isid/ms/2014/15
October 31, 2014
http://www.isid.ac.in/~statmath/index.php?module=Preprint

Positive linear maps and spreads of matrices-II

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Abstract. It is shown how positive linear maps can be used to obtain lower bounds for spreads of normal matrices. Some known results and some new ones are obtained in this way.

AMS classification: 15A45, 15A60, 47A12,

Keywords : Positive unital linear map, Spread, Trace, Numerical range

1 Introduction

This is continuation of our earlier work in [3] and [4]. Its goal is to show how positive linear maps can be used to obtain matrix inequalities. Our focus in this paper is on lower bounds for the spread of a matrix, a topic much studied since the work of L. Mirsky [6] in 1956.

Let $\mathbb{M}(n)$ be the space of $n \times n$ complex matrices. Associated with any element A of $\mathbb{M}(n)$ are the following quantities of interest to us:

(i) The spread of A, denoted $\operatorname{spd}(A)$, is defined as

$$\operatorname{spd}(A) = \max_{1 \leq i,j \leq n} \left| \lambda_i(A) - \lambda_j(A) \right|,$$

where $\lambda_1(A), ..., \lambda_n(A)$ are the eigenvalues of A.

(ii) $r(A)$ is the radius of the smallest disk in the plane that includes all eigenvalues of A.

(iii) The numerical range of A is the set

$$W(A) = \{ \langle x, \ Ax \rangle : \|x\| = 1 \}.$$

This is a convex subset of the plane. Its diameter is denoted as $\operatorname{diam} W(A)$.

(iv) The distance of A to scalar matrices

$$\Delta(A) = \inf_{z \in \mathbb{C}} \|A - zI\|.$$

Here $\|A\|$ stands for the operator norm of A.

There are some relationships between these quantities:

If all eigenvalues of A are real, then $\operatorname{spd}(A) = 2r(A)$. In particular, this is so if A is Hermitian. In the general case, $\operatorname{spd}(A) \geq \sqrt{3}r(A)$.

For every A we have $\Delta(A) \geq r(A)$, and there is equality here if A is normal. Likewise $\operatorname{diam} W(A) \geq \operatorname{spd}(A)$ for every A, and there is equality here if A is normal. (See, [3], [4]).

In particular, when A is Hermitian, we have

$$\Delta(A) = r(A) = \frac{1}{2}\operatorname{spd}(A) = \frac{1}{2}\operatorname{diam} W(A).$$

A linear map $\Phi : \mathbb{M}(n) \to \mathbb{M}(k)$ is called positive if $\Phi(A)$ is positive semidefinite (psd) whenever A has that property, and unital if $\Phi(I) = I$. When $k = 1$, such a map is called a positive, unital, linear functional and is denoted by the lower case letter φ.

In [2] Bhatia and Davis showed that if Φ is any positive unital linear map, and A is any Hermitian matrix, then

$$\Phi(A^2) - \Phi(A)^2 \leq \frac{1}{4}\operatorname{spd}(A)^2 . \tag{1.1}$$

In [3] we extended this by showing that for every matrix A we have the inequality

$$\Phi(A^*A) - \Phi(A)^*\Phi(A) \leq \Delta(A)^2 . \tag{1.2}$$

The quantities on the left-hand sides of (1.1) and (1.2) are analogues of "variance" in classical probability, and the motivation for Bhatia and Davis was to find an upper bound for this. This can be turned around and we can use these inequalities to obtain lower bounds for spd(A) and $\Delta(A)$. In [3] we showed how judicious choices of Φ lead to interesting bounds, some old and some new.

In our more recent work [4] we have augmented this technique with another use of positive unital linear maps. We showed that if Φ_1, Φ_2 are positive unital linear maps from $\mathbb{M}(n)$ into $\mathbb{M}(k)$, then for every Hermitian A in $\mathbb{M}(n)$ we have

$$\|\Phi_1(A) - \Phi_2(A)\| \leq \operatorname{diam} W(A). \tag{1.3}$$

This inequality does not hold in general, except that when $n = 2$, it is valid also for every normal matrix A. However, we do have the following: if φ_1, φ_2 are positive unital linear functional on $\mathbb{M}(n)$, then for every matrix A in $\mathbb{M}(n)$

$$|\varphi_1(A) - \varphi_2(A)| \leq \operatorname{diam} W(A). \tag{1.4}$$

As we have remarked, for normal matrices diam $W(A) = \operatorname{spd}(A)$. So using (1.3) and (1.4) we can obtain lower bounds for spreads of normal matrices. The efficacy of this method was illustrated in [4]. In this paper we carry this further.

2 Lower bounds for the spread

Among the earliest results on this problem is one of Mirsky's theorems [6] that says that if $A = [a_{ij}]$ is a Hermitian matrix, then

$$\operatorname{spd}(A)^2 \geq \max_{i \neq j} \left((a_{ii} - a_{jj})^2 + 4\,|a_{ij}|^2 \right) . \tag{2.1}$$

We remark here that for every scalar c, the matrices A and $A + cI$ have the same spread. So, a "right" bound for spd(A) should involve not the magnitudes of the diagonal elements but only their differences, whereas the off-diagonal elements may be represented by their magnitudes. In [4] we have shown that the inequality (2.1) can be derived from (1.3) upon choosing

$$\Phi_1(A) = \begin{bmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{bmatrix} \quad \text{and} \quad \Phi_2(A) = \begin{bmatrix} a_{jj} & -a_{ij} \\ -a_{ji} & a_{ii} \end{bmatrix}, \quad i \neq j.$$

A stronger inequality than (2.1) was obtained by Barnes and Hoffman [1]. This says that for every normal matrix A,

$$4\, r(A)^2 \geq \max_{i,j} \left\{ |a_{ii} - a_{jj}|^2 + 2\sum_{k\neq i} |a_{ki}|^2 + 2\sum_{k\neq j} |a_{kj}|^2 \right\}. \tag{2.2}$$

In [3] we have obtained a version of this for all matrices A, in which the left-hand side of (2.2) is replaced by $4\,\Delta(A)^2$. This can be obtained from (1.2) upon choosing

$$\varphi(A) = \frac{1}{2}(a_{ii} + a_{jj}).$$

In [5] Jiang and Zhan have obtained several lower bounds for the spread of Hermitian matrices, among which is a stronger inequality than (2.2). We now show how these bounds, and improvements thereof, can be obtained by our technique.

To save space, let us introduce the quantity

$$h_{ij}(A) = |a_{ii} - a_{jj}|^2 + 2\sum_{k\neq i} |a_{ki}|^2 + 2\sum_{k\neq j} |a_{kj}|^2. \tag{2.3}$$

For Hermitian matrices, the inequality (2.2) says that

$$\operatorname{spd}(A)^2 \geq \max_{i,j} h_{ij}(A). \tag{2.4}$$

Let $i \neq j$, and consider the linear map Φ from $\mathbb{M}(n)$ into $\mathbb{M}(2)$ defined as

$$\Phi(A) = \begin{bmatrix} \frac{a_{ii}+a_{jj}}{2} & a_{ij} \\ a_{ji} & \frac{a_{ii}+a_{jj}}{2} \end{bmatrix}.$$

Then Φ is positive and unital. Let A be any Hermitian matrix. A calculation shows that

$$\Phi(A^2) - \Phi(A)^2 = \begin{bmatrix} \frac{h_{ij}(A)}{4} - |a_{ij}|^2 & \sum\limits_{k\neq i,j} a_{ik}\overline{a_{jk}} \\ \sum\limits_{k\neq i,j} \overline{a_{ik}} a_{jk} & \frac{h_{ij}(A)}{4} - |a_{ij}|^2 \end{bmatrix}.$$

This is a positive semidefinite matrix of the form $\begin{bmatrix} b & c \\ \bar{c} & b \end{bmatrix}$. The norm of such a matrix is $b + |c|$. Thus

$$\left|\left|\Phi(A^2) - \Phi(A)^2\right|\right| = \frac{h_{ij}(A)}{4} - |a_{ij}|^2 + \left|\sum_{k \neq i,j} a_{ik}\overline{a_{jk}}\right|.$$

So, from (1.1) we obtain

$$\operatorname{spd}(A)^2 \geq h_{ij}(A) - 4|a_{ij}|^2 + 4\left|\sum_{k \neq i,j} a_{ik}\overline{a_{jk}}\right|.$$

If $a_{ij} = 0$, we have an improvement of the inequality (2.4), which says

$$\operatorname{spd}(A)^2 \geq h_{ij}(A) + 4\left|\sum_{k \neq i,j} a_{ik}\overline{a_{jk}}\right|. \tag{2.5}$$

If $a_{ij} \neq 0$, we proceed in another way. Let α be any complex number with $|\alpha| \leq 2$, and define

$$\varphi(A) = \frac{a_{ii} + a_{jj}}{2} + i\frac{\overline{\alpha}a_{ji} - \alpha a_{ij}}{4}, \quad i \neq j.$$

Then φ is a linear functional on $\mathbb{M}(n)$, and unital. If A is Hermitian, then

$$\varphi(A) = \frac{a_{ii} + a_{jj} + \operatorname{Im}(\alpha a_{ij})}{2}, \quad i \neq j.$$

It follows that φ is positive. A calculation shows that for every Hermitian A

$$\Phi(A^2) - \Phi(A)^2 = \frac{1}{4}\left\{h_{ij}(A) - (\operatorname{Im}(\alpha a_{ij}))^2 + 2\left|\operatorname{Im}\left(\alpha \sum_{k \neq i,j} a_{ik}a_{kj}\right)\right|\right\}.$$

If we choose $\alpha = \frac{2\overline{a_{ij}}}{|a_{ij}|}$, then $\operatorname{Im}(\alpha a_{ij}) = 0$, and the equality above combined with (1.1) gives

$$\operatorname{spd}(A)^2 \geq h_{ij}(A) + 4\frac{\left|\operatorname{Im} \overline{a_{ij}} \sum_{k \neq i,j} a_{ik}a_{kj}\right|}{|a_{ij}|}. \tag{2.6}$$

Again, this is an improvement of the inequality (2.4), valid in the case $a_{ij} \neq 0$.

Together, (2.5) and (2.6) may be written as

$$\operatorname{spd}(A)^2 \geq \max_{i \neq j}(h_{ij}(A) + e_{ij}(A)), \tag{2.7}$$

where $e_{ij}(A)$ is the positive quantity represented by the last term in (2.5) in case $a_{ij} = 0$, and the last term in (2.6) in case $a_{ij} \neq 0$. This is Theorem 3 of Jiang and Zhan [5].

Theorem 4 in [5] is proved for real symmetric matrices. We now show how to extend this in two different directions. We derive an inequality valid for all Hermitian matrices (not necessarily real) and another for all real matrices (not necessarily symmetric).

Theorem 1. *Let A be a Hermitian matrix, and let*

$$\alpha_1 = \frac{\left| \sum\limits_{k \neq i,j} a_{ik} a_{kj} \right|}{2 \left| a_{ij} \right|^2},$$

and

$$g_{ij}(A) = \begin{cases} 4 \left| \sum\limits_{k \neq i,j} a_{ik} a_{kj} \right| & \text{if } \ a_{ij} = 0 \\ 4\alpha_1^2 \left| a_{ij} \right|^2 & \text{if } \ \alpha_1 \leq 1 \\ 4 \left(2\alpha_1 - 1 \right) \ \left| a_{ij} \right|^2 & \text{if } \ \alpha_1 \geq 1. \end{cases}$$

Then

$$\operatorname{spd}(A)^2 \geq \max_{i \neq j} \left(h_{ij}(A) + g_{ij}(A) \right). \tag{2.8}$$

Proof. Let $-1 \leq \beta \leq 1$, and define

$$\Phi(A) = \begin{bmatrix} \frac{a_{ii}+a_{jj}}{2} & \beta a_{ij} \\ \beta a_{ji} & \frac{a_{ii}+a_{jj}}{2} \end{bmatrix}, \ i \neq j \ .$$

Then Φ is a positive unital linear map. A simple calculation shows that for $\beta \geq 0$,

$$\left|\left| \Phi(A^2) - \Phi(A)^2 \right|\right| = \frac{h_{ij}}{4} + \beta \left| \sum_{k \neq i,j} a_{ik} a_{kj} \right| - \beta^2 \left| a_{ij} \right|^2 = f(\beta) \ \text{(say)}. \tag{2.9}$$

It follows from (1.1) and (2.9) that

$$\operatorname{spd}(A)^2 \geq 4f(\beta). \tag{2.10}$$

For $\alpha_1 \leq 1$, $f(\beta)$ has its maximum at $\beta = \alpha_1$ and for $\alpha_1 > 1$ or $a_{ij} = 0$, $f(\beta)$ has its maximum at $\beta = 1$. So, (2.10) gives (2.8). ■

Note that if $\alpha_1 \geq 1$, then

$$\left|a_{ij}\right|^2 + \left|2\left|a_{ij}\right|^2 - \left|\sum_{k\neq i,j} a_{ik}a_{kj}\right|\right| = \left(2\alpha_1 - 1\right)\left|a_{ij}\right|^2 \leq \frac{\left|\sum_{k\neq i,j} a_{ik}a_{kj}\right|^2}{\left(2\left|a_{ij}\right|\right)^2}.$$

Also, if $\alpha_1 \leq 1$, then

$$\left|a_{ij}\right|^2 + \left|2\left|a_{ij}\right|^2 - \left|\sum_{k\neq i,j} a_{ik}a_{kj}\right|\right| \geq \frac{\left|\sum_{k\neq i,j} a_{ik}a_{kj}\right|^2}{\left(2\left|a_{ij}\right|\right)^2} = \alpha_1^2\left|a_{ij}\right|^2.$$

So an equivalent expression for g_{ij}, as obtained in Theorem 4 in [5] for real symmetric matrix is

$$g_{ij} = \begin{cases} 4\left|\sum_{k\neq i,j} a_{ik}a_{kj}\right| & \text{if } a_{ij} = 0 \\ 4\min\left(\left|a_{ij}\right|^2 + \left|2\left|a_{ij}\right|^2 - \left|\sum_{k\neq i,j} a_{ik}a_{kj}\right|\right|, \frac{\left|\sum_{k\neq i,j} a_{ik}a_{kj}\right|^2}{\left(2\left|a_{ij}\right|\right)^2}\right) & \text{otherwise.} \end{cases}$$

Theorem 2. *Let A be any matrix, and let*

$$\alpha_2 = \frac{2\left|\operatorname{Re}\left(2\sum_{k\neq i,j} a_{ki}a_{kj} + \left(\overline{a_{ii}} - \overline{a_{jj}}\right)\left(a_{ij} - a_{ji}\right)\right)\right|}{\left|a_{ij} + a_{ji}\right|^2},$$

and

$$\eta_{ij}(A) = \begin{cases} 2\left|\operatorname{Re}\left(2\sum_{k\neq i,j} a_{ki}a_{kj} + \left(\overline{a_{ii}} - \overline{a_{jj}}\right)\left(a_{ij} - a_{ji}\right)\right)\right| & \text{if } a_{ij} + a_{ji} = 0 \\ \frac{1}{4}\alpha_2^2\left|a_{ij} + a_{ji}\right|^2 & \text{if } \alpha_2 \leq 2 \\ \left(\alpha_2 - 1\right)\left|a_{ij} + a_{ji}\right|^2 & \text{if } \alpha_2 \geq 2. \end{cases}$$

Then

$$4\,\Delta(A)^2 \geq \max_{i\neq j}\left(h_{ij}(A) + \eta_{ij}(A)\right). \tag{2.11}$$

Proof. Let

$$\varphi(A)=\frac{a_{ii}+a_{jj}}{2}+\beta\frac{a_{ij}+a_{ji}}{4}\ ,\ i\neq j\ .$$

Then φ is a positive unital linear functional whenever $-2\leq\beta\leq 2$. So, we have

$$\begin{aligned}\varphi(A^*A)-\varphi(A)^*\varphi(A) &= \frac{h_{ij}}{4}-\frac{\beta^2}{16}\left|a_{ij}+a_{ji}\right|^2\\ &\quad+\frac{\beta}{4}\left|\operatorname{Re}\left(2\sum_{k\neq i,j}\overline{a_{ki}}a_{kj}+\left(\overline{a_{ii}}-\overline{a_{jj}}\right)\left(a_{ij}-a_{ji}\right)\right)\right| \quad (2.12)\\ &= f(\beta)\ \ (say).\end{aligned}$$

From (1.2) and (2.12),

$$\Delta(A)^2\geq f(\beta)\,. \quad (2.13)$$

For $\alpha_2\leq 2$, $f(\beta)$ has its maximum at $\beta=\alpha_2$; and if $\alpha_2\geq 2$ or $a_{ij}+a_{ji}=0$, $f(\beta)$ has its maximum at $\beta=2$. So, the inequality (2.11) follows from (2.13). ■

Our next theorem generalises Theorem 5 of [5] from Hermitian to arbitrary matrices.

Theorem 3. *Let A be any matrix. If*

$$\left|\sum_{k\neq i}\left|a_{ki}\right|^2-\sum_{k\neq j}\left|a_{kj}\right|^2\right|\leq\left|a_{ii}-a_{jj}\right|^2, \quad (2.14)$$

then

$$4\ \Delta(A)^2\geq\max_{i,j}\left(h_{ij}+\frac{1}{\left|a_{ii}-a_{jj}\right|^2}\left(\sum_{k\neq i}\left|a_{ki}\right|^2-\sum_{k\neq j}\left|a_{kj}\right|^2\right)^2\right). \quad (2.15)$$

If the reverse of inequality (2.14) *holds , then*

$$\Delta(A)^2\geq\max_i\sum_{k\neq i}\left|a_{ki}\right|^2. \quad (2.16)$$

Proof. Let $0\leq p\leq 1$ and put

$$\varphi(A)=pa_{ii}+(1-p)\,a_{jj}\ .$$

Then φ is a positive unital linear functional. We have

$$\begin{aligned}\varphi(A^*A)-\varphi(A)^*\varphi(A) &= p(1-p)\left|a_{ii}-a_{jj}\right|^2\\ &\quad+p\sum_{k\neq i}\left|a_{ki}\right|^2+(1-p)\sum_{k\neq j}\left|a_{kj}\right|^2 \quad (2.17)\\ &= f(p)\quad \text{(say)}.\end{aligned}$$

From (1.2) and (2.17),

$$\Delta(A)^2 \geq f(p). \tag{2.18}$$

If (2.14) holds, then the function $f(p)$ has its maximum at

$$p = \frac{1}{2} + \frac{\sum\limits_{k\neq i} |a_{ki}|^2 - \sum\limits_{k\neq j} |a_{kj}|^2}{2|a_{ii} - a_{jj}|^2}.$$

In the opposite case, $f(p)$ has its maximum at $p = 1$. The assertions of the theorem now follow from (2.18). ■

If A is Hermitian and (2.14) holds then

$$(a_{ii} - a_{jj})^2 + 2\left|(a_{ii} - a_{jj})^2 - f_{ij}\right| \geq \frac{(f_{ij})^2}{(a_{ii} - a_{jj})^2},$$

where $f_{ij} = \left|\sum\limits_{k\neq i} |a_{ki}|^2 - \sum\limits_{k\neq j} |a_{kj}|^2\right|$. In opposite case,

$$(a_{ii} - a_{jj})^2 + 2\left|(a_{ii} - a_{jj})^2 - f_{ij}\right| = 2f_{ij} - (a_{ii} - a_{jj})^2 \leq \frac{(f_{ij})^2}{(a_{ii} - a_{jj})^2}.$$

From this we can see that if A is Hermitian then the above theorem gives Theorem 5 in [5] when $a_{ii} \neq a_{jj}$. If $a_{ii} = a_{jj}$ then Theorem 5 in [5] says that $\mathrm{Spd}(A)^2 \geq 2f_{ij}$ while our theorem gives better bound (2.16).

Jiang and Zhan [5] prove four theorems (Theorems 3-6 in their paper). We have shown how their Theorems 3-5 can be proved and strengthened using our methods. A stronger and more general version of their Theorem 6 was obtained in our earlier paper [3]; see Theorem 3.4 there.

We now turn to another well-known bound. Choosing $\varphi(A) = \frac{\mathrm{tr}A}{n}$, we obtain from (1.2)

$$\Delta(A)^2 \geq \frac{1}{n}\mathrm{tr}A^*A - \frac{1}{n^2}|\mathrm{tr}A|^2. \tag{2.19}$$

For Hermitian A, this specializes to

$$\frac{1}{4}\mathrm{spd}(A)^2 \geq \frac{\mathrm{tr}A^2}{n} - \left(\frac{\mathrm{tr}A}{n}\right)^2. \tag{2.20}$$

The next theorem is a strengthening of this in the spirit of the discussion above.

Theorem 4. *Let A be an $n \times n$ Hermitian matrix. Let*

$$\alpha_3 = \frac{\left| \sum_{k=1}^{n} a_{ik} a_{kj} - 2 a_{ij} \frac{\mathrm{tr} A}{n} \right|}{2 \left| a_{ij} \right|^2},$$

and

$$w_{ij}(A) = \begin{cases} \frac{2}{n} \left| \sum_{k=1}^{n} a_{ik} a_{kj} \right| & \text{if } a_{ij} = 0 \\ \\ \alpha_3^2 \left| a_{ij} \right|^2, & \text{if } \alpha_3 \leq \frac{2}{n} \\ \\ \frac{4}{n^2} \left(n\alpha_3 - 1 \right) \left| a_{ij} \right|^2, & \text{if } \alpha_3 \geq \frac{2}{n} \ . \end{cases}$$

Then

$$\frac{1}{4} \mathrm{spd}\,(A)^2 \geq \frac{\mathrm{tr} A^2}{n} - \left(\frac{\mathrm{tr} A}{n} \right)^2 + \max_{i \neq j} w_{ij} \ . \tag{2.21}$$

Proof. Let

$$\Phi(A) = \begin{bmatrix} \frac{\mathrm{tr} A}{n} & \beta a_{ij} \\ \beta a_{ji} & \frac{\mathrm{tr} A}{n} \end{bmatrix} \ , \ \ i \neq j \ .$$

Then Φ is a positive unital linear map whenever $-\frac{2}{n} \leq \beta \leq \frac{2}{n}$. For $\beta \geq 0$, we have

$$\left|\left| \Phi(A^2) - \Phi(A)^2 \right|\right| = \frac{\mathrm{tr} A^2}{n} - \left(\frac{\mathrm{tr} A}{n} \right)^2 - \beta^2 \left| a_{ij} \right|^2 + \beta \left| \sum_{k=1}^{n} a_{ik} a_{kj} - 2 a_{ij} \frac{\mathrm{tr} A}{n} \right| = f(\beta) \ \text{(say)}. \tag{2.22}$$

It follows from (1.1) and (2.22) that

$$\mathrm{spd}\,(A)^2 \geq 4 f(\beta) \, . \tag{2.23}$$

For $\alpha_3 \leq \frac{2}{n}$, $f(\beta)$ has its maximum at $\beta = \alpha_3$, and for $\alpha_3 > \frac{2}{n}$ or $a_{ij} = 0$, $f(\beta)$ has its maximum at $\beta = \frac{2}{n}$. So, (2.23) implies (2.21). ■

Another inequality, independent of (2.19), follows on choosing

$$\varphi(A) = \frac{\mathrm{tr} A}{n} \pm \frac{a_{ij} + a_{ji}}{n} \ , \ i \neq j,$$

in (1.2). This says

$$\Delta(A)^2 \geq \max_{i \neq j} \left\{ \frac{\mathrm{tr}\,(A^* A)}{n} \pm \frac{2}{n} \mathrm{Re} \left(\sum_{k=1}^{n} a_{ik} a_{kj} \right) - \left| \frac{\mathrm{tr} A}{n} \pm \frac{a_{ij} + a_{ji}}{n} \right|^2 \right\}. \tag{2.24}$$

Complementary to (1.2) we have the inequality

$$\Phi(AA^*) - \Phi(A)^* \Phi(A) \leq \Delta(A)^2 .$$

See [3]. Using this instead of (1.2) we can obtain different versions of some of the subsequent inequalities.

Our next theorem gives a variant of (1.2) in the special case when A is normal and φ is a linear functional.

Theorem 5. *Let φ be a positive unital linear functional. Let A be any normal matrix. Then*

$$\varphi(A^*A) - |\varphi(A)|^2 + \left|\varphi\left(A^2\right) - \varphi(A)^2\right| \leq \frac{\operatorname{spd}(A)^2}{2}. \tag{2.25}$$

Proof. If A is normal, then $\operatorname{spd}(A) \geq \operatorname{spd}(\frac{A+A^*}{2})$. Also, if z is any complex number with $|z| = 1$, then $\operatorname{spd}(A) = \operatorname{spd}(zA)$. Hence

$$\operatorname{spd}(A) \geq \operatorname{spd}\left(\frac{zA + \overline{z}A^*}{2}\right). \tag{2.26}$$

A little calculation shows that for $B = \frac{zA+\overline{z}A^*}{2}$, we have

$$\varphi\left(B^2\right) - \varphi(B)^2 = \frac{1}{2}\operatorname{Re}\left\{\left(z^2\left(\varphi\left(A^2\right) - \varphi(A)^2\right)\right) + \varphi(A^*A) - |\varphi(A)|^2\right\}. \tag{2.27}$$

For any complex number a there is a complex number z with $|z| = 1$ such that $\operatorname{Re}(z^2 a) = |a|$. Therefore we can choose a complex number z with $|z| = 1$ such that

$$\operatorname{Re}\left(z^2\left(\varphi\left(A^2\right) - \varphi(A)^2\right)\right) = \left|\varphi\left(A^2\right) - \varphi(A)^2\right|. \tag{2.28}$$

From (2.27) and (2.28), we have

$$2\left(\varphi\left(B^2\right) - \varphi(B)^2\right) = \varphi(A^*A) - |\varphi(A)|^2 + \left|\varphi\left(A^2\right) - \varphi(A)^2\right|. \tag{2.29}$$

The inequality (2.25) now follows from the inequality (1.1). ■

Choosing $\varphi(A) = a_{jj}$, we obtain from (2.25)

$$\operatorname{spd}(A)^2 \geq 2 \max_j \left(\left|\sum_{k\neq j}^{n} a_{jk}a_{kj}\right| + \sum_{k\neq j}^{n} |a_{jk}|^2\right).$$

This is Theorem 9(i) of Merikoski and Kumar [7]. Theorem 9(ii) in [7] also follows from (2.25) on choosing $\varphi(A) = \frac{1}{n}\sum_{i,j}^{n} a_{ij}$. Likewise, the choice $\varphi(A) = \frac{a_{ii}+a_{jj}}{2}$ in (2.25) gives Theorem 14 of [7].

Corollary 6. *Let A be a normal matrix. Then*

$$\frac{\operatorname{spd}(A)^2}{2} \geq \frac{\operatorname{tr}A^*A}{n} - \left|\frac{\operatorname{tr}A}{n}\right|^2 + \left|\frac{\operatorname{tr}A^2}{n} - \left(\frac{\operatorname{tr}A}{n}\right)^2\right|. \tag{2.30}$$

Proof. The inequality (2.30) is the particular case of (2.25) with $\varphi(A) = \frac{\operatorname{tr}A}{n}$. ∎

Acknowledgements. *The first author is a J. C. Bose National Fellow. He was a Fellow Professor at Sungkyunkwan University in August 2014 when the results of this paper were presented to the 19th Conference of the ILAS. The second author is supported by the UGC-SAP. He thanks the Indian Statistical Institute for a visit in January 2014.*

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