isid/ms/2014/10
August 25, 2014
http://www.isid.ac.in/ statmath/eprint

Quantum Stochastic Calculus and Quantum Gaussian Processes

K. R. Parthasarathy

Indian Statistical Institute, Delhi Centre
7, SJSS Marg, New Delhi–110 016,India

Quantum Stochastic Calculus and Quantum Gaussian Processes

K. R. Parthasarathy

Abstract In this lecture we present a brief outline of boson Fock space stochastic calculus based on the creation, conservation and annihilation operators of free field theory, as given in the 1984 paper of Hudson and Parthasarathy [9]. We show how a part of this architecture yields Gaussian fields stationary under a group action. Then we introduce the notion of semigroups of quasifree completely positive maps on the algebra of all bounded operators in the boson Fock space $\Gamma(\mathbb{C}^n)$ over $\mathbb{C}^n$. These semigroups are not strongly continuous but their preduals map Gaussian states to Gaussian states. They were first introduced and their generators were shown to be of the Lindblad type by Vanheuverzwijn [19]. They were recently investigated in the context of quantum information theory by Heinosaari, Holevo and Wolf [7]. Here we present the exact noisy Schrödinger equation which dilates such a semigroup to a quantum Gaussian Markov process.

1 Introduction

Consider a system whose state at any time is described by n real coordinates $(\xi_1(t), \xi_2(t), \ldots, \xi_n(t))$. As an example one may look at the system of a single particle moving in the space $\mathbb{R}^3$ and its state consisting of six coordinates, three for its position and three for its velocity components. Suppose a characteristic of the system described by a function f of the state is being studied. Then such a characteristic changes with time and we write

$$X(t) = f(\xi_1(t), \ldots, \xi_n(t)). \tag{1}$$

K. R. Parthasarathy
Indian Statistical Institute, Delhi Centre, 7, S. J. S. Sansanwal Marg, New Delhi 110016, e-mail: krp@isid.ac.in

Such a change is described by its differential

$$dX(t) = \sum_{j=1}^{n} \frac{\partial f}{\partial \xi_j}(\xi_1(t), \ldots, \xi_n(t))\, d\xi_j(t). \tag{2}$$

If $Y(\cdot)$ is another characteristic such that

$$Y(t) = g(\xi_1(t), \ldots, \xi_n(t)) \tag{3}$$

then

$$dX(t)Y(t) = X(t)dY(t) + Y(t)dX(t). \tag{4}$$

Note that (2) is interpreted as

$$X(t+h) - X(t) = \int_t^{t+h} \sum_{j=1}^{n} \frac{\partial f}{\partial \xi_j}(\xi_1(t), \ldots, \xi_n(t))\, \frac{d\xi_j}{dt}\, dt \tag{5}$$

In two seminal papers in 1944 and 1951, K. Ito [11], [12] developed a method for such a differential description when the path

$$\boldsymbol{\xi}(t) = (\xi_1(t,\omega), \xi_2(t,\omega), \ldots, \xi_n(t,\omega))$$

is random, i.e., subject to the laws of chance and the randomness is described by points ω in a probability space. Two paradigm examples of such random paths or trajectories come to our mind: the continous Brownian motion executed by a small particle suspended in a fluid and the jump motion of the number of radioactive particles emitted during the time interval $[0,t]$ by a radioactive substance undergoing radioactive decay. The first example yields a Gaussian process with independent increments and the second yields a Poisson jump process.

To begin with we consider a standard Brownian motion process $\boldsymbol{B}(t) = (B_1(t), B_2(t), \ldots, B_n(t))$ where B_i's are independent and each $\{B_j(t), t \geq 0\}$ is a standard Brownian motion process with continuous sample paths and independent increments, $B_j(0) = 0$ and $B_j(t)$ being normally distributed with mean 0 and variance t. To do a differential analysis of functionals of such a Brownian motion one has to define stochastic integrals with respect to such Brownian paths which are known to be of unbounded variation. Integrands in such a theory of stochastic integration are the *Ito functionals* or *nonanticipating functionals*. They are of the form

$$f(t, \boldsymbol{B}(\cdot)) = f(t, \{B(s), 0 \leq s \leq t\}), \quad 0 \leq t < \infty$$

which take real or complex values. In other words the random variable $f(t, \boldsymbol{B}(\cdot))$ depends on t and the whole Brownian path in the interval $[0,t]$. For $n+1$ such Ito functionals $f_1, f_2, \ldots, f_n$, g the Ito theory associates an integral of the form

$$X(t) = X(0) + \sum_{j=1}^{n} \int_0^t f_j(s, \boldsymbol{B}) dB_j(s) + \int_0^t g(s, \boldsymbol{B}) ds \tag{6}$$

and shows that for a large linear space of such vector-valued functionals $(f_1, f_2, \ldots, f_n, g)$, such integrals are well-defined and have many interesting properties. If (6) holds for every t in an interval $[0, T]$ we write

$$dX(t) = \sum_{j=1}^{n} f_j(t, \boldsymbol{B}) dB_j(t) + g(t, \boldsymbol{B}) dt \tag{7}$$

for $0 \leq t \leq T$ with the prescribed initial value $X(0)$. If we have another relation of the form (7), say,

$$dY(t) = \sum_{j=1}^{n} h_j(t, \boldsymbol{B}) dB_j(t) + k(t, \boldsymbol{B}) dt \tag{8}$$

in $[0, T]$ with an initial value $Y(0)$, then both $X(t) = X(t, \boldsymbol{B})$, $Y(t) = Y(t, \boldsymbol{B})$ are again Ito functionals and one can ask what is the differential of the product Ito functional $X(t)Y(t), 0 \leq t \leq T$. The famous *Ito's formula* states that

$$dX(t)Y(t) = X(t)dY(t) + Y(t)dX(t) + dX(t)dY(t) \tag{9}$$

where

$$dX(t)dY(t) = \left\{ \sum_{j=1}^{n} f_j(t, \boldsymbol{B}) h_j(t, \boldsymbol{B}) \right\} dt. \tag{10}$$

This is best expressed in the form of a multiplication table for the fundamental differentials dB_j, $1 \leq j \leq n$ and dt :

$$\begin{array}{c|cccccc} & dB_1 & dB_2 & \cdots & dB_n & dt \\ \hline dB_1 & dt & 0 & \cdots & 0 & 0 \\ dB_2 & 0 & dt & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ dB_n & 0 & 0 & \cdots & dt & 0 \\ dt & 0 & 0 & \cdots & 0 & 0 \end{array} \tag{11}$$

Here the last diagonal entry and all the nondiagonal entries are 0. This multiplication table also implies that for any twice continuously differentiable function $\varphi(t, x_1, x_2, \ldots, x_n)$ on $\mathbb{R} \times \mathbb{R}^n$, the Ito functional $X(t) = \varphi(t, B_1(t), \ldots, B_n(t))$ satisfies

$$dX(t) = \sum_{j=1}^{n} \frac{\partial \varphi}{\partial x_j}(t, B_1(t), \ldots, B_n(t))dB_j(t) + \left\{ \frac{\partial \varphi}{\partial t} + \frac{1}{2}\sum_{j=1}^{n} \frac{\partial^2 \varphi}{\partial x_j^2} \right\}(t, \boldsymbol{B}(t))dt. \tag{12}$$

Formulae (9)-(12) constitute the backbone of the Ito stochastic calculus and its diverse applications.

Ito calculus has been extended to all local semimartingales and it is an extraordinarily rich theory with applications to many areas ranging from physics and biology to economics and social sciences. For the Poisson process $\{N_\lambda(t), t \geq 0\}$ with intensity parameter λ the multiplication table for differentials has the form

$$\begin{array}{c|cc} & dN_\lambda & dt \\ \hline dN_\lambda & dN_\lambda & 0 \\ dt & 0 & 0 \end{array} \tag{13}$$

For a comprehensive account of classical stochastic calculus we refer to [10].

Coming to quantum theory we observe that both chance and noncommutativity of observables play an important role. A quantum stochastic process may be roughly described by a family $\{X_t\}$ of observables or, equivalently, selfadjoint operators in a complex and separable Hilbert space $\mathcal{H}$ together with a state ρ which is a nonnegative operator of unit trace in $\mathcal{H}$. The operators X_t at different time points may not commute with each other. However, one would like to have a 'differential' description of $\{X_t\}$ in terms of the differentials of some fundamental processes which may be viewed as quantum analogues of processes like Brownian motion and Poisson process. Then the differential description will depend on integrals of operator-valued processes with respect to the fundamental processes. Borrowing from the fact that there is a close connection between infinitely divisible distributions, classical stochastic processes with independent increments and free quantum fields on a boson Fock space as outlined in the papers of H. Araki [1], R. F. Streater [18], K. R. Parthasarathy and K. Schmidt [17] we search for observables from free field theory to provide us the fundamental processes and their differentials. Our aim would then be to describe a quantum Ito's formula or an Ito table similar to (11) and (13). This goal is achieved in the paper [9] by Hudson and Parthasarathy. We shall follow [9] and use it to construct examples of quantum Gaussian processes of the quasifree type [19], [7] as solutions of quantum stochastic differential equations.

2 Boson Fock space and Weyl operators

All the Hilbert spaces we deal with will be assumed to be complex and separable and scalar products will be expressed in the Dirac notation. To any Hilbert space $\mathcal{H}$ we associate its *boson Fock space* $\Gamma(\mathcal{H})$ defined by

$$\Gamma(\mathcal{H}) = \mathcal{C} \oplus \mathcal{H} \oplus \mathcal{H}^{\circledS^2} \oplus \cdots \oplus \mathcal{H}^{\circledS^n} \oplus \cdots . \tag{1}$$

where $\mathcal{C}$ is the 1-dimensional Hilbert space of complex numbers and $\mathcal{H}^{\circledS^n}$ is the n-fold symmetric tensor product of copies of $\mathcal{H}$. The subspace $\mathcal{H}^{\circledS^n}$ in $\Gamma(\mathcal{H})$ is called the *n-particle* subspace and $\mathcal{C}$ is called the *vacuum* subspace. For any $u \in \mathcal{H}$, define the *exponential vector* $e(u)$ associated with u by

$$e(u) = 1 \oplus u \oplus \frac{u^{\otimes^2}}{\sqrt{2!}} \oplus \cdots \oplus \frac{u^{\otimes^n}}{\sqrt{n!}} \oplus \cdots \tag{2}$$

and observe that

$$\langle e(u)|e(v)\rangle = \exp\langle u|v\rangle \quad \forall\ u, v \in \mathcal{H}. \tag{3}$$

Denote by $\mathcal{E}$ the linear manifold generated by the set of all exponential vectors in $\Gamma(\mathcal{H})$ and call $\mathcal{E}$ the *exponential domain.* Then the following properties hold:

(i) $\mathcal{E}$ is dense in $\Gamma(\mathcal{H})$;
(ii) Any finite set of exponential vectors is linearly independent;
(iii) The correspondence $u \to e(u)$ is strongly continuous.
(iv) Any bounded operator A in $\Gamma(\mathcal{H})$ is uniquely determined by the map $u \to Ae(u)$, $u \in \mathcal{H}$.
(v) Any correspondence $e(u) \to Ae(u)$, $u \in \mathcal{H}$ extends to a densely defined operator with the dense domain $\mathcal{E}$.
(vi) $e(0) = 1 \oplus 0 \oplus 0 \oplus \cdots$ is the vacuum vector and

$$\psi(u) = e^{-\|u\|^2/2} e(u) \tag{4}$$

is a unit vector. The pure state with density operator $|\psi(u)\rangle\langle\psi(u)|$ is called the *coherent state* associated with u.
(vi) The correspondence

$$W(u)e(v) = e^{-1/2\|u\|^2 - \langle u|v\rangle} e(u+v) \quad \forall\ v \tag{5}$$

is scalar product preserving on the set of all exponential vectors for every fixed u and therefore extends uniquely to a unitary operator $W(u)$ on $\Gamma(\mathcal{H})$. The operator $W(u)$ is called the *Weyl operator* associated with u.

The Weyl operators play a central role in quantum theory and particularly in quantum stochastic calculus. They obey the following multiplication relations:

$$W(u)W(v) = e^{-i\ \mathrm{Im}\langle u|v\rangle} W(u+v), \tag{6}$$
$$W(u)W(v)W(u)^{-1} = e^{-2i\,\mathrm{Im}\langle u|v\rangle} W(v) \tag{7}$$

for all u, v in $\mathcal{H}$. The correspondence $u \to W(u)$ is strongly continuous. There is no proper subspace of $\Gamma(\mathcal{H})$ invariant under all the Weyl operators. We summarize by saying that the correspondence $u \to W(u)$ is a projective unitary and irreducible representation of the additive group $\mathcal{H}$. Furthermore, the correspondence $t \to W(tu)$, $t \in \mathbb{R}$ is a strongly continuous unitary representation of $\mathbb{R}$ and hence by Stone's theorem there exists a unique selfadjoint operator $p(u)$ such that

$$W(tu) = e^{-it\,p(u)}, t \in \mathbb{R}, u \in \mathcal{H}. \tag{8}$$

If $\mathcal{H}$ is finite dimensional and $u \to W'(u)$ is a strongly continuous map from $\mathcal{H}$ into the unitary group of a Hilbert space K obeying the multiplication relations (6) with W replaced by W' then there exists a Hilbert space h and a Hilbert space isomorphism $U : K \to \Gamma(\mathcal{H}) \otimes h$ such that

$$UW'(u)U^{-1} = W(u) \otimes I_h \quad \forall\, u \in \mathcal{H}.$$

In particular, if $W'(\cdot)$ is also irreducible then h is one dimensional and $W(\cdot)$ and $W'(\cdot)$ are unitarily equivalent through U. This is the well-known Stone-von Neumann theorem.

From (6) and (8) one obtains the following commutation relations:

$$[p(u), p(v)] = 2i\,\mathrm{Im}\langle u|v\rangle \quad \forall\, u, v \in \mathcal{H} \tag{9}$$

on the domain $\mathcal{E}$. The domain $\mathcal{E}$ is a common core for all the selfadjoint operators $p(u)$ and $\mathcal{E}$ is contained in the domain of all products of the form $p(u_1)p(u_2)\ldots p(u_n)$, $u_i \in \mathcal{H}$. Writing

$$q(u) = -p(iu) \tag{10}$$

$$a(u) = \frac{1}{2}(q(u) + ip(u)) \tag{11}$$

$$a^\dagger(u) = \frac{1}{2}(q(u) - ip(u)) \tag{12}$$

we obtain closable operators in the domain $\mathcal{E}$ obeying the following commutation relations:

$$\begin{aligned} [a(u), a(v)] &= 0, \\ [a^\dagger(u), a^\dagger(v)] &= 0, \\ [a(u), a^\dagger(v)] &= \langle u|v\rangle. \end{aligned} \tag{13}$$

Furthermore the correspondence $u \to a(u)$ is antilinear, $u \to a^\dagger(u)$ is linear,

$$\begin{aligned} a(u)e(v) &= \langle u|v\rangle e(v), \\ \langle e(v)|a(u)e(w)\rangle &= \langle a^\dagger(u)e(v)|e(w)\rangle \end{aligned}$$

for all u, v, w in $\mathcal{H}$. If we denote the closures of $a(u)$ and $a^\dagger(v)$ on $\mathcal{E}$ by the same symbols one obtains their actions on n-particle vectors as follows:

$$\begin{aligned} a(u)e(0) &= 0, \\ a(u)v^{\otimes^n} &= \sqrt{n}\langle u|v\rangle v^{\otimes^{n-1}} \\ a^\dagger(u)v^{\otimes^n} &= (n+1)^{-1/2}\sum_{r=0}^{n} v^{\otimes^r}\otimes u\otimes v^{\otimes^{n-r}}. \end{aligned} \tag{14}$$

These relations show that $a(u)$ maps $\mathcal{H}^{ⓢ^n}$ into $\mathcal{H}^{ⓢ^{n-1}}$ whereas $a^\dagger(u)$ maps $\mathcal{H}^{ⓢ^n}$ into $\mathcal{H}^{ⓢ^{n+1}}$. In view of this property $a(u)$ and $a^\dagger(u)$ are called *annihilation* and *creation* operators associated with u.

For any selfadjoint operator H in $\mathcal{H}$ with domain $D(H)$ denote by $\mathcal{E}(D(H))$ the linear manifold generated by $\{e(u)|u \in D(H)\}$. For any unitary operator U in $\mathcal{H}$ define the operator $\Gamma(U)$ in $\Gamma(\mathcal{H})$ by putting

$$\Gamma(U)e(u) = e(Uu), \quad u \in \mathcal{H},$$

noting that it is scalar product preserving and hence extending it to a unitary operator on $\Gamma(\mathcal{H})$. Then for any two unitary operators U, V in $\mathcal{H}$ one has the relation $\Gamma(U)\Gamma(V) = \Gamma(UV)$. Now we see that $\Gamma(e^{-itH})$, $t \in \mathbb{R}$ is a strongly continuous one parameter unitary group and therefore, by Stone's theorem, there exists a selfadjoint operator $\lambda(H)$ in $\Gamma(\mathcal{H})$ such that

$$\Gamma(e^{-itH}) = e^{-it\ \lambda(H)}, t \in \mathbb{R}. \tag{15}$$

$\Gamma(U)$ is called the *second quantization* of U and $\lambda(H)$ is called the *differential second quantization* of the selfadjoint operator H. For $v \in D(H)$ one has the relation

$$\lambda(H)v^{\otimes^n} = \sum_{r=0}^{n-1} v^{\otimes^r}\otimes \lambda(H)v\otimes v^{\otimes^{n-r-1}} \tag{16}$$

Thus $\lambda(H)$ sends an n-particle vector to an n-particle vector and hence $\lambda(H)$ is called the *conservation operator* associated with H.

If H is any bounded operator in $\mathcal{H}$ one can express

$$H = \frac{H+H^\dagger}{2} + i\frac{H-H^\dagger}{2i}$$

and put

$$\lambda(H) = \lambda\left(\frac{H+H^\dagger}{2}\right) + i\lambda\left(\frac{H-H^\dagger}{2i}\right).$$

If we denote by $\mathcal{B}(\mathcal{H})$ the algebra of all bounded operators on $\mathcal{H}$ with the adjoint operation $\dagger$ as involution then the following hold:

(i) The map $H \to \lambda(H)$ is linear on $\mathcal{B}(\mathcal{H})$;
(ii) $\lambda(H)^\dagger = \lambda(H^\dagger)$;
(iii) On the domain $\mathcal{E}$ the following commutation relations hold:

$$\begin{aligned}&\text{(a)}\ \ [\lambda(H_1), \lambda(H_2)] = \lambda([H_1, H_2]),\\&\text{(b)}\ \ [a(u), \lambda(H)] = a(H^\dagger u),\\&\text{(c)}\ \ [\lambda(H), a^\dagger(u)] = a^\dagger(Hu)\end{aligned} \tag{17}$$

In terms of the Weyl operators and second quantization of unitary operators in $\mathcal{H}$ one has the following relations:

$$\begin{aligned}W(u)\Gamma(U) &= e^{\frac{1}{2}\|u\|^2} e^{a^\dagger(u)} \Gamma(U) e^{-a(U^{-1}u)},\\ W(u) &= e^{a(u)-a^\dagger(u)}\\ \Gamma(U)W(u)\Gamma(U)^{-1} &= W(Uu)\end{aligned}$$

for all $u \in \mathcal{H}$ and unitary operators in $\mathcal{H}$. The first two identities are to be interpreted in a weak sense on the domain $\mathcal{E}$.

It is to be emphasized that the Weyl operators and second quantization yield a rich harvest of interestig observables $p(u)$ and $\lambda(H)$ as u varies in $\mathcal{H}$ and H varies in the set of all selfadjoint operators. In the next section we shall examine their statistical properties in the vacuum state.

3 Statistics of observables arising from Weyl operators and secod quantization

First, we begin with a general remark about the mechanism by which classical stochastic processes arise in the quantum framework. Suppose $\mathcal{H}_0$ is the Hilbert space of a quantu system and ρ is a state in $\mathcal{H}_0$, i.e., a positive operator of unit trace. Let $\{X_t, t \in T\}$ be a commuting family of selfadjoint operators in $\mathcal{H}_0$ or, equivalently, observables. Then write

$$\varphi_{t_1,t_2,\ldots,t_n}(x_1, x_2, \ldots, x_n) = \operatorname{Tr} \rho \ \exp\ i \sum_{j=1}^{n} x_j X_{t_j} \tag{1}$$

for any $x_1, x_2, \ldots, x_n$ in $\mathbb{R}$ and $\{t_1, t_2, \ldots, t_n\} \subset T$. Then $\varphi_{t_1,t_2,\ldots,t_n}$ is the characteristic function (or Fourier transform of a probability measure $\mu_{t_1,t_2,\ldots,t_n}$ in $\mathbb{R}^n$ or, more precisely, $\mathbb{R}^{\{t_1,t_2,\ldots,t_n\}}$, so that

$$\varphi_{t_1,t_2,\ldots,t_n}(x_1, x_2, \ldots, x_n) = \int_{\mathbb{R}^n} e^{i\boldsymbol{x}\cdot\boldsymbol{y}} \mu_{t_1,\ldots,t_n}(d\boldsymbol{y}). \tag{2}$$

The family $\{\mu_{t_1,\ldots,t_n}, \{t_1, t_2, \ldots, t_n\} \subset T\}$ of finite dimensional probability distributions is consistent in the sense of Kolmogorov and therefore, by Kolmogorov's consistency theorem, there exists a unique probability measure μ on the product Borel space $\mathbb{R}^T$ whose projection on any $\mathbb{R}^{t_1,\ldots,t_n}$ is $\mu_{t_1,\ldots,t_n}$. In other words, one obtains a classical stochastic process described by μ. If T

is a real vector space and the map $t \to X_t$ is linear we see that, in any state ρ, one obtains a classical random field over T.

Now we choose a Hilbert space $\mathcal{H}$ and fix an orthonormal basis $\{e_1, e_2, \ldots\}$ in $\mathcal{H}$. Define $\mathcal{A}_{\mathbb{R}}$ to be the real linear space of all bounded selfadjoint operators in $\mathcal{H}$ and fix a subspace $\mathcal{A}_0 \subset \mathcal{A}_{\mathbb{R}}$ where any two elements of $\mathcal{A}_0$ commute with each other. In the boson Fock space $\Gamma(\mathcal{H})$ defined by (1) in section 2 consider the following three families of observables:

(i) $\{p(u)|u \in \mathcal{H}_{\mathbb{R}}\}$,
(ii) $\{q(u)|u \in \mathcal{H}_{\mathbb{R}}\}$,
(iii) $\{\lambda(H)|H \in \mathcal{A}_0\}$,

where $\mathcal{H}_{\mathbb{R}}$ is the closed real linear subspace spanned by $e_i, e_2, \ldots$.

Then each of the three families above is commutative, thanks to (8), (9), (10), (15) and (17) in section 2. Furthermore, the correspondence $u \to p(u)$, $u \to q(u)$ and $H \to \lambda(H)$ are all real linear. Of course, one must take care of the fact that the operators involved are unbounded. By our general remarks at the beginning of this section each of the three families (i)-(iii) of observables determines a classical random field in any fixed state ρ in $\Gamma(\mathcal{H})$.

We now specialize to the case when $\rho = |\psi(u_0)\rangle\langle\psi(u_0)|$ is the coherent state corresponding to u_0 defined by (2) and (4) in Section 2. Note that

$$\begin{aligned}&\langle\psi(u_0)|e^{-itp(u)}|\psi(u_0)\rangle\\ &= \langle\psi(u_0)|W(tu)|\psi(u_0)\rangle\\ &= e^{2it\,\mathrm{Im}\langle u_0|u\rangle - \frac{1}{2}t^2\|u\|^2}\end{aligned}$$

which is the characteristic function of the normal distribution with mean $2\,\mathrm{Im}\langle u_0|u\rangle$ and variance $\|u\|^2$. This also implies that

$$\begin{aligned}&\mathrm{Tr}\, e^{-i\sum_{j=1}^{n} t_j p(u_j)}|\psi(u_0)\rangle\langle\psi(u_0)|\\ &= \exp\left\{2i\sum_{j=1}^{n} t_j \mathrm{Im}\langle u_0|u_j\rangle - \frac{1}{2}\sum_{i,j} t_i t_j\langle u_i|u_j\rangle\right\}.\end{aligned}$$

In other words $\{p(u)|u \in \mathcal{H}_{\mathbb{R}}\}$ is a classical Gaussian random field in the state $|\psi(u_0)\rangle\langle\psi(u_0)|$ with mean functional $m(\cdot)$ and covariance kernel $K(.,.)$ given by

$$m(u) = 2\mathrm{Im}\langle u_0|u\rangle, K(u,v) = \langle u|v\rangle$$

for all $u, v \in \mathcal{H}_{\mathbb{R}}$. By the same arguments $\{q(u)|u \in \mathcal{H}_{\mathbb{R}}\}$ is again a classical Gaussian random field with the same covariance kernel but its mean functional m' is given by $m'(u) = 2\mathrm{Re}\langle u_0|u\rangle$. Note that $[q(u), p(v)] = 2i\langle u|v\rangle$ for all $u, v \in \mathcal{H}_{\mathbb{R}}$ and therefore $q(u)$ and $p(v)$ need not commute with each other.

Let us now examine the case of the third family. We have for any fixed $u \in \mathcal{H}$ and any selfadjoint operator H in $\mathcal{H}$

$$
\begin{aligned}
&\langle\psi(u)|e^{-it\lambda(H)}|\psi(u)\rangle \\
&= e^{-\|u\|^2}\langle e(u)|\Gamma(e^{-itH})|e(u)\rangle \\
&= e^{-\|u\|^2}\langle e(u)|e(e^{-itH}u)\rangle \\
&= \exp\langle u|e^{-itH}-1|u\rangle.
\end{aligned}
$$

If P^H is the spectral measure of H so that

$$H=\int_{\mathbb{R}} xP^H(dx)$$

then

$$
\begin{aligned}
&\langle\psi(u)|e^{-it\lambda(H)}|\psi(u)\rangle \\
&= \exp\int(e^{itx}-1)\langle u|P^H(dx)|u\rangle.
\end{aligned}
$$

In other words the distribution of the observable $\lambda(H)$ in the coherent state $|\psi(u)\rangle\langle\psi(u)|$ is the infinitely divisible distribution with characteristic function having the Lévy-Khinchine representation

$$\exp\int(e^{itx}-1)\mu_{H,u}(dx)$$

where

$$\mu_{H,u}(E)=\langle u|P^H(E)|u\rangle$$

for any Borel set $E\subset\mathbb{R}$. Thus $\{\lambda(H)|H\in\mathcal{A}_0\}$ in the coherent state $|\psi(u)\rangle\langle\psi(u)|$ realizes a classical random field over the vector space $\mathcal{A}_0\subset\mathcal{A}_{\mathbb{R}}$ for which $\lambda(H)$ has the infinitely divisible distribution with Lévy measure $\mu_{H,u}$ described above.

We observe that $\{p(u),u\in\mathcal{H}\}$ is a real linear but noncommutative family of observables such that, in a fixed coherent state, each $p(u)$ has a Gaussian distribution. It is natural to call the pair $\{\{p(u),u\in\mathcal{H}\},|\psi(u_0)\rangle\langle\psi(u_0)|\}$ a *quantum Gaussian field*. Similarly, $\{\lambda(H)|H\in\mathcal{A}_{\mathbb{R}}\}$ where $\mathcal{A}_{\mathbb{R}}$ is the real linear space of all bounded selfadjoint operators in $\mathcal{H}$, is a real linear but noncommutative space of observables such that in the state $|\psi(u)\rangle\langle\psi(u)|$, $\lambda(H)$ has the infinitely divisible distribution with Lévy measure $\langle u|P^H(\cdot)|u\rangle$ for every H. Thus it is natural to call the pair $\{\{\lambda(H)|H\in\mathcal{A}_{\mathcal{R}}\},|\psi(u)\rangle\langle\psi(u)|\}$ a *quantum Lévy field.* It looks like an interesting problem to examine the nature of the most general quantum Gaussian and Lévy fields.

Our discussions in this section also show that the observable fields $\{p(u),u\in\mathcal{H}\}$ and $\{\lambda(H),H\in\mathcal{A}_{\mathbb{R}}\}$ constitute a natural ground for constructing a theory of stochastic integration and paving the way for introducing a quantum stochastic calculus.

4 *G*-stationary quantum Gaussian processes

Let G be a second countable metric group acting continuously on a second countable metric space A and let $K(\alpha, \beta)$, $\alpha, \beta \in A$ be a continuous positive definite G-invariant kernel so that

$$K(g\alpha, g\beta) = K(\alpha, \beta) \quad \forall\, \alpha, \beta \in A, g \in G. \tag{1}$$

Then by the GNS principle there exists a Gelfand triple $(\mathcal{H}, \lambda, \pi)$ consisting of a Hilbert space $\mathcal{H}$, a map $\lambda : A \to \mathcal{H}$ and a strongly continuous unitary representation $g \to \pi(g)$ of G such that the following properties hold:

(i) λ is continuous and $\mathcal{H}$ is the closed linear span of $\{\lambda(\alpha), \alpha \in A\}$;
(ii) $K(\alpha, \beta) = \langle \lambda(\alpha) | \lambda(\beta) \rangle \quad \forall\, \alpha, \beta \in A$;
(iii) $\pi(g)\lambda(\alpha) = \lambda(g\alpha) \quad \forall\, g \in G, \alpha \in A$;
(iv) If there is another triple $(\mathcal{H}', \lambda', \pi')$ satisfying (i), (ii) and (iii) then there exists a Hilbert space isomorphism $U : \mathcal{H} \to \mathcal{H}'$ such that $U\lambda(\alpha) = \lambda'(\alpha) \forall \alpha$ and $U\pi(g)U^{-1} = \pi'(g) \forall\, g \in G$.
For a proof we refer to [17].

Now consider the boson Fock space $\Gamma(\mathcal{H})$ where $(\mathcal{H}, \lambda, \pi)$ is a Gelfand triple described above and associated with $K(.,.)$. Define the observables

$$q(\alpha) = \frac{1}{\sqrt{2}}(a(\lambda(\alpha)) + a^\dagger(\lambda(\alpha))),$$
$$p(\alpha) = \frac{1}{i\sqrt{2}}(a(\lambda(\alpha)) - a^\dagger(\lambda(\alpha))$$

$a(u)$ and $a^\dagger(u)$ are the annihilation and creation operators associated with $u \in \mathcal{H}$. Then

$$[q(\alpha), q(\beta)] = [p(\alpha), p(\beta)] = i \operatorname{Im} K(\alpha, \beta)$$
$$[q(\alpha), p(\beta)] = i \operatorname{Re} K(\alpha, \beta)$$

for all $\alpha, \beta \in A$. We shall now examine the probability distribution of an arbitrary finite real linear combination of the form

$$Z = \sum_i (x_j q(\alpha_j) + y_j p(\alpha_j))$$

where $\alpha_j \in A$, and x_j, y_j are real scalars. If we write $z_j = x_j + iy_j$ then

$$itZ = a^\dagger \left(it \frac{\sum z_j \lambda(\alpha_j)}{\sqrt{2}} \right) - a \left(\frac{it \sum z_j \lambda(\alpha_j)}{\sqrt{2}} \right)$$

and

$$e^{itZ} = W\left(\frac{it}{\sqrt{2}}\sum_j z_j\alpha_j\right)$$

is the unitary Weyl operator for any $t \in \mathbb{R}$. Thus

$$\langle e(0)|e^{itZ}|e(0)\rangle = e^{-\frac{t^2}{4}\sum \bar{z}_j z_k K(\alpha_j,\alpha_k)}.$$

This shows that, in the vacuum state, the observable Z has the normal distribution $N(0, \frac{1}{2}\sum_{j,k}\bar{z}_j z_k K(\alpha_j,\alpha_k))$.

Furthermore

$$\Gamma(\pi(g))\begin{bmatrix} q(\alpha) \\ p(\alpha) \end{bmatrix}\Gamma(\pi(g))^{-1} = \begin{bmatrix} q(g\alpha) \\ p(g\alpha) \end{bmatrix},$$

where $\Gamma(\pi(g))$ is the second quantization of $\pi(g)$. Thus the observable Z and $\Gamma(\pi(g))Z\,\Gamma(\pi(g))^{-1}$ have the same normal distribution.

This shows that the family of observables $\{q(\alpha), p(\alpha), \alpha \in A\}$ constitute a G-invarinat quantum Gaussian process in the vacuum state.

If K is a real positive definite kernel then each of the families $\{q(\alpha), \alpha \in A\}$ and $\{p(\alpha), \alpha \in A\}$ is commutative and therefore each of them executes a classical G-stationary Gaussian process.

5 Quantum stochastic calculus and a noisy Schrödinger equation

Consider a quantum system S whose states are described by density operators in a Hilbert space $\mathcal{H}_S$. Suppose that this system is coupled to a bath or an extermal environment whose states are described by density operators in a boson Fock space of the form

$$\mathcal{H} = \Gamma(L^2(0,\infty)\otimes\mathbb{C}^d) \tag{1}$$

where $e_i = (0,\ldots,0,1,0\ldots,0)$ with 1 in the i-th place, $i = 1,2,\ldots,d$ is fixed as a canonical orthonormal basis in $\mathbb{C}^d$. We write

$$\widetilde{\mathcal{H}} = \mathcal{H}_S \otimes \mathcal{H} \tag{2}$$
$$\widetilde{\mathcal{H}}_t = \mathcal{H}_S \otimes \Gamma(L^2[0,t]\otimes\mathbb{C}^d) \tag{3}$$
$$\widetilde{\mathcal{H}}^t = \Gamma(L^2([t,\infty]\otimes\mathbb{C}^d) \tag{4}$$
$$\widetilde{\mathcal{H}}^{(s,t)} = \Gamma(L^2[s,t]\otimes\mathbb{C}^d), 0 \le s \le t < \infty \tag{5}$$

Then for any $0 < t_1 < t_2 < \cdots < t_n < \infty$,

$$\mathcal{H}^{(0,\infty)} = \mathcal{H}^{(0,t_1)} \otimes \mathcal{H}^{(t_1,t_2)} \otimes \cdots \otimes \mathcal{H}^{(t_{n-1},t_n)} \otimes \mathcal{H}^{t_n}$$

where the identification of both sides can be achieved through products of exponential vectors from different components. The noise accumulated from the bath during the time period (s,t) admits a description through observables in the Hilbert space $\mathcal{H}^{(s,t)}$. This also suggests that noise can be described by observables in a general continuous tensor product Hilbert space but the boson Fock space $\mathcal{H}$ is the simplest such model in which $\mathbb{C}^d$ means that there are d degrees of freedom in the selection of noise.

We introduce the following family of noise processes:

$$\Lambda_0^i(t) = I_S \otimes a\left(1_{[0,t]} \otimes e_i\right) \tag{6}$$

$$\Lambda_i^0(t) = I_S \otimes a^\dagger\left(1_{[0,t]} \otimes e_i\right) \tag{7}$$

$$\Lambda_j^i(t) = I_S \otimes \lambda(I_{[0,t]} \otimes |e_j\rangle\langle e_i|), \tag{8}$$

$$\Lambda_0^0(t) = t\ I_{\widetilde{\mathcal{H}}} \tag{9}$$

where the indices i,j vary in $\{1,2,\ldots,d\}$, $1_{[0,t]}$ is the indicator function of $[0,t]$ as an element of $L^2([0,\infty])$, $1_{[0.t]}$ is the operator of multiplication by $1_{[0,t]}$ in $L^2([0,\infty))$ and I_S, $I_{\widetilde{\mathcal{H}}}$ denote respectively the identity operators in $\mathcal{H}_S$, $\widetilde{\mathcal{H}}$. We shall use Greek letters $\alpha,\beta,\ldots$ to indicate indices in $\{0,1,2,\ldots,d\}$. All the operators $\Lambda_\beta^\alpha(t)$ in (6)-(9) are well-defined on the linear manifold generated by elements of the form $\psi \otimes e(f)$, $\psi \in \mathcal{H}_S$ and $f \in L^2(\mathbb{R}_+) \otimes \mathbb{C}^d$, a Hilbert space which can be viewed as the space of $\mathbb{C}^d$-valued functions on $\mathbb{R}_+$ which are norm square integrable. We interpret $\{\Lambda_0^0(t)\}$ as *time process*, $\{\Lambda_0^i(t)\}$ and $\{\Lambda_i^0(t)\}$ as *annihilation* and *creation processes* of *type* (or *colour*) i and $\{\Lambda_j^i(t)\}$ as a *conservation process* which *changes the type i to type j*. It is interesting to note that, for $0 < s < t < \infty$, $\Lambda_\beta^\alpha(t) - \Lambda_\beta^\alpha(s)$ as an operator is active only in the sector $\mathcal{H}^{(s,t)}$ of the continuous tensor product $\mathcal{H}$ defined in (1)-(5). The processes $\{\Lambda_\beta^\alpha(t)\}$ will be the fundamental processes with respect to which stochastic integrals can be defined.

A family $X = \{X(t), 0 \le t < \infty\}$ of operators in $\widetilde{\mathcal{H}}$ is said to be *adapted* if, for each t, there exists an operator X_t in $\widetilde{\mathcal{H}}_t$ such that $X(t) = X_t \otimes I^t$ where I^t is the identity operator in $\mathcal{H}^t$. An adapted process X is said to be *simple* if there exists a partition $0 < t_1 < t_2 < \cdots < t_n < \cdots < \infty$ of $[0,\infty)$ so that $t_n \to \infty$ as $n \to \infty$ and

$$X(t) = \sum_{j=0}^{\infty} X(t_j) 1_{[t_j,t_{j+1})}(t) \quad \forall\ t \tag{10}$$

where $t_0 = 0$. If $X_1, X_2, \ldots, X_k$ are simple adapted processes it is clear that by taking the intersection of their respective partitions one can express X_j in the form (10) in terms of a partition independent of j.

Now suppose $\{E_\beta^\alpha\}$ is a family of simple adapted processes with respect to a single partition $0 < t_1 < t_2 < \cdots$. Then define the (quantum) stochastic integral

$$\int_0^t E^\alpha_\beta(s)d\Lambda^\beta_\alpha(s) = \sum_j E^\alpha_\beta(t_j)\left\{\Lambda^\alpha_\beta(t_{j+1}\wedge t) - \Lambda^\alpha_\beta(t_j \wedge t)\right\} \tag{11}$$

where we adopt the Einstein convention that repeated index means summation with respect to that index. Here $a \wedge b$ denotes the minimum of a and b.

The stochastic integral (11) as a function of t is again an adapted process. Quantum stochastic calculus tells us that the definition (11) can be completed to extend the notion of stochastic integral to a rich class of adapted processes. For details we refer to the paper by R. L. Hudson and K. R. Parthasarathy (1984) and also the book [14]. With this completed definition we consider adapted processes of the form

$$X(t) = X(0) + \int_0^t E^\alpha_\beta(s)d\Lambda^\beta_\alpha(s) \tag{12}$$

where $X(0) = X_0 \otimes I$ with X_0 an operator in $\mathcal{H}_S$ and I, the identity operator in $\mathcal{H}$. We express (12) as

$$dX(t) = E^\alpha_\beta(t)d\Lambda^\beta_\alpha(t) \tag{13}$$

with initial value X_0. It is interesting to note that $E^\alpha_\beta(t)$ is active in $\widetilde{\mathcal{H}}_t$ whereas $d\Lambda^\beta_\alpha(t)$ is active in $\mathcal{H}^{(t,t+dt)}$ and therefore $E^\alpha_\beta(t)$ and $d\Lambda^\beta_\alpha(t)$ commute with each other.

Now suppose $Y(t)$ is another adapted process with initial value Y_0 and

$$dY(t) = F^\alpha_\beta(t)d\Lambda^\beta_\alpha(t). \tag{14}$$

Then it is natural to ask, in the light of our initial discussions in Section 1, what is the diffeential of $X(t)Y(t)$. The answer to this question was provided by Hudson and Parthasarathy [9]: In fact

$$d(X(t)Y(t)) = X(t)dY(t) + (dX(t))Y(t) + dX(t)dY(t) \tag{15}$$

where

$$X(t)dY(t) = X(t)F^\alpha_\beta(t)d\Lambda^\beta_\alpha(t), \tag{16}$$

$$(dX(t))Y(t) = E^\alpha_\beta(t)Y(t)d\Lambda^\beta_\alpha(t), \tag{17}$$

$$dX(t)dY(t) = E^\alpha_\beta(t)F^\gamma_\varepsilon(t)d\Lambda^\beta_\alpha(t).d\Lambda^\varepsilon_\gamma(t) \tag{18}$$

$$d\Lambda^\beta_\alpha(t)d\Lambda^\varepsilon_\gamma(t) = \widehat{\delta}^\beta_\gamma d\Lambda^\varepsilon_\alpha(t), \tag{19}$$

$$\widehat{\delta}^\beta_\gamma = \begin{cases} \delta^\beta_\gamma & \text{if} \quad \beta \neq 0, \gamma \neq 0 \\ 0 & \text{otherwise} \end{cases} \tag{20}$$

Equations (15)-(20) completely describe how the classical Leibnitz formula gets corrected by the product of the differentials of $X(t)$ and $Y(t)$ in terms of the products of the differentials of the fundamental processes $\{\Lambda^{\alpha}_{\beta}, \alpha, \beta \in \{0, 1, \ldots, d\}$. This is the '*quantum Ito's formula*'. Equations (19) and (20) constitute the boson Fock quantum stochastic calculus version of the classical Ito's tables (11) and (13) of Section 1 for Brownian motion and Poisson process. If we write

$$Q_i(t) = (\Lambda^i_0(t) + \Lambda^0_i(t))^{\sim}, 1 \leq i \leq d, \tag{21}$$

$\sim$ denoting closure over $\mathcal{E}$ then $(Q_1, Q_2, \ldots, Q_d)$ is a collection of commuting observables and they execute the d-dimensional Brownian motion process in the vacuum state $|\psi(0)\rangle\langle\psi(0)|$ and equations (19) and (20) imply that $dQ_i(t)dQ_j(t) = \delta_{ij}dt$, which is the classical Ito correction formula for Brownian motion.

If we define

$$N_i(t) = \sqrt{\lambda_i}Q_i(t) + \Lambda^i_i(t) + \lambda_i t \tag{22}$$

then $\{N_i(t)\}$ is a commutative process which executes a classical Poisson process with intensity λ_i in the Fock vacuum state and

$$dN_i(t)dN_j(t) = \delta_{ij}dN_j(t) \tag{23}$$

for any i, j. This is Ito's formula for the classical Poisson process.

Suppose L^{α}_{β}, $\alpha, \beta \in \{0, 1, \ldots, d\}$ are bounded operators in the system Hilbert space $\mathcal{H}_S$ and

$$L^{\alpha}_{\beta}(t) = L^{\alpha}_{\beta} \otimes I_{\mathcal{H}} \quad \forall\ t \geq 0$$

where $I_{\mathcal{H}}$ is the identity operator in the bath Hilbert space $\mathcal{H}$. Thus we obtain a family of constant adapted processes. We now write a quantum stochastic differential equation of the exponential type:

$$dU(t) = \{L^{\alpha}_{\beta}(t)d\Lambda^{\beta}_{\alpha}\}U(t), t \geq 0 \tag{24}$$

with the initial condition $U(0) = I$. Then we have the following theorem:

Theorem 1 (Hudson and Parthasarathy [9]). *Equation* (24) *with the initial condition $U(0) = I$ has a unique unitary operator-valued adapted process as a solution if and only if*

$$L^{\alpha}_{\beta} + (L^{\beta}_{\alpha})^{\dagger} + (L^i_{\alpha})^{\dagger}L^i_{\beta} = L^{\alpha}_{\beta} + (L^{\beta}_{\alpha})^{\dagger} + L^{\alpha}_i(L^{\beta}_i)^{\dagger} = 0 \tag{25}$$

for each α, β in $\{0, 1, \ldots, d\}$.

Remark 1. Suppose $L^{\alpha}_{\beta} = 0$ whenever $\alpha \neq 0$ or $\beta \neq 0$. Then condition (25) becomes $L^0_0 + L^{0\dagger}_0 = 0$, i.e., $L^0_0 = -iH$ where H is a bounded selfadjoint operator and equation (24) takes the form

$$dU(t) = -iHU(t) \tag{26}$$

which is the familiar Schrödinger equation with energy operator H. In view of this property we say that (24) with condition (25) is a *noisy Schrödinger equation.*

Note that (26) has a unique unitary solution with the initial condition $U(0) = I$ even when H is any unbounded selfadjoint operator. Thus it is natural to examine (24) for a unique unitary solution even when L^α_β are unbounded operators and (25) holds on a dense domain or in the form sense. There is some recent progress in this direction in the paper [4].

A case of special interest arises when $L^i_j = 0$ for all i, j in $\{1, 2, \ldots, d\}$. Writing $L^i_0 = L_i$ and

$$\Lambda^i_0 = A_i, \quad \Lambda^0_i = A^\dagger_i$$

and using (25), equation (24) takes the form

$$dU(t) = \left\{ \sum_{j=1}^{n} (L_j dA^\dagger_j - L^\dagger_j dA_j) - (iH + \frac{1}{2}\sum_{j=1}^{n} L^\dagger_j L_j) \right\} U(t) \tag{27}$$

where A_j and $A^\dagger_j$ are the annihilation and creation processes of type j. Here the operators L_j and the selfadjoint operator H may all be unbounded.

A further specialization to the case when the initial Hilbert space $\mathcal{H}_S$ is $\Gamma(\mathbb{C}^n)$, each L_j is of the form $a(u_j) + a^\dagger(v_j)$ with u_j, v_j in $\mathbb{C}^n$ and H can be expressed as a second degree polynomial of creation and annihilation operators in $\Gamma(\mathbb{C}^n)$ is of great interest. F. Fagnola [4] has shown the existence of unique unitary solutions in the case $n = 1$ and has communicated to me that his proofs in [4] go through for every finite n. In the next section we shall show how this can be used to construct quantum Gaussian Markov processes of the quasifree type.

We conclude this section with a note on the *Heisenberg equation in the presence of noise.* Consider the unitary solution $\{U(t)\}$ of (24) under conditions (25). For any bounded operator X in the system Hilbert space $\mathcal{H}_S$ define

$$j_t(X) = U(t)^\dagger X \otimes IU(t), \quad t \geq 0 \tag{28}$$

where I is the identity operator in the boson Fock space $\Gamma(L^2 \otimes \mathbb{C}^d)$. From conditions (25) it follows that there exist bounded operators L_i, $1 \leq i \leq d$, H and S^i_j, $i, j \in \{1, 2, \ldots, d\}$ such that

(i) H is selfadjoint

(ii) the matrix operator $((S^i_j))$, $i, j \in (1 \leq i \leq d)$ in $\mathcal{H}_S \otimes \mathbb{C}^d$ is unitary and

$$L^i_j = \begin{cases} S^i_j - \delta^i_j & \text{if } i, j \in \{1, 2, \ldots, d\}, \\ L_i & \text{if } 1 \le i \le d, j = 0 \\ -\sum\limits_k L^\dagger_k S^k_j & \text{if } 1 \le j \le d, i = 0, \\ -\{iH + \frac{1}{2}\sum_{k=1}^d L^\dagger_k L_k\} & \text{if } i = 0, j = 0. \end{cases} \tag{29}$$

Then it follows from quantum Ito's formula that

$$dj_t(X) = j_t(\theta^\alpha_\beta(X)) d\Lambda^\beta_\alpha \tag{30}$$

where

$$\theta^\alpha_\beta(X) = \begin{cases} \sum_{k=1}^d (S^k_j)^\dagger X S^k_j - \delta^i_j X \quad \text{if} \quad \alpha = i, \beta = j, \\ \sum_{k=1}^d (S^k_i)^\dagger [X, L_k] \quad \text{if} \quad \alpha = i, \beta = 0, \\ \sum_{k=1}^d [L^\dagger_k, X] S^k_j \quad \text{if} \quad \alpha = 0, \beta = j, \\ i[H, X] - \frac{1}{2}\sum_{k=1}^d (L^\dagger_k L_k X + X L^\dagger_k L_k - 2 L^\dagger_k X L_k) \quad \text{if} \quad \alpha = \beta = 0 \end{cases} \tag{31}$$

Equations (26), (28) and (29) describe the dynamics of the system observable X in the presence of quantum noise generated by the fundamental processes $\Lambda^\alpha_\beta(t)$ which includes time. Suppose $S^i_j = \delta^i_j$ and $L_k = 0 \, \forall \, k$. Then (30) reduces to

$$dj_t(X) = j_t(i[H, X])$$

which is the classical Heisenberg equation with energy operator H. Thus (28) and (29) together deserve to be called a *Heisenberg equation in the presence of noise* or *a Heisenberg-Langevin equation.*

Suppose we take the Fock vacuum conditional expectation of $j_t(X)$ and write

$$\langle f | T_t(X) | g \rangle = \langle f \otimes \psi(0) | j_t(X) | g \otimes \psi(0) \rangle.$$

Then $T_t(X)$ is an operator in $\mathcal{H}_S$ and the fact that all the true noise operators $\Lambda^\alpha_\beta(s)$ with $\alpha \neq 0$ or $\beta \neq 0$ annihilate the vacuum vector $|\psi(0)\rangle$, it follows that

$$dT_t(X) = T_t(\theta^0_0(X)) dt$$

where θ^0_0 is given by the last equation in (31), namely

$$\theta^0_0(X) = i[H, X] - \frac{1}{2}\sum_{k=1}^d (L^\dagger_k L_k X + X L^\dagger_k L_k - 2 L^\dagger_k X L_k).$$

In other words θ^0_0 is the generator of the semigroup of completely positive maps $\{T_t, t \ge 0\}$. The expression θ^0_0, rather remarkably, coincides with the well-known generator in the form obtained by Gorini, Kassakowski, Sudarshan [6] and Lindblad [13] in 1976. This shows a way to construct stationary quantum Markov processes mediated by quantum dynamical semigroups of completely positive maps.

6 Quantum Gaussian Markov processes and stochastic differential equations

Consider the boson Fock space $\Gamma(\mathbb{C}^n)$ where $\mathbb{C}^n$ is equipped with the canonical orthonormal basis $e_j = (0, 0, \ldots, 0, 1, 0, \ldots, 0)^T$, 1 being the j-th coordinate, $j = 1, 2, \ldots, n$. Define

$$a_j = a(e_j), \quad a_j^\dagger = a(e_j)^\dagger$$
$$q_j = \frac{a_j + a_j^\dagger}{\sqrt{2}}, \quad p_j = \frac{a_j - a_j^\dagger}{i\sqrt{2}}$$

Then $p_1, \ldots, p_n$ and $q_1, \ldots, q_n$ satisfy the canonical Heisenberg commutation relations and therefore can be viewed as momentum and position observables of a quantum system with n degrees of freedom. A state ρ in $\Gamma(\mathbb{C})$ is called *Gaussian* if its quantum Fourier transform $\widehat{\rho}(\boldsymbol{u}) = \operatorname{Tr} \rho W(\boldsymbol{u})$ has the form

$$\widehat{\rho}(\boldsymbol{u}) = \exp\left\{-i\sqrt{2}\left(\boldsymbol{\ell}^T\boldsymbol{x} - \boldsymbol{m}^T\boldsymbol{y}\right) - \left(\boldsymbol{x}^T, \boldsymbol{y}^T\right) S \begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{pmatrix}\right\} \quad \forall\ \boldsymbol{u} \in \mathbb{C}^n \tag{1}$$

where $\boldsymbol{x} = \mathrm{Re}\boldsymbol{u}$, $\boldsymbol{y} = \mathrm{Im}\boldsymbol{u}$ and S is a real $2n \times 2n$ matrix satisfying the matrix inequality

$$2S + i\begin{bmatrix} 0 & -\mathrm{I_n} \\ \mathrm{I_n} & 0 \end{bmatrix} \geq 0, \tag{2}$$

I_n being the identity matrix of order n. In such a case one has $\boldsymbol{\ell} = \operatorname{Tr} \boldsymbol{p}\rho$, $\boldsymbol{m} = \operatorname{Tr} \boldsymbol{q}\rho$ and S is the covariance matrix of the observables $(X_1, X_2, \ldots, X_{2n}) = (p_1, p_2, \ldots, p_n, -q_1, \ldots, -q_n)$ so that the ij-the entry of S is $\operatorname{Tr} \frac{X_iX_j + X_jX}{2}\rho - \mathrm{Tr}X_i\rho\ \mathrm{Tr}X_j\rho$. So we call a state ρ satisfying (1), a Gaussian state with *momentum mean* vector $\boldsymbol{\ell}$, *position mean* vector $\boldsymbol{m}$ and *covariance matrix S*. For a detailed account of the properties of such Gaussian states with a finite degree of freedom we refer to Holevo [8] and Parthasarathy [15], [16].

Now we look at some transformations of Gaussian states. Define the correspondence

$$T(W(\boldsymbol{z})) = W(R^{-1}AR\boldsymbol{z}) \exp -\frac{1}{2}(R\boldsymbol{z})^T B(R\boldsymbol{z}), \boldsymbol{z} \in \mathbb{C}^n \tag{3}$$

where A and B are real $2n \times 2n$ matrices, $B \geq 0$

$$R\boldsymbol{z} = \begin{bmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{bmatrix}, \boldsymbol{x} = \mathrm{Re}\boldsymbol{z}, \boldsymbol{y} = \mathrm{Im}\ \boldsymbol{z}. \tag{4}$$

For any mean zero Gaussian state ρ with covariance matrix S we have from (1)

$$\operatorname{Tr} \rho\, T(W(\boldsymbol{z})) = \exp -(R\boldsymbol{z})^T(A^TSA + \frac{1}{2}B)R\boldsymbol{z}.$$

The right hand side will be the quantum Fourier transform of a Gaussian state with covariance matrix $A^T SA + \frac{1}{2}B$ if and only if

$$2\left(A^T SA + \frac{1}{2}B\right) + iJ_{2n} \geq 0 \tag{5}$$

where

$$J_{2n} = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix} \tag{6}$$

Note that (5) holds for every Gaussian covariance matrix S of order $2n$ if

$$B + i(J_{2n} - A^T J_{2n} A) \geq 0. \tag{7}$$

It is known from a result of Vanheuerzweijn [19] and the discussions at the beginning of the paper [7] by Heinosaari, Holevo and Wolf that for any pair (A, B) of $2n \times 2n$ matrices satisfying (7), the correspondence defined by (3) extends uniquely to a completely positive map T on the algebra of all bounded operators on $\Gamma(\mathbb{C}^n)$.

Now consider a one parameter family $\{T_t\}$ of such completely positive maps determined by a family of pairs (A_t, B_t) obeying (7). Simple algebra shows that the semigroup condition $T_t T_s = T_{t+s}$ holds if and only if the pairs (A_t, B_t) obey the relations

$$\begin{aligned} A_t A_s &= A_{t+s} \\ B_s + A_s^T B_t A_s &= B_{t+s} \quad \forall\ s, t \geq 0. \end{aligned}$$

If we write $A_t = e^{tK}$ then it turns out that any continous solution B_t has the form

$$B_t = \int_0^t e^{sK^T} C e^{sK} ds$$

for some matrix C. It is clear that B_t is positive if and only if C is positive. Condition (7) for the pairs (A_t, B_t) can be written as

$$\int_0^t e^{sK^T} \left(C - i\left\{K^T J_{2n} + J_{2n} K\right\}\right) e^{sK} ds \geq 0$$

for every $t \geq 0$. This is equivalent to the condition

$$C + i\left(K^T J_{2n} + J_{2n} K\right) \geq 0,$$

since C, K and J_{2n} are all real. Thus we have the following proposition.

Proposition 1. *(Vanheuerzweijn) Let (K, C) be a pair of $2n \times 2n$ real matrices satisfying the matrix inequality*

$$C + i\left(K^T J_{2n} + J_{2n} K\right) \geq 0 \tag{8}$$

where J_{2n} is defined by (6). *Then there exists a one parameter semigroup $\{T_t\}$ of completely positive maps on the algebra of all operators in $\varGamma(\mathbb{C}^n)$ satisfying*

$$T_t(W(\boldsymbol{z})) = W(R^{-1}e^{tK}R\boldsymbol{z}) \exp -\frac{1}{2}\left(R(\boldsymbol{z})^T \int_0^t e^{sK^T} C e^{sK} ds\right) R\boldsymbol{z} \tag{9}$$

for all $\boldsymbol{z} \in \mathbb{C}^n$.

The semigroup $\{T_t\}$ in Proposition 1 is not strongly continuous. However, we can analyse the time derivatives of expressions of the form $\langle e(\boldsymbol{u})|T_t(W(\boldsymbol{z}))|e(\boldsymbol{v})\rangle$. The analysis of such derivatives leads to a formal quantum stochastic differential equation of the noisy Heisenberg type.

Proposition 2. *Let $\{T_t\}$ be the semigroup defined by* (9). *Then*

$$\begin{aligned}&\frac{d}{dt}\langle e(\boldsymbol{u})\,|T_t(W(\boldsymbol{z}))|e(\boldsymbol{v})\rangle|_{t=0}\\ &= \langle e(\boldsymbol{u})\,|\mathcal{L}(W(\boldsymbol{z}))|\, e(\boldsymbol{v})\rangle \quad \forall\, \boldsymbol{z}, \boldsymbol{u}, \boldsymbol{v} \in \mathbb{C}^n,\end{aligned}$$

where

$$\begin{aligned}\mathcal{L}(W(\boldsymbol{z})) = \big\{&a^\dagger(R^{-1}KR\boldsymbol{z}) - a(R^{-1}KR\boldsymbol{z})\\ &+\frac{1}{2}\left\{\langle R^{-1}KR\boldsymbol{z}\rangle - \langle \boldsymbol{z}|R^{-1}KR\boldsymbol{z}\rangle - (R\boldsymbol{z})^T CR\boldsymbol{z}\right\} W(\boldsymbol{z}).\end{aligned}$$

Proof. This is straightforward differentiation using the definitions of the operators $a(\cdot)$, $a^\dagger(\cdot)$, $W(\cdot)$ and the commutation relation $W(\boldsymbol{u})a(\boldsymbol{v})W(\boldsymbol{u})^{-1} = a(\boldsymbol{v}) - \langle \boldsymbol{v}|\boldsymbol{u}\rangle$ for all $\boldsymbol{u}, \boldsymbol{v}$ in $\mathbb{C}^n$.

Proposition 3. *Let $\{T_t\}$ be the completely positive semigroup determined by the pair (K, C) in Proposition 1 by* (9). *For any state ρ in $\varGamma(\mathbb{C}^n)$ let $T_t'(\rho)$ be the state defined by*

$$\operatorname{Tr}\, T_t'(\rho)W(\boldsymbol{z}) = \operatorname{Tr} \rho\, T_t(W(\boldsymbol{z})) \quad \forall\, \boldsymbol{z} \in \mathbb{C}^n.$$

Denote by $\rho_g(\boldsymbol{\ell}, \boldsymbol{m}, S)$ the Gaussian state with momentum mean vector $\boldsymbol{\ell}$, position mean vector $\boldsymbol{m}$ and covariance matrix S. Then

$$T_t'\left(\rho_g(\boldsymbol{\ell}, \boldsymbol{m}, S)\right) = \rho_g(\boldsymbol{\ell}_t, \boldsymbol{m}_t, S_t)$$

where

$$\begin{aligned}\begin{bmatrix}\boldsymbol{\ell}_t\\ -\boldsymbol{m}_t\end{bmatrix} &= e^{tK^T}\begin{bmatrix}\boldsymbol{\ell}\\ -\boldsymbol{m}\end{bmatrix}\\ S_t &= e^{tK^T} S e^{tK} + \frac{1}{2}\int_0^t e^{sK^T} C e^{sK} ds,\ t \geq 0.\end{aligned}$$

Remark 2. It is known from a general theory [3] that $\{T_t\}$ can be dilated to a quantum Markov process. If the initial state of the Markov process is Gaussian it follows that the state of the system at any time t is Gaussian. We may call the dilation of $\{T_t\}$ a quantum Gaussian Markov process.

We shall now present examples when the dilated process can be obtained by a unitary evolution driven by a quantum stochastic differential equation.

Consider the Hilbert space

$$\widehat{\mathcal{H}} = \Gamma(\mathbb{C}^n) \otimes \Gamma(L^2(\mathbb{R}_+)).$$

Let

$$L = a(\boldsymbol{u}) + a^{\dagger}(\boldsymbol{v}), \tag{10}$$

where $\boldsymbol{u}, \boldsymbol{v}$ are fixed in $\mathbb{C}^n$. Then L is an operator in the initial Fock space $\Gamma(\mathbb{C}^n)$. Let

$$\begin{aligned} A(t) &= a(1_{[0,t]}), \\ A^{\dagger}(t) &= a^{\dagger}(1_{[0,t]}), \quad t \geq 0 \end{aligned}$$

be the annihilation and creation processes in the Fock space $\Gamma(L^2(\mathbb{R}_+))$. Following the notations of Section 5 consider the specialized form of the stochastic differential equation (27), Section 5:

$$dU(t) = (LdA^{\dagger} - L^{\dagger}dA - \frac{1}{2}L^{\dagger}Ldt)U(t), t \geq 0 \tag{11}$$

with $U(0) = I$ where L is now the unbounded operator given by (10). Then, by Fagnola's theorem [4] it follows that there exists a unique adapted unitary operator-valued process satisfying (11) and therefore one can define the operators

$$j_t(X) = U(t)^{\dagger}(X \otimes 1)U(t), t \geq 0$$

for any bounded operator X in $\Gamma(\mathbb{C}^n)$. If we choose $X = W(\boldsymbol{z}), \boldsymbol{z} \in \mathbb{C}^n$, then the Markovian dynamical semigroup $\{T_t^L, t \geq 0\}$ determined by the flow $\{j_t\}$ satisfies the Lindblad equation

$$\begin{aligned} \frac{d}{dt}T_t^L\ (W(\boldsymbol{z}))|_{t=0} &= -\frac{1}{2}\left\{L^{\dagger}LW(\boldsymbol{z}) + W(\boldsymbol{z})L^{\dagger}L - 2L^{\dagger}W(\boldsymbol{z})L\right\} \\ &= -\frac{1}{2}\left\{L^{\dagger}[L, W(\boldsymbol{z})] + [W(\boldsymbol{z}), L^{\dagger}]L\right\}. \end{aligned}$$

The commutation relations between $a(\boldsymbol{u})$ and $W(\boldsymbol{z})$ lead to the relations:

$$\begin{aligned} &\frac{d}{dt}\ \left(T_t^L(W(\boldsymbol{z}))\right|_{t=0} \\ &= \left\{a^{\dagger}\left(\frac{\overline{\lambda(\boldsymbol{z})}\boldsymbol{v} - \lambda(\boldsymbol{z})\boldsymbol{u}}{2}\right) - a\left(\frac{\overline{\lambda(\boldsymbol{z})}\boldsymbol{v} - \lambda(\boldsymbol{z})u}{2}\right) - \frac{1}{2}|\lambda(\boldsymbol{z})|^2\right\} W(\boldsymbol{z}) \end{aligned} \tag{12}$$

for all $\boldsymbol{z} \in \mathbb{C}^n$, where

$$\lambda(\boldsymbol{z}) = \langle \boldsymbol{z}|\boldsymbol{u}\rangle + \langle \boldsymbol{v}|\boldsymbol{z}\rangle \tag{13}$$

and the equations are to be understood in the weak sense on the exponential domain $\mathcal{E}$, the linear manifold generated by exponential vectors in $\Gamma(\mathbb{C}^n)$. Comparing (12) and the equation for the generator $\mathcal{L}$ in Proposition 2 we see that $\mathcal{L}(W(\boldsymbol{z}))$ coincides with the right hand side of (12) if the following hold for all $\boldsymbol{z} \in \mathbb{C}^n$:

$$R^{-1}KR\boldsymbol{z} = \frac{\overline{\lambda(\boldsymbol{z})}\boldsymbol{v} - \lambda(\boldsymbol{z})\boldsymbol{u}}{2}, \tag{14}$$

$$\frac{1}{2}\left\{\langle R^{-1}KR\boldsymbol{z}|\boldsymbol{z}\rangle \ - \ \langle \boldsymbol{z}|R^{-1}KR\boldsymbol{z}\rangle - (R\boldsymbol{z})^T CR\boldsymbol{z}\right\} = -\frac{1}{2}\left|\lambda(\boldsymbol{z})\right|^2 \tag{15}$$

where $\lambda(\boldsymbol{z})$ is given by (13) and (K, C) is the pair of $2n \times 2n$ real matrices in Proposition 1. Solving for the pair (K, C) in (14) and (15) for a given pair $\boldsymbol{u}, \boldsymbol{v}$ in $\mathbb{C}^n$ one obtains the following

$$K = \frac{1}{2}\begin{bmatrix} -\mathrm{Re}(\overline{\boldsymbol{u}} - \boldsymbol{v})(\boldsymbol{u} + \overline{\boldsymbol{v}})^T & \mathrm{Im}(\overline{\boldsymbol{u}} - \boldsymbol{v})(\boldsymbol{u} + \overline{\boldsymbol{v}})^T \\ \mathrm{Im}(\overline{\boldsymbol{u}} - \boldsymbol{v})(\boldsymbol{u} + \overline{\boldsymbol{v}})^T & -\mathrm{Re}(\overline{\boldsymbol{u}} - \boldsymbol{v})(\boldsymbol{u} + \overline{\boldsymbol{v}})^T \end{bmatrix} \tag{16}$$

$$C = \mathrm{Re}\begin{bmatrix} \overline{\boldsymbol{u}} + \boldsymbol{v} \\ i(\overline{\boldsymbol{u}} - \boldsymbol{v}) \end{bmatrix}\left[(\boldsymbol{u} + \overline{\boldsymbol{v}})^T, -i(\boldsymbol{u} - \overline{\boldsymbol{v}})^T\right] \tag{17}$$

and

$$C + i\left(K^T J_{2n} + J_{2n}K\right) = \begin{bmatrix} \overline{\boldsymbol{u}} + \boldsymbol{v} \\ i(\overline{\boldsymbol{u}} - \boldsymbol{v}) \end{bmatrix}\begin{bmatrix} \overline{\boldsymbol{u}} + \boldsymbol{v} \\ i(\overline{\boldsymbol{u}} - \boldsymbol{v}) \end{bmatrix}^\dagger \geq 0. \tag{18}$$

We denote by $(K(\boldsymbol{u}, \boldsymbol{v}), C(\boldsymbol{u}, \boldsymbol{v}))$ the pair (K, C) determined by (16) and (17). Thus the Markov process determined by the pair $(K(u, v), C(u, v))$ according to the semigroup $\{T_t\}$ in (9) of Proposition 1 is completely described by the noisy Schrödinger equation

$$\begin{aligned} dU(t) = \big\{&(a(\boldsymbol{u}) + a^\dagger(\boldsymbol{v}))dA^\dagger - (a^\dagger(\boldsymbol{u}) + a(\boldsymbol{v}))dA \\ &-\frac{1}{2}(a^\dagger(\boldsymbol{u}) + a(\boldsymbol{v}))(a(\boldsymbol{u}) + \boldsymbol{a}^\dagger(\boldsymbol{v}))dt\Big\} U(t). \end{aligned} \tag{19}$$

with $U(0) = I$.

We shall now consider the general case of a pair (K, C) of $2n \times 2n$ real matrices satisfying the condition (8) and show how the semigroup $\{T_t\}$ in (9) can be dilated to a quantum Markov process driven by a noisy Schrödinger equation. To this end we write

$$C + i(KJ_{2n} + J_{2n}K) = \sum_{j=1}^{r}\begin{bmatrix} \boldsymbol{u}_j \\ \boldsymbol{v}_j \end{bmatrix}\begin{bmatrix} \boldsymbol{u}_j \\ \boldsymbol{v}_j \end{bmatrix}^\dagger$$

where $\left\{ \begin{bmatrix} \boldsymbol{u}_j \\ \boldsymbol{v}_j \end{bmatrix}, 1 \leq j \leq r \right\}$ is a set of mutually orthogonal vectors in $\mathbb{C}^{2n}$ so that $\boldsymbol{u}_j, \boldsymbol{v}_j$ belong to $\mathbb{C}^n$ and r is the rank of the left hand side when it is not the zero matrix. If the left hand side vanishes then $C = 0$ and K is an element of the Lie algebra $sp(2n)$. Now write

$$L_j = a(\boldsymbol{u}_j) + a^\dagger(\boldsymbol{v}_j), 1 \leq j \leq r \quad \text{if } r \neq 0. \tag{20}$$

and define the matrices $K(\boldsymbol{u}_j, \boldsymbol{v}_j)$, $C(\boldsymbol{u}_j, \boldsymbol{v}_j)$ as the right hand side of (16) and (17) respectively with $\boldsymbol{u}, \boldsymbol{v}$ replaced by $\boldsymbol{u}_j, \boldsymbol{v}_j$ respectively. Then

$$C = \sum_{j=1}^{r} C(\boldsymbol{u}_j, \boldsymbol{v}_j) \tag{21}$$

$$K = \sum_{j=1}^{r} K(\boldsymbol{u}_j, \boldsymbol{v}_j) + K' \tag{22}$$

where $K' \in sp(2n)$. Then $\{e^{tK'}, t \in \mathbb{R}\}$ is a one parameter group in $sp(2n)$. By Stone-von Neumann theorem we know that there exists a one parameter group $\{e^{-itH}, t \in \mathbb{R}\}$ of unitary operators in $\Gamma(\mathbb{C}^n)$ satisfying the relations

$$e^{-itH} W(\boldsymbol{z}) e^{itH} = W(R^{-1} e^{tK'} R \boldsymbol{z}) \quad \forall \boldsymbol{z} \in \mathbb{C}^n, t \in \mathbb{R} \tag{23}$$

where R is defined by (4). Now observe from [2] that the selfadjoint operator H can be expressed as a second degree polynomial in the annihilation and creation operators $a_j, a_j^\dagger$, $1 \leq j \leq n$ in $\Gamma(\mathbb{C}^n)$. Conversely, given any selfadjoint operator H of this form one obtains the relation (23) for some element K' in $sp(2n)$.

Now consider the noisy Schrödinger equation

$$dU(t) = \left\{ \sum_{j=1}^{r} (L_j dA_j^\dagger - L_j^\dagger dA_j) - (iH + \frac{1}{2} \sum_{j=1}^{r} L_j^\dagger L_j) \right\} U(t)$$

where L_j is given by (20) and H is the selfadjoint operator satisfying (23) with K' as in (22). Then by Fagnola's theorem for the case of finite number of degrees of freedom we observe that

$$T_t(X) = \mathbb{E}_{0]} U(t)^\dagger X \otimes 1 U(t), \quad t \geq 0$$

where $\mathbb{E}_{0]}$ is the Fock vacuum conditional expectation of quantum stochastic calculus. Thus we have described completely the dilation of the completely positive semigroup in Proposition 1. In view of Proposition 3, such a dilation is a quantum Gaussian Markov process.

References

[1] Araki, H, *Factorizable representations of current algebra,* Publ. RIMS, Kyoto Univ. Ser. A, 5 (1970) 361-422.

[2] Arvind, Dutta B, Mukunda N. and Simon R, *The real symplectic groups in quantum mechanics and optics,* Pramana - J. Phys. 45 (1995) 471-497.

[3] Bhat B. V. R. and Parthasarathy K. R. *Kolmogorov's existence theorem for Markov processes in C^*-algebras,* Proc. Ind. Acad. Sci. (Math. Sci.) 104 (1994) 253-262.

[4] Fagnola F. *On Quantum stochastic differential equations with unbounded coefficients,* Probab. Th. and Rel. Fields 86 (1990) 501-516.

[5] Fagnola F. *Private communication.*

[6] Gorini V. Kossakowski A. and Sudarshan E. C. G., *Completely positive dynamical semigroups of n-level systems,* J. Math. Phys. 17 (1976) 821-825.

[7] Heinosaari T., Holevo A. S. and Wolf M. M. *The semigroup structure of Gaussian channels,* J. Quantum Inf. Comp. 10 (2010) 0619-0635.

[8] Holevo A. S. *Probablistic and Statistical Aspects of Quantum Theory* (1982) (Amsterdam: North Holland).

[9] Hudson R. L. and Parthasarathy K. R., *Quantum Ito's formula and stochastic evolutions,* Commun. Math. Phys. 93(1984) 301-323.

[10] Ikeda N. and Watanabe S. *Stochastic Differential Equations and Diffusion Processes* (1989) (Amsterdam: North Holland).

[11] Ito K., *Stochastic integral,* Proc. Imp. Acad. Tokyo 20 (1944) 519-524.

[12] Ito K., *On a fomula concerning stochastic differentials,* Nagoya Math. J. 3 (1951) 55-65.

[13] Lindblad G., *On the generators of quantum dynamical semigroups,* Commun. Math. Phys. 48 (1976) 119-130.

[14] Parthasarathy K. R., *An Introduction to Quantum Stochastic Calculus* (1992) (Basel: Birkhauser).

[15] Parthasarathy K. R., *What is a Gaussian state?* Commun. Stoch. Anal. 4 (2010) 143-160.

[16] Parthasarathy K. R. *The symmetry group of Gaussian states in $L^2(\mathbb{R}^n)$, in Prokhorov and Contemporary Probability* (2013) 349-369 (Eds. Shiryaev A. N., Varadhan S. R. S. and Pressman E. L., Berlin: Springer).

[17] Parthasarathy K. R. and Schmidt K., *Positive Definite kernels,* Continuous Tensor Products and Central Limit Theorems of Probability Theory, Springer LNM 272 (1972) (Berlin).

[18] Streater R. F., *Current commutation relations,Continuous tensor products and infinitely divisible group representations,* Rendiconti di sc. Inst. di Fisica E. Fermi, Vol x1 (1969) 247-263.

[19] Vanheuverzwijn P., *Generators of completely positive semigroups,* Ann. Inst. H. Poincaré Sect. A (N.S.) 29 (1978) 123-138.