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U-STATISTICS APPROACH TO HOLLANDER-PROSCHAN TEST FOR NBUE ALTERNATIVES

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Abstract. We develop a simple non-parametric test based on U-statistics for testing exponentiality against NBUE alternative. The proposed test is asymptotically equivalent to that of Hollander and Proschan (1975). Since the test is based on U-statistics, the study of asymptotic theory is very simple. The test statistic is shown to be asymptotically normal and consistent against the alternatives under consideration.

1. Introduction

The problem of testing exponentiality against New Better Than Used in Expectation (NBUE) alternatives has received considerable attention during the last four decades. In fact this test procedure enables engineers to develop a better replacement policies for efficient running of several systems. Test for exponentiality against NBUE alternatives was first considered by Hollander and Proschan (1975). Subsequently, various authors used different types of approaches in deriving the test statistics, see Koul (1978), Borges et al. (1984), Fernandez-Ponce et al. (1996), Belzunce et al. (2000) and Belzunce et al. (2001).

Recently, Anis and Mitra (2011) have genralized the Hollander-Proschan approach to propose a family of tests for NBUE alternatives. Both Anis and Mitra (2011) and Hollander and Proschan (1975) have shown that the asymptotic null distribution of their statistics is normal. Anis and Basu (2011) obtained an exact null distribution of the generalized Hollander-Proschan

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type test developed by Anis and Mitra (2011). Anis and Basu (2012) conducted a monte carlo study to compare the different approaches for testing exponentiality against NBUE alternative.

Motivated from these recent works, we develop a new test procedure for testing exponentiality against NBUE alternatives. We organize the paper as follows. In Section 2, based on U-statistics, we propose a non-parametric test and show that it is asymptotically equivalent to that of Hollander and Proschan (1975). Using U-statistics theory, the asymptotic properties of the test statistic are studied in Section 3. Finally, in Section 4 we give conclusions of our study.

2. A Simple Non-parametric test

Let X be a non-negative random variable with cumulative distribution function $F(.)$ and reliability function $\bar{F}(x) = P(X > x) = 1 - F(x)$. Also let $\mu = E(X) = \int_0^\infty y dF(y) < \infty$. Then the mean residual life function denoted by $m(x)$ is defined as

$$\begin{aligned} m(x) &= E(X - x | X > x), \\ &= \frac{1}{\bar{F}(x)} \int_x^\infty y dF(y) - x. \end{aligned} \tag{2.1}$$

Definition 2.1. The random variable X is said to be NBUE if

$$\mu \geq m(x), \quad \forall x \geq 0. \tag{2.2}$$

We are interested to test the hypothesis

$$H_0 : F \quad \text{is} \quad \text{exponential}$$

versus

$$H_1 : F \quad \text{is} \quad \text{NBUE} \quad \text{(and not exponential)},$$

on the basis of a random sample $X_1, X_2, \ldots, X_n$ from F. Constant mean residual life function characterizes the exponential distribution. It can be seen that $\mu - m(x)$ is zero under H_0 and is positive under the alternative hypothesis . Hence the quantity defined by

$$\gamma(F) = \int_0^\infty \bar{F}(t)(\mu - m(t))dF(t), \tag{2.3}$$

is a measure of deviation between H_0 and H_1. This measure was first considered by Hollander and Proschan (1975). They substituted the unknown distribution function F by its empirical distribution function F_n in (2.3) and obtained the following test statistics

$$\widehat{\gamma}_1(F) = \frac{1}{n^2} \sum_{i=1}^{n} (\frac{3n}{2} - 2i + \frac{1}{2}) X_{(i)}, \tag{2.4}$$

where $X_{(i)}$, $i = 1, 2, ...n$, are the order statistics. Note that this is a linear function of the total time on test statistic. Hollander and Proschan (1975) used the asymptotic normality of the total time on test statistic to develop the test procedure.

Next we obtain a simple non-parametric test based on U-statistics. One can express $\gamma(F)$ as follows :

$$\begin{aligned} \gamma(F) &= \int_0^\infty \bar{F}(t)\mu dF(t) - \int_0^\infty \bar{F}(t)m(t)dF(t) \\ &= \frac{\mu}{2} - \int_0^\infty \bar{F}(t)\Big\{\frac{1}{\bar{F}(t)} \int_t^\infty x dF(x)\Big\} dF(t) \\ &\quad + \int_0^\infty t\bar{F}(t)dF(t) \quad \text{(using} \quad \text{(2.1))}. \end{aligned}$$

Use Fubini's theorem , we get

$$\gamma(F) = \int_0^\infty 2t\bar{F}(t)dF(t) - \frac{\mu}{2} \tag{2.5}$$

Note that the distribution function of $X_{(1:n)} = min(X_1, X_2, ..., X_n)$ is given by

$$F_{x_{(1:n)}}(x) = (\bar{F}(x))^n.$$

Hence

$$E(X_{(1:n)}) = \int_0^\infty ny(\bar{F}(y))^{n-1} dF(y).$$

In Particular, when $n = 2$

$$E(X_{(1:2)}) = 2\int_0^\infty y\bar{F}(y)dF(y). \tag{2.6}$$

Substituting (2.6) in (2.5), we find

$$\gamma(F) = E(X_{(1:2)}) - \frac{\mu}{2}.$$

$U_1 = \frac{1}{n}\sum_{i=1}^{n} X_i$ is an unbiased estimator of μ.

Taking the symmetric kernel $h(X_1, X_2) = min(X_1, X_2)$, an estimator of $E(X_{(1:2)})$ based on U-statistic is given by

$$U_2 = \binom{n}{2}^{-1} \sum_{1\le i<j\le n} h(X_i, X_j).$$

Hence an unbiased estimator of $\gamma(F)$ is

$$\widehat{\gamma}(F) = U_2 - \frac{U_1}{2}. \tag{2.7}$$

After simplification we can write the above expression as

$$\widehat{\gamma}(F) = \frac{1}{n(n-1)}\sum_{i=1}^{n}(\frac{3n}{2} - 2i + \frac{1}{2})X_{(i)}, \tag{2.8}$$

Note that the expression (2.8) is different from (2.4) only by a multiplicative factor appeared in the denominator. As n^2 and $n(n-1)$ are asymptotically equivalent, the test statistics developed here is asymptotically equivalent to that of Hollander and Proschan (1975). To make the test scale invariant, we consider

$$\gamma^*(F) = \frac{\gamma(F)}{\mu}, \tag{2.9}$$

which can be estimated by

$$\widehat{\gamma}^*(F) = \frac{\widehat{\gamma}(F)}{\bar{X}}. \tag{2.10}$$

Hence the test procedure is to reject the null hypothesis in favour of the alternative H_1 for large values of $\widehat{\gamma}^*(F)$.

Next we investigate the asymptotic properties of the test statistics using the asymptotic theory of U-statistics.

3. **Asymptotic properties of the estimator**

Hollander and Proschan (1975) used the asymptotic properties of the total time on test statistic to prove the asymptotic normality and the consistency of the test statistics. Since the proposed test is based on the U-statistics, proving asymptotic properties are quite simple. The consistency of the test statistics is due to Lehmann(1951).

Theorem 3.1. *$\widehat{\gamma}(F)$ is a consistent estimator of $\gamma(F)$ against the alternatives H_1.*

The next theorem proves the asymptotic normality of the test statistics.

Theorem 3.2. *The distribution of $\sqrt{n}(\hat{\gamma}(F)-\gamma(F))$, as $n \to \infty$, is Gaussian with mean zero and variance $4\sigma_1^2+\frac{1}{4}\sigma^2-2\sigma_{12}$, where σ^2 is the variance of X, σ_1^2 is the asymptotic variance of U_2 and σ_{12} is the asymptotic covariance between $\sqrt{n}(U_1-\mu)$ and $\sqrt{n}(U_2-E(X_{1:2}))$ and the values of σ_{12} and σ_1^2 are given in equations (3.2) and (3.3) respectively.*

Proof: Consider

$$\begin{aligned} \sqrt{n}(\hat{\Delta}(F)-\Delta(F)) &= \sqrt{n}\Big(U_2-\frac{1}{2}U_1-E(X_{(2)})+\frac{1}{2}\mu\Big) \\ &= \sqrt{n}\Big(U_2-E(X_{(2)})-\frac{1}{2}(U_1-\mu)\Big). \qquad (3.1) \end{aligned}$$

Using the central limit theorem for U-statistics (Serfling (2001)), $\sqrt{n}(U_2-E(X_{(1:2)}))$ has limiting distribution

$$N\big(0,4\sigma_1^2\big) \quad \text{as} \quad n \to \infty,$$

where σ_1^2 is the asymptotic variance of U_2.

The limiting distribution of $\frac{\sqrt{n}}{2}\Big(U_1 - \mu\Big)$ is

$$N(0, \frac{1}{4}\sigma^2) \quad \text{as} \quad n \to \infty.$$

Hence

$$\sqrt{n}(\hat{\Delta}(F) - \Delta(F)) \to N\big(0, 4\sigma_1^2 + \frac{1}{4}\sigma^2 - 2\sigma_{12}\big) \quad \text{as} \quad n \to \infty,$$

where σ_{12} is specified in the theorem.

Denoting $k(x) = \int_0^x y dF(y)$, we have

$$\sigma_1^2 \quad = \quad V(X - XF(X) + k(X)) \tag{3.2}$$

The value of σ_{12} is given by

$$\begin{aligned} \sigma_{12} \quad &= \quad \sigma^2 - \int_0^\infty x^2 F(x) dF(x) + \mu \int_0^\infty x F(x) dF(x) \\ &\quad + \int_0^\infty x k(x) dF(x) - \mu \int_0^\infty k(x) dF(x). \end{aligned} \tag{3.3}$$

Next we obtain the limiting distribution of the test statistics under the null hypothesis of exponentiality.

Corollary 3.1. Let X be continuous non-negative random variable with $\bar{F}(x) = e^{-\lambda x}$, then under H_0, as $n \to \infty$, $\sqrt{n}(\widehat{\gamma}(F) - \gamma(F))$ is Normal random variable with mean zero and variance $\sigma_0^2 = \frac{1}{12\lambda^2}$.

Proof: Under H_0, it can be easily verify that

$$\sigma^2 = \frac{1}{\lambda^2}, \quad \sigma_1^2 = \frac{1}{12\lambda^2} \quad \text{and} \quad \sigma_{12} = \frac{1}{4\lambda^2}.$$

Then the result follows from Theorem 3.1.

Using Slutsky's theorem, the following result can be proved easily.

Corollary 3.2. Let X be continuous non-negative random variable with $\bar{F}(x) = e^{-\lambda x}$, then under H_0, as $n \to \infty$, $\sqrt{n}(\widehat{\gamma}^*(F) - \gamma^*(F))$ is a Normal random variable with mean zero and variance $\sigma_0^2 = \frac{1}{12}$.

Hence, for large values of n, we reject the null hypothesis of exponentiality in favour of H_1, if

$$\sqrt{12n}\widehat{\gamma}^*(F) > Z_\alpha$$

where Z_α is the upper α-percentile of $N(0,1)$.

Remark 3.1. One can also look at the problem of testing exponentiality against the dual concept called new worse than in expectation (NWUE). We reject H_0 in favour of NWUE alternative, if

$$\sqrt{12n}\widehat{\gamma}^*(F) < -Z_\alpha$$

4. **Conclusion**

We obtained a simple non-parametric test for testing exponential against NBUE alternatives. The proposed test is asymptotically equivalent to the test proposed by Hollander and Proschan (1975). Using theory of U-statistics, we showed that the test statistics is unbiased, consistent and has limiting normal distribution. One can refer to Anis and Basu (2012) for a numerical study on comparison of power of various tests available in literature for testing exponentiality against NBUE alternatives.

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