledi
Department of Statistics
Razi University
Kermanshah, IRAN

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[1]Corresponding author
[2]Partially supported by Ordered and Spatial Data Center of Excellence of Ferdowsi University of Mashhad

Abstract

Let X_1, X_2, Θ and Θ' be independent non-negative random variables. The residual life of X_i at random time Θ, that is, $X_i^{\Theta} = X_i - \Theta | X_i > \Theta$ is considered. Some sufficient conditions which lead to the likelihood ratio ordering, the failure rate ordering, the reverse failure rate ordering and the mean residual life ordering between X_1^{Θ} and X_2^{Θ} are obtained and an application in queuing theory is explained. A set of conditions which lead to the same stochastic orderings between X_1^{Θ} and $X_1^{\Theta'}$ are also derived.

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1 Introduction

Let X be a non-negative random variable with distribution function F. The residual lifetime of X at t, $t > 0$, denoted by X^t, is a random variable whose distribution is the same as the distribution of $X - t$ given that $X > t$, that is $X^t =_{st} (X - t | X > t)$. Stochastic characteristics like survival function and mean of X^t are great tools to evaluate the stochastic behavior of X. For more details about residual random variable the reader is referred to Guess and Proschan (1988), Shaked and Shanthikumar (2007, Chapters 1 and 2), Nanda, Bhattacharjee and Balakrishnan (2010) and Cai and Zheng (2012).

If we replace t with a random variable Θ, independent of X, then the residual lifetime of X at Θ, denoted by X^{Θ}, is defined to be residual lifetime of X at random time Θ. Stochastic aging properties and stochastic comparisons of residual lifetimes at random time have been investigated by Yue and Cao (2000), Yue and Cao (2001), Li and Zuo (2004), Misra, Gupta and Dhariyal (2008) and Eryilmaz (2013). The aim of this paper is to obtain some new stochastic orderings results among residual lifetimes at random time in one sample as well as two sample problems and give simpler proofs of some known results in the literature.

First let us recall some definitions of stochastic orders that are used later in this paper. Assume the positive random variables X and Y have distribution functions F and G, survival functions $\bar{F} = 1 - F$ and $\bar{G} = 1 - G$, density functions f and g, reverse failure rate functions $\tilde{r}_X = f/F$ and $\tilde{r}_Y = g/G$ and failure rate functions $r_X = f/\bar{F}$ and $r_Y = g/\bar{G}$, respectively. The following stochastic orders are usually used to compare the random variables X and Y.

Definition 1.1. X is said to be smaller than Y in the

(i) likelihood ratio order (denoted by $X \leq_{lr} Y$) if $g(x)/f(x)$ is increasing in x;

(ii) failure rate order (denoted by $X \leq_{hr} Y$) if $\bar{G}(x)/\bar{F}(x)$ is increasing in x;

(iii) reverse failure rate order (denoted by $X \leq_{rh} Y$) if $G(x)/F(x)$ is increasing in x;

(iv) stochastic ordering (denoted by $X \leq_{st} Y$) if $\overline{F}(x) \leq \overline{G}(x)$ for every x;

(v) mean residual life order, denoted by $X \leq_{mrl} Y$, if

$$\frac{\int_t^{\infty} \bar{F}(x)dx}{\bar{F}(t)} \leq \frac{\int_t^{\infty} \bar{G}(x)dx}{\bar{G}(t)};$$

(vi) increasing convex order (denoted by $X \leq_{icx} Y$) if

$$\int_t^{\infty} \bar{F}(x)dx \leq \int_t^{\infty} \bar{G}(x)dx.$$

It is well known that $X \leq_{st} Y$ is equivalent to that

$$E[\phi(X)] \leq (\geq) \, E[\phi(Y)] \tag{1.1}$$

for all increasing (decreasing) functions $\phi : \mathcal{R} \to \mathcal{R}$, for which the expectations exist. It is also known that (cf. Shaked and Shanthikumar (2007)),

$$X \leq_{lr} Y \Rightarrow X \leq_{hr} Y \Rightarrow X \leq_{mrl} Y \Rightarrow \mathrm{E}X \leq \mathrm{E}Y$$

and

$$X \leq_{lr} Y \Rightarrow X \leq_{hr} Y \Rightarrow X \leq_{st} Y \Rightarrow X \leq_{icx} Y \Rightarrow \mathrm{E}X \leq \mathrm{E}Y.$$

We shall also use the notions of totally positive of order 2 and reverse regular of order 2. Karlin (1968) is a comprehensive reference for TP_2 and RR_2 functions.

Definition 1.2. (i) A non-negative function $h(x, y)$ is said to be totally positive of order 2 (TP_2) if

$$h(x, y)h(x', y') \geq h(x', y)h(x, y')$$

whenever $x \leq x'$ and $y \leq y'$.

(ii) A non-negative function $h(x, y)$ is said to be reverse regular of order 2 (RR_2) if

$$h(x, y)h(x', y') \leq h(x', y)h(x, y')$$

whenever $x \leq x'$ and $y \leq y'$.

Let X_1, X_2, Θ_1 and Θ_2 be independent non-negative random variables. Yue and Cao (2000) considered stochastic comparisons between $X_1^{\Theta_1}$ and $X_1^{\Theta_2}$, the residual lifetime of X_1 at two different random times Θ_1 and Θ_2 . They proved that if $\Theta_1 \leq_{rh} \Theta_2$ and X is DFR (decreasing failure rate), then $X^{\Theta_1} \leq_{st} X^{\Theta_2}$. The inequality is reversed if X is IFR (increasing failure rate). Misra, Gupta and Dhariyal (2008) in their Theorem 3.1 gave a lengthy proof for extending this result from the usual stochastic ordering to the failure rate ordering. We give a simpler proof of their result (Theorem 2.2 (c)).

Yue and Cao (2000) also showed that if $\Theta_1 \leq_{rh} \Theta_2$ and X is IMRL (increasing mean residual life), then

$$E(X^{\Theta_1}) \leq E(X^{\Theta_2}). \tag{1.2}$$

The inequality in (1.2) is reversed if X is DMRL (decreasing mean residual life). Li and Zuo (2004) extended the above expectation order result to the increasing convex order. That is

$$X^{\Theta_1} \leq_{icx} X^{\Theta_2}. \tag{1.3}$$

Misra, Gupta and Dhariyal (2008) further considered this problem and extended (1.2) to the mean residual life order. That is, they showed that if $\Theta_1 \leq_{rh} \Theta_2$ and X is IMRL (DMRL), then

$$X^{\Theta_1} \leq_{mrl} (\geq_{mrl}) X^{\Theta_2}. \tag{1.4}$$

We also give a simpler proof of the above result (Theorem 2.2 (d)).

In Section 2, we make stochastic comparisons between X_1^{Θ} and X_2^{Θ}, the residual lifetimes of X_1 and X_2 at the same random time Θ. We provide some sufficient conditions under which X_1^{Θ} is comparable with X_2^{Θ} according to the likelihood ratio order, the failure rate order, the reverse failure rate order and the mean residual order. We also make stochastic comparisons between X^{Θ_1} and X^{Θ_2}, the residual life time of X at two different random times Θ_1 and Θ_2 according to the likelihood ratio order and the reverse failure rate order. An application in queuing theory is explained in Section 3.

2 Main Results

We need the following lemma, which might be of independent interest, to prove the main results in this section.

Lemma 2.1. *Let $h_i(x,\theta)$, $i = 1, 2$, be a non-negative real valued function on $\mathbb{R} \times \mathbb{X}$, where $\mathbb{X}$ is a subset of real line. If*

(i) $h_2(x,\theta)/h_1(x,\theta)$ is increasing in x and θ and

(ii) if either $h_1(x,\theta)$ or $h_2(x,\theta)$ is TP_2 in (x,θ),

then

$$s_i(x) = \int_{\mathbb{X}} h_i(x,\theta)l(\theta)d\theta \tag{2.1}$$

is TP_2 in (x,θ), where l is a continuous function with $\int_{\mathbb{X}} l(\theta)d\theta < \infty$.

Proof. First, we prove the required result when $h_1(x,\theta)$ is TP_2 in (x,θ).

Let $\Theta^*(x)$ denote a random variable with density function given by

$$\frac{h_1(x,\theta)l(\theta)}{\int_{\mathbb{X}} h_1(x,\theta)l(\theta)d\theta}.$$

Then the assumption (ii) is equivalent to the fact that for $x_1 \leq x_2$, $\Theta^*(x_1) \leq_{lr} \Theta^*(x_2)$, which in turn implies that $\Theta^*(x_1) \leq_{st} \Theta^*(x_2)$.

Let $x_1 \leq x_2$. Then

$$\begin{aligned} \frac{s_2(x_2)}{s_1(x_2)} &= \frac{\int_{\mathbb{X}} h_2(x_2,\theta)l(\theta)d\theta}{\int_{\mathbb{X}} h_1(x_2,\theta)l(\theta)d\theta} \\ &= \int_{\mathbb{X}} \frac{h_2(x_2,\theta)}{h_1(x_2,\theta)} \frac{h_1(x_2,\theta)l(\theta)}{\int_{\mathbb{X}} h_1(x_2,\theta)l(\theta)d\theta} d\theta \\ &\geq \int_{\mathbb{X}} \frac{h_2(x_2,\theta)}{h_1(x_2,\theta)} \frac{h_1(x_1,\theta)l(\theta)}{\int_{\mathbb{X}} h_1(x_1,\theta)l(\theta)d\theta} d\theta \qquad (2.2) \\ &\geq \int_{\mathbb{X}} \frac{h_2(x_1,\theta)}{h_1(x_1,\theta)} \frac{h_1(x_1,\theta)l(\theta)}{\int_{\mathbb{X}} h_1(x_1,\theta)l(\theta)d\theta} d\theta \qquad (2.3) \\ &= \frac{s_2(x_1)}{s_1(x_1)}. \end{aligned}$$

Note that inequality (2.2) follows from the assumption (i) that $h_2(x,\theta)/h_1(x,\theta)$ is increasing in θ for each $x \in \mathbb{R}$ and the inequality (1.1). The inequality (2.3) follows from the assumption (i) that $h_2(x,\theta)/h_1(x,\theta)$ is increasing x for each $\theta \in \mathbb{X}$.

Next assume that $h_2(x,\theta)$ is TP_2 in (x,θ). Let $x_1 \le x_2$. Then

$$\begin{aligned}
\frac{s_1(x_2)}{s_2(x_2)} &= \frac{\int_{\mathbb{X}} h_1(x_2,\theta)l(\theta)d\theta}{\int_{\mathbb{X}} h_2(x_2,\theta)l(\theta)d\theta} \\
&= \int_{\mathbb{X}} \frac{h_1(x_2,\theta)}{h_2(x_2,\theta)} \frac{h_2(x_2,\theta)l(\theta)}{\int_{\mathbb{X}} h_2(x_2,\theta)l(\theta)d\theta} d\theta \\
&\le \int_{\mathbb{X}} \frac{h_1(x_2,\theta)}{h_2(x_2,\theta)} \frac{h_2(x_1,\theta)l(\theta)}{\int_{\mathbb{X}} h_2(x_1,\theta)l(\theta)d\theta} d\theta \qquad (2.4) \\
&\le \int_{\mathbb{X}} \frac{h_1(x_1,\theta)}{h_2(x_1,\theta)} \frac{h_2(x_1,\theta)l(\theta)}{\int_{\mathbb{X}} h_2(x_1,\theta)l(\theta)d\theta} d\theta \qquad (2.5) \\
&= \frac{s_1(x_1)}{s_2(x_1)}.
\end{aligned}$$

The inequalities (2.4) and (2.5) follow using arguments similar to the ones used to show inequalities (2.2) and (2.3). □

Let X and Θ be two independent non-negative random variables with distribution functions F and H, survival functions $\overline{F}$ and $\overline{H}$, density functions f and h, respectively. The residual life time of X at Θ, denoted by X^{Θ}, is defined to be a random variable with a distribution function equal to that of $X-\Theta$ given that $X > \Theta$, that is $X^{\Theta} =_{st} (X-\Theta | X > \Theta)$. Then, the density function, distribution function, survival function and mean residual life (mrl) function of X^{Θ}, are respectively given by

$$g_{X^{\Theta}}(x) = \frac{\int_0^{\infty} f(x+\theta)h(\theta)d\theta}{P(X > \Theta)}. \qquad (2.6)$$

$$G_{X^{\Theta}}(x) = \frac{\int_0^{\infty} F(x+\theta)h(\theta)d\theta}{P(X > \Theta)}, \qquad (2.7)$$

$$\overline{G}_{X^{\Theta}}(x) = \frac{\int_0^{\infty} \overline{F}(x+\theta)h(\theta)d\theta}{P(X > \Theta)}, \qquad (2.8)$$

and

$$m_{X^{\Theta}}(x) = \frac{\int_x^{\infty} \overline{G}_{X^{\Theta}}(u)du}{P(X^{\Theta} > x)}. \qquad (2.9)$$

Theorem 2.2. *Let X_i, $i=1,2$ be two independent random variables with X_i, $i=1,2$ having density function f_i, distibution function F_i, survival function $\overline{F}_i$ and mrl function m_i. Let Θ be a random variable with density function h and distribution function H. Θ is independent of X_1 and X_2.*

(a) If $X_1 \le_{lr} X_2$ and either X_1 or X_2 is ILR, then

$$X_1^{\Theta} \le_{lr} X_2^{\Theta}.$$

(b) If $X_1 \le_{rh} X_2$ and either X_1 or X_2 is $IRFR$, then

$$X_1^{\Theta} \le_{rh} X_2^{\Theta}.$$

(c) *If* $X_1 \leq_{hr} X_2$ *and either* X_1 *or* X_2 *is* DFR*, then*

$$X_1^{\Theta} \leq_{hr} X_2^{\Theta}.$$

(d) *If* $X_1 \leq_{mrl} X_2$ *and either* X_1 *or* X_2 *is* $IMRL$*, then*

$$X_1^{\Theta} \leq_{mrl} X_2^{\Theta}.$$

Proof. (a) From (2.6), the density function of X_i^{Θ} is

$$g_{X_i^{\Theta}}(x) = \frac{\int_0^{\infty} f_i(x+\theta)h(\theta)d\theta}{P(X > \Theta)}, \quad i = 1, 2.$$

In Lemma 2.1, replace $l(\theta)$ with $h(\theta)$ and $h_i(x,\theta)$ with $f_i(x+\theta)$ for $i = 1, 2$. The random variable X_i is ILR if and only if $f_i(x+\theta)$ is TP_2 in x and θ. On the other hand, $X_1 \leq_{lr} X_2$ if and only if $f_2(u)/f_1(u)$ is increasing in u which in turn implies that $f_2(x+\theta)/f_1(x+\theta)$ is increasing in x as well as θ. Combining these observations, the required result of part (a) follows from Lemma 2.1.

(b) X_i is $IRFR$ if and only if $F_i(x+\theta)$ is TP_2 in x and θ. On the other hand, $X_1 \leq_{rh} X_2$ if and only if $F_2(u)/F_1(u)$ is increasing in u which in turn implies that $F_2(x+\theta)/F_1(x+\theta)$ is increasing in x as well as θ. That is, the conditions of Lemma 2.1 (b) are satisfied by replacing the function $l(\theta)$ with $h(\theta)$ and $h_i(x,\theta)$ with $F_i(x+\theta)$, $i = 1, 2$. This proves part (b).

(c) X_i is DFR if and only if $\overline{F}_i(x+\theta)$ is TP_2 in x and θ. On the other hand, $X_1 \leq_{hr} X_2$ if and only if $\overline{F}_2(u)/\overline{F}_1(u)$ is increasing in u which in turn implies that $\overline{F}_2(x+\theta)/\overline{F}_1(x+\theta)$ is increasing in x as well as θ. That is, the conditions of Lemma 2.1 (c) are satisfied by replacing the function $l(\theta)$ with $h(\theta)$ and $h_i(x,\theta)$ with $\overline{F}_i(x+\theta)$, $i = 1, 2$. This proves part (c).

(d) Using (2.9), the mrl function of X_i^{Θ}, $i = 1, 2$ can be written as

$$\begin{aligned} m_{X_i^{\Theta}}(x) &= \frac{\int_x^{\infty} \overline{G}_{X_i^{\Theta}}(u)du}{P(X_i^{\Theta} > x)} \\ &= \frac{\int_0^{\infty} \{\int_x^{\infty} \overline{F}_i(u+\theta)du\} h(\theta)d\theta}{P(X_i^{\Theta} > x)} \\ &= \frac{\int_0^{\infty} \{\int_{x+\theta}^{\infty} \overline{F}_i(u)du\} h(\theta)d\theta}{P(X_i^{\Theta} > x)} \end{aligned}$$

X_i is $IMRL$ if and only if $\int_{x+\theta}^{\infty} \overline{F}_i(u)du$ is TP_2 in x and θ. On the other hand, $X_1 \leq_{mrl} X_2$ implies that $\int_{x+\theta}^{\infty} \overline{F}_2(u)du / \int_{x+\theta}^{\infty} \overline{F}_1(u)du$ is increasing in x and θ. That is, the conditions of Lemma 2.1 (d) are satisfied by replacing the function $l(\theta)$ with $h(\theta)$ and $h_i(x,\theta)$ with $\int_{x+\theta}^{\infty} \overline{F}_i(u)du$, $i = 1, 2$. This proves part (d). $\square$

Example 2.3. Let X_i, $i = 1, 2$ be a gamma random variable with density function

$$f(x; \alpha_i, \beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha_i)} x^{\alpha_i - 1} e^{-x\beta}, \ x > 0; \, \alpha_i > 0, \, \beta > 0.$$

If $\alpha_1 < 1$ and $\alpha_1 \leq \alpha_2$, then it is easy to see that $X_1 \leq_{lr} X_2$ and X_1 is ILR. Therefore it follows from Theorem 2.2 (a) that for any non-negative random variable Θ, $X_1^{\Theta} \leq_{lr} X_2^{\Theta}$.

Example 2.4. Let X_1 be a random variable with density function

$$f_{X_1}(x) = \left(\frac{1}{\sqrt{x}} + 1 \right) exp(-2\sqrt{x} - x), \ \ x > 0$$

and X_2 be another random variable with density function

$$f_{X_2}(x) = \left(\frac{1}{\sqrt{x}} + \frac{1}{2} \right) exp(-2\sqrt{x} - \frac{x}{2}), \ \ x > 0.$$

It is easy to see that $X_1 \leq_{hr} X_2$ and both X_1 and X_2 are DFR. Therefore it follows from Theorem 2.2 (c) that for any non-negative random variable Θ, $X_1^{\Theta} \leq_{hr} X_2^{\Theta}$. Note that in this example $X_1 \nleq_{lr} X_2$.

The following three lemmas will be used below to obtain stochastic orderings between X^{Θ_1} and X^{Θ_2}

Lemma 2.5. *(Karlin (1968), p.99) Let $g_1 : \mathbb{R} \to \mathbb{R}$ and $g_2 : \mathbb{R} \to \mathbb{R}$ be two continuous functions and f_1 and f_2 be two density functions. Suppose that*

$$\int_{\mathbb{R}} g_k(s) \, f_i(s) ds \ \textit{exists and is finite}, \quad k = 1, 2, \ i = 1, 2$$

and

(i) *$f_i(s)$ is* TP_2 (RR_2) *in $(i, s) \in \{1, 2\} \times \mathbb{R}$,*

(ii) *$g_k(s)$ is* TP_2 *in $(k, s) \in \{1, 2\} \times \mathbb{R}$,*

Then $\int g_k(s) f_i(s) \, ds$ is TP_2 (RR_2) *in $(i, k) \in \{1, 2\} \times \{1, 2\}$.*

Lemma 2.6. *(Joag-Dev, Kochar and Proschan (1995), p. 115) Let $g_1 : \mathbb{R} \to \mathbb{R}$ and $g_2 : \mathbb{R} \to \mathbb{R}$ be two differentiable functions with derivatives g_1' and g_2', and let F_1 and F_2 be two distribution functions with respective density functions f_1 and f_2, and respective survival functions $\overline{F}_1$ and $\overline{F}_2$. Suppose that*

$$\int_{\mathbb{R}} g_k(s) \, dF_i(s) \ \textit{exists and is finite}, \quad k = 1, 2, \ i = 1, 2$$

and

(i) $\overline{F}_i(s)$ *is* TP_2 *in* $(i,s) \in \{1,2\} \times \mathbb{R}$,

(ii) $g_k(s)$ *is* TP_2 *in* $(k,s) \in \{1,2\} \times \mathbb{R}$,

(iii) $g_1(s)$ *is increasing in* $s \in \mathbb{R}$, *for* $k = 1,2$.

Then $\int g_k(s) f_i(s)\, ds$ *is* TP_2 *in* $(i,k) \in \{1,2\} \times \{1,2\}$.

Lemma 2.7. *(Khaledi and Shaked (2010), p. 2490) Let* $g_1 : \mathbb{R} \to \mathbb{R}$ *and* $g_2 : \mathbb{R} \to \mathbb{R}$ *be two differentiable functions with derivatives* g_1' *and* g_2'*, and let* F_1 *and* F_2 *be two distribution functions with respective density functions* f_1 *and* f_2*, and respective survival functions* $\overline{F}_1$ *and* $\overline{F}_2$*. Suppose that*

$$\int_{\mathbb{R}} g_k(s)\, dF_i(s) \text{ exists and is finite,} \quad k = 1,2,\ i = 1,2$$

and

(i) $F_i(s)$ *is* TP_2 *in* $(i,s) \in \{1,2\} \times \mathbb{R}$,

(ii) $g_k(s)$ *is* RR_2 *in* $(k,s) \in \{1,2\} \times \mathbb{R}$,

(iii) $g_k(s)$ *is decreasing in* $s \in \mathbb{R}$, *for* $k = 1,2$.

Then $\int g_k(s) f_i(s)\, ds$ *is* RR_2 *in* $(i,k) \in \{1,2\} \times \{1,2\}$.

Theorem 2.8. *Let* Θ_i, $i = 1,2$ *be two independent random variables with* Θ_i, $i = 1,2$ *having density function* h_i*, distibution function* H_i*, survival function* $\overline{H}_i$ *and mrl function* μ_i*. Let also* X *be a random variable with density function* f*, distribution function* F*, survival function* $\overline{F}$ *.* X *is independent of* Θ_1 *and* Θ_2*.*

(a) If X *is* ILR (DLR) *and* $\Theta_1 \leq_{lr} \Theta_2$*, then*

$$X^{\Theta_1} \leq_{lr} (\geq_{lr}) X^{\Theta_2}.$$

(b) X *is* $IRFR$ *and* $\Theta_1 \leq_{hr} \Theta_2$*, then*

$$X^{\Theta_1} \leq_{rh} X^{\Theta_2}.$$

(c) (Theorem 3.1 of Misra et al. (2008)) If X *is* IFR *and* $\Theta_1 \leq_{rh} \Theta_2$*, then*

$$X^{\Theta_1} \geq_{hr} X^{\Theta_2}.$$

(d) (Theorem 3.2 of Misra et al. (2008)) If X *is* $DMRL$ *and* $\Theta_1 \leq_{rh} \Theta_2$*, then*

$$X^{\Theta_1} \geq_{mrl} X^{\Theta_2}.$$

Proof. (a) From (2.6), the density function of X^{Θ_i} is

$$g_{X^{\Theta_i}}(x) = \frac{\int_0^\infty f(x+\theta)h_i(\theta)d\theta}{P(X > \Theta_i)}, \quad i = 1, 2.$$

$f(x+\theta)$ is TP_2 (RR_2) in (x, θ), since X is ILR (DLR). $h_i(\theta)$ is TP_2 in (i, θ), since $\Theta_1 \leq_{lr} \Theta_2$. Using these results, it follows from Lemma 2.5 that the function

$$\int_0^\infty f(x+\theta)h_i(\theta)d\theta$$

is TP_2 (RR_2) in (i, x) which proves the required results of part (a).

(b) From (2.7), the distribution function of X^{Θ_i}, $\quad i = 1, 2$, is

$$G_{X^{\Theta_i}}(x) = \frac{\int_0^\infty F(x+\theta)h_i(\theta)d\theta}{P(X > \Theta_i)}.$$

$F(x+\theta)$ is TP_2 in (x, θ), since X is $IRFR$. $\overline{H}_i(\theta)$ is TP_2 in (i, θ), since $\Theta_1 \leq_{hr} \Theta_2$. Hence, it follows from Lemma 2.6 that

$$\int_0^\infty F(x+\theta)h_i(\theta)d\theta$$

is TP_2 in (i, x) which is the required result of part (b).

(c) From (2.8), the survivial function of X^{Θ_i} , $i = 1, 2$, is

$$\overline{G}_{X^{\Theta_i}}(x) = \frac{\int_0^\infty \overline{F}(x+\theta)h_i(\theta)d\theta}{P(X > \Theta_i)}.$$

$\overline{F}(x+\theta)$ is RR_2 in (x, θ), since X is IFR. $H_i(\theta)$ is TP_2 in (i, θ), since $\Theta_1 \leq_{rh} \Theta_2$. The function $\overline{F}(x+\theta)$ is decreasing in θ. Therefore, it follows from Lemma 2.7 that the function

$$\int_0^\infty \overline{F}(x+\theta)h_i(\theta)d\theta$$

is (RR_2) in (i, x) which proves part (c).

(d) The mrl function of X^{Θ_i} , $i = 1, 2$, is

$$m_{X^{\Theta_i}}(x) = \frac{\int_0^\infty \{\int_{x+\theta}^\infty \overline{F}(u)du\}h_i(\theta)d\theta}{P(X^{\Theta_i} > x)}$$

X is $DMRL$ is equivalent to $\int_{x+\theta}^\infty \overline{F}(u)du$ is RR_2 in (x, θ). $H_i(\theta)$ is TP_2 in (i, θ), since $\Theta_1 \leq_{rh} \Theta_2$. The function is decreasing in θ. Combining these results, it follows from Lemma 2.7 that $m_{X^{\Theta_i}}(x)$ is RR_2 in (i, x) which proves part (d). $\square$

3 An Application in Queuing Theory

In a $GI/G/1$ queue, let T_n with distribution F_T, denote the time between nth and $(n+1)$th arrival, W_n with distribution F_W, denote the waiting time in the queue for the nth customer, S_n with distribution F_S, denote service time of the nth customer and I with distribution H, denote the length of idle period between busy periods. It is well known that $W_{n+1} = max\{0, W_n + S_n - T_n\}$ and $I =_{st} (T - (W + S)|W + S > T)$, where st stands for equal in distribution (cf. Marshall (1968)).

Suppose that distributions of T, W and S are not completely known and only we know that the hazard rate of T is bounded by some positive known constants. That is suppose that for $0 < \lambda_1 < \lambda_2$,

$$\lambda_1 \leq r_T(t) \leq \lambda_2, \tag{3.1}$$

then it follows from Theorem 2.2 (c) that

$$\lambda_1 \leq r_I(t) \leq \lambda_2. \tag{3.2}$$

To prove this observation, let E_{λ_i}, $i = 1, 2$, be an exponential random variable with hazard rate λ_i independent of S and W. Then (3.1) is equivalent to that

$$E_{\lambda_2} \leq_{hr} T \leq_{hr} E_{\lambda_1}. \tag{3.3}$$

On the other hand, from (2.6), it is easy to see that $I_i =_{st} E_{\lambda_i}$, $i = 1, 2$, where I_i is the length of idle period between busy periods of a queue with inter-arrival time E_{λ_i} and an arbitrary servicing time. Using this observation, (3.3), the fact that exponential random variable is DFR and Theorem 2.2 (c), we obtain that

$$I_2 =_{st} E_{\lambda_2} =_{st} E_{\lambda_2}^{S+W} \leq_{hr} T^{S+W} \leq_{hr} E_{\lambda_1}^{S+W} =_{st} E_{\lambda_1} =_{st} I_1,$$

which is equivalent to (3.2). Inequalities (3.2) gives a lower bound and upper bound on the hazard rate of r_I without any IFR assumption on T. Therefore it is comparable to the ones given in Theorem 6 of Marshall (1968) and it is a generalization of Theorem 4 in Marshall (1968) which is discussed next.

If instead of (3.1) it is known that the mean residual life function of T is bounded with some known constants, that is

$$\gamma_1 \leq m_T(t) \leq \gamma_2, \tag{3.4}$$

then, using similar kind of arguments, it follows from Theorem 2.2 (d) that

$$\gamma_1 \leq m_I(t) \leq \gamma_2. \tag{3.5}$$

Inequalities (3.5) were directly proved in Marshall (1968).

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