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Positive linear maps and spreads of matrices

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POSITIVE LINEAR MAPS AND SPREADS OF MATRICES

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There is an interesting theorem in linear algebra which says that the eigenvalues of a normal matrix are more spread out than its diagonal entries; i.e., if $A = [a_{ij}]$ is an $n \times n$ normal matrix with eigenvalues $\lambda_1(A), \ldots, \lambda_n(A)$, then

$$\max_{i,j} \ |a_{ii} - a_{jj}| \ \leq \max_{i,j} \ |\lambda_i(A) - \lambda_j(A)| \, . \tag{1}$$

It is customary to call the quantity on the right-hand side of (1) the *spread* of A, and denote it by $\mathrm{spd}(A)$. Then the inequality (1) can be stated as

$$\mathrm{spd}(\mathrm{diag}(A)) \leq \mathrm{spd}(A). \tag{2}$$

One proof of this goes as follows. Let $\langle x, y\rangle$ be the standard inner product on $\mathbb{C}^n$ defined as $\langle x, y\rangle = \sum\limits_{i=1}^{n} \overline{x_i} \ y_i$, and let $\|x\| = \langle x, x\rangle^{1/2}$ be the associated norm. The set

$$W(A) = \{\langle x, Ax\rangle : \|x\| = 1\} \, , \tag{3}$$

is called the *numerical range* of the matrix A. If A is normal, then using the spectral theorem, one can see that $W(A)$ is the convex polygon spanned by the eigenvalues of A. So $\mathrm{spd}(A)$ is equal to the diameter $\mathrm{diam}(W(A))$. The diagonal entry $a_{ii} = \langle e_i, Ae_i\rangle$ evidently is in $W(A)$. So, we have the inequality (1).

The *Toeplitz-Hausdorff Theorem* is the statement that for every matrix A, the numerical range $W(A)$ is a convex set. It contains all the eigenvalues of A (in (3) choose x to be an eigenvector of A). So, we always have $\mathrm{diam}\, W(A) \geq \mathrm{spd}(A)$. Chapter 1 of [4] contains a comprehensive discussion of the numerical range, and all these facts can be found there.

In the special case when A is Hermitian, we can arrange its eigenvalues in decreasing order as $\lambda_1^{\downarrow}(A) \geq \cdots \geq \lambda_n^{\downarrow}$. Then $W(A)$ is the interval $\left[\lambda_n^{\downarrow}(A), \lambda_1^{\downarrow}(A)\right]$, and the inequality (1) says

$$\max_{i,j} \ |a_{ii} - a_{jj}| \ \leq \lambda_1^{\downarrow}(A) - \lambda_n^{\downarrow}(A). \tag{4}$$

The inequality (2) is not always true for arbitrary matrices. For example, the 2×2 matrix $\begin{bmatrix} 1 & 1/4 \\ -1 & 0 \end{bmatrix}$ has eigenvalue $1/2$ with multiplicity 2. In this case $\operatorname{spd}(A) = 0$, but $\operatorname{spd}(\operatorname{diag}(A)) = 1$.

It is not always easy to find the eigenvalues of a matrix, and the importance of relations like (1) lies in the information they give about eigenvalues in terms of matrix entries. Many authors have found different lower bounds for $\operatorname{spd}(A)$ in which the left-hand side of (1) is replaced by a larger quantity or by some other function of entries of A. The aim of this note is to propose a method by which many of the known results, and some new ones, can be obtained.

Let $\mathbb{M}(n)$ be the space of all $n \times n$ complex matrices. A linear map Φ from $\mathbb{M}(n)$ to $\mathbb{M}(k)$ is said to be *positive* if $\Phi(A)$ is positive semidefinite whenever A is. It is said to be *unital* if $\Phi(I) = I$. In the special case when $k = 1$, such a Φ is called a positive, unital, linear functional, and it is customary to represent it by the lower case letter φ. We refer the reader to [1] for properties of such maps.

The space $\mathbb{M}(n)$ is a Hilbert space with the inner product $\langle A, B\rangle = \operatorname{tr} A^*B$. As a consequence, every linear functional on $\mathbb{M}(n)$ has the form $\varphi(A) = \operatorname{tr} AX$ for some matrix X. This functional is positive if and only if X is positive semidefinite, and unital if and only if $\operatorname{tr} X = 1$. (Positive semidefinite matrices with trace 1 are called *density matrices* in the physics literature.) Let $\alpha_1, \ldots, \alpha_n$ be the (necessarily real and nonnegative) eigenvalues of X and let $u_1, \ldots, u_n$ be a corresponding orthonormal set of eigenvectors. If T is any $n \times n$ matrix, and $u_1, \ldots, u_n$ is an orthonormal basis of $\mathbb{C}^n$, then $\operatorname{tr} T = \sum_{j=1}^{n} \langle u_j, Tu_j\rangle$. Hence

$$\varphi(A) = \operatorname{tr} AX = \sum_{j=1}^{n} \langle u_j, AXu_j\rangle = \sum_{j=1}^{n} \alpha_j \langle u_j, Au_j\rangle.$$

Since $\sum \alpha_j = 1$, this shows that $\varphi(A)$ is a convex combination of the complex numbers $\langle u_j, Au_j\rangle$, each of which is in $W(A)$. So, by the Toeplitz-Hausdorff Theorem $\varphi(A)$ is also in $W(A)$. So, there exists a unit vector y (depending on A) such that $\varphi(A) = \langle y, Ay\rangle$. Thus the numerical range $W(A)$ is also the collection of all complex numbers $\varphi(A)$ as φ varies over positive unital linear functionals. So, if φ_1 and φ_2 are any two such functionals, then

$$|\varphi_1(A) - \varphi_2(A)| \le \operatorname{diam} W(A). \tag{5}$$

The following theorem, which is of independent interest, is an extension of this observation.

We use the notation $\|A\|$ for the operator norm of A defined as $A = \sup\{\|Ax\| : \|x\| = 1\}$. If $s_1(A) \geq \cdots \geq s_n(A)$ are the decreasingly ordered singular values of A, then $\|A\| = s_1(A)$. If A is normal, then this means that $\|A\| = \max_j |\lambda_j(A)|$. If A is Hermitian, then $\|A\| = \max_{\|x\|=1} |\langle x, Ax\rangle|$. These facts about the norm $\|\cdot\|$ are used in the following discussion. If A and B are two Hermitian matrices, we say $A \geq B$ if $A - B$ is positive semidefinite.

Theorem. *Let Φ_1, Φ_2 be any two positive unital linear maps from $\mathbb{M}(n)$ into $\mathbb{M}(k)$. Then*

(i) *For every Hermitian A in $\mathbb{M}(n)$*

$$\|\Phi_1(A) - \Phi_2(A)\| \leq \mathrm{diam} W(A). \tag{6}$$

(ii) *If $n = 2$, then the inequality* (6) *holds also for all normal matrices A.*

Proof. If A is an $n \times n$ Hermitian matrix, then $\lambda_n^{\downarrow}(A)I \leq A \leq \lambda_1^{\downarrow}(A)I$. The linear maps Φ_j, $j = 1, 2$, preserve order and take the identity I in $\mathbb{M}(n)$ to I in $\mathbb{M}(k)$. So we have $\lambda_n^{\downarrow}(A)I \leq \Phi_j(A) \leq \lambda_1^{\downarrow}(A)I$, $j = 1, 2$. From this we obtain

$$\Phi_1(A) - \Phi_2(A) \leq \left(\lambda_1^{\downarrow}(A) - \lambda_n^{\downarrow}(A)\right) I$$

and

$$\Phi_2(A) - \Phi_1(A) \leq \left(\lambda_1^{\downarrow}(A) - \lambda_n^{\downarrow}(A)\right) I.$$

Now if X is a Hermitian matrix and $\pm X \leq \alpha I$, then $|\lambda_j(X)| \leq \alpha$ for all j, and hence $\|X\| \leq \alpha$. So, we have the inequality (6).

Now suppose $n = 2$. If A is a 2×2 normal matrix, then $A = \lambda P + \mu Q$, where λ, μ are the eigenvalues of A and P, Q are the corresponding eigenprojections. We have $P + Q = I$, and hence $\Phi_j(P) + \Phi_j(Q) = I$. Hence

$$\begin{aligned}
\Phi_1(A) - \Phi_2(A) &= \lambda\Phi_1(P) + \mu\Phi_1(Q) - \lambda\Phi_2(P) - \mu\Phi_2(Q) \\
&= \lambda(I - \Phi_1(Q)) + \mu\Phi_1(Q) - \lambda(I - \Phi_2(Q)) - \mu\Phi_2(Q) \\
&= (\lambda - \mu)(\Phi_2(Q) - \Phi_1(Q)).
\end{aligned}$$

Hence,

$$\|\Phi_1(A) - \Phi_2(A)\| \leq |\lambda - \mu| \; \|\Phi_2(Q) - \Phi_1(Q)\|. \tag{7}$$

Since $0 \leq Q \leq I$, we have $0 \leq \Phi_j(Q) \leq I$, and hence $\|\Phi_j(Q)\| \leq 1$ for $j = 1, 2$. If X, Y are positive semidefinite, then $\|X - Y\| \leq \max(\|X\|, \|Y\|)$. So, the inequality (7) shows that $\|\Phi_1(A) - \Phi_2(A)\| \leq |\lambda - \mu|$. This proves part (ii) of the Theorem. ∎

When $n = 3$ the inequality (6) is not valid for all normal matrices. For nonnormal matrices it need not hold even when $n = 2$. Let Φ_1 be the map that takes a 3×3 matrix A to its top left 2×2 block, and let Φ_2 be the map that takes A to its bottom right 2×2 block. Then Φ_1, Φ_2 are positive unital linear maps from $\mathbb{M}(3)$ into $\mathbb{M}(2)$. Let

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Then A is normal, and its eigenvalues are the three cube roots of -1. So $\operatorname{diam} W(A) = \sqrt{3}$, but $\|\Phi_1(A) - \Phi_2(A)\| = \left\| \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \right\| = 2$, and the inequality (6) breaks down. Let $\Phi_1 : \mathbb{M}(2) \to \mathbb{M}(2)$ be the map defined as $\Phi_1(A) = \left(\frac{1}{2} \sum\limits_{i,j} a_{ij} \right) I$, and let $\Phi_2(A) = A$. If X is a positive semidefinite matrix of any order n and e the all-ones n-vector, then $\sum\limits_{i,j} x_{ij} = \langle e, Xe \rangle \geq 0$. So Φ_1 defined above is a positive unital linear map. Choose $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. A little calculation shows that $W(A)$ is the disk of radius $1/2$ centred at the origin. So, $\operatorname{diam} W(A) = 1$. On the other hand the matrix $\Phi_1(A) - \Phi_2(A) = \begin{bmatrix} 1/2 & -1 \\ 0 & 1/2 \end{bmatrix}$, and its norm is bigger than 1. (The norm $\|X\|$ can not be smaller than the Euclidean norm of any column of X.)

Interesting lower bounds for $\operatorname{spd}(A)$ of normal and Hermitian matrices can be obtained from (5) and (6). We illustrate this with a few examples.

Let φ_1, φ_2 be linear functionals on $\mathbb{M}(n)$ defined for $i \neq j$ as

$$\begin{aligned} \varphi_1(A) &= \frac{1}{2} \left(a_{ii} + a_{jj} + a_{ij}\, e^{i\theta} + a_{ji}\, e^{-i\theta} \right) \\ \varphi_2(A) &= \frac{1}{2} \left(a_{ii} + a_{jj} - a_{ij}\, e^{i\theta} - a_{ji}\, e^{-i\theta} \right). \end{aligned}$$

Both φ_1 and φ_2 are positive and unital. (Positivity is a consequence of the fact that if A is positive semidefinite, then $|a_{ij}| \leq \sqrt{a_{ii} a_{jj}} \leq \frac{1}{2}(a_{ii} + a_{jj})$.) So, from (5) we see that for every normal matrix A

$$\operatorname{spd}(A) \geq \left| a_{ij}\, e^{i\theta} + a_{ji}\, e^{-i\theta} \right|.$$

This is true for every θ. The maximum value of the right-hand side over θ is $|a_{ij}| + |a_{ji}|$. Thus for every normal matrix A we have

$$\operatorname{spd}(A) \geq \max_{i \neq j} \; (|a_{ij}| + |a_{ji}|). \tag{8}$$

This was first proved by L. Mirsky. See Theorem 3 (iii) in [7]. When A is Hermitian, this says
$$\operatorname{spd}(A) \geq 2 \max_{i \neq j} |a_{ij}|. \tag{9}$$
Another result of Mirsky, Theorem 2 in [7], subsumes both the inequalities (4) and (9). It says that for every Hermitian matrix A, we have
$$\operatorname{spd}(A)^2 \geq \max_{i \neq j} \left((a_{ii} - a_{jj})^2 + 4\, |a_{ij}|^2 \right). \tag{10}$$
This can be obtained from (6) as follows. Let
$$\Phi_1(A) = \begin{bmatrix} a_{ii} & a_{ij} \\ \overline{a_{ij}} & a_{jj} \end{bmatrix}, \quad \Phi_2(A) = \begin{bmatrix} a_{jj} & -a_{ij} \\ \overline{-a_{ij}} & a_{ii} \end{bmatrix}.$$
Then Φ_1 and Φ_2 are positive, unital, linear maps, and
$$\Phi_1(A) - \Phi_2(A) = \begin{bmatrix} a_{ii} - a_{jj} & 2\, a_{ij} \\ 2\, \overline{a_{ij}} & a_{jj} - a_{ii} \end{bmatrix}.$$
This is a Hermitian matrix with trace 0. Its eigenvalues are $\pm\alpha$, where $\alpha = (a_{ii} - a_{jj})^2 + 4\, |a_{ij}|^2$. So $\|\Phi_1(A) - \Phi_2(A)\| = \alpha$. The inequality (10) then follows from (6).

Next let $\varphi_1(A) = \frac{1}{n}\sum\limits_{i,j} a_{ij}$, and $\varphi_2(A) = \frac{1}{n}\left(\operatorname{tr} A - \frac{1}{n-1} \sum\limits_{i \neq j} a_{ij} \right)$. Both are unital linear functionals. We have already observed φ_1 is positive. We claim φ_2 is also positive. If A is any Hermitian matrix, then
$$\sum_{i \neq j} a_{ij} = 2 \operatorname{Re} \sum_{i \neq j} a_{ij} \leq 2 \sum_{i \neq j} |a_{ij}|.$$
If further A is positive semidefinite, then we have $|a_{ij}| \leq \frac{1}{2}(a_{ii} + a_{jj})$. Combining these two inequalities, one sees that $\sum\limits_{i \neq j} a_{ij} \leq (n-1) \operatorname{tr} A$. So the linear functional φ_2 is positive. The inequality (5) now shows that for every normal matrix A we have
$$\operatorname{spd}(A) \geq \frac{1}{n-1} \left| \sum_{i \neq j} a_{ij} \right|. \tag{11}$$
This inequality is stated as Theorem 2.1 in [5] and as Theorem 5 in [6], and is proved there by other arguments.

Many more inequalities, some of them stronger and more intricate than the ones we have discussed, can be obtained choosing other positive maps. Enhancing this technique, we have the inequality of Bhatia and Davis [2]. This says that if Φ is a positive unital linear map, then for every Hermitian matrix A
$$\Phi(A^2) - \Phi(A)^2 \leq \frac{1}{4} \operatorname{spd}(A)^2. \tag{12}$$

Again choosing different Φ a variety of inequalities can be obtained. This is demonstrated in [3].

References

[1] R. Bhatia, *Positive Definite Matrices,* Princeton University Press, 2007.

[2] R. Bhatia, and C. Davis, *A better bound on the variance,* Amer. Math. Monthly, 107 (2000) 353-357.

[3] R. Bhatia, and R. Sharma, *Some inequalities for positive linear maps,* Linear Algebra Appl., 436 (2012) 1562-1571.

[4] R. A. Horn, and C. R. Johnson, *Topics in Matrix Analysis,* Cambridge University Press, 1991.

[5] C. R. Johnson, R. Kumar and H. Wolkowicz, *Lower bounds for the spread of a matrix,* Linear Algebra Appl., 71 (1985) 161-173.

[6] J. K. Merikoski and R. Kumar, *Characterizations and lower bounds for the spread of a normal matrix,* Linear Algebra Appl., 364 (2003) 13-31.

[7] L. Mirsky *Inequalities for normal and Hermitian matrices,* Duke Math. J., 24 (1957) 591-599.

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