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Symmetrizing and Variance Stabilizing Transformations of Sample Coefficient of Variation from Inverse Gaussian Distribution

YOGENDRA P. CHAUBEY, MURARI SINGH AND DEBARAJ SEN

Indian Statistical Institute, Delhi Centre
7, SJSS Marg, New Delhi–110016, India

Symmetrizing and Variance Stabilizing Transformations of Sample Coefficient of Variation from Inverse Gaussian Distribution

Yogendra P. Chaubey,[*] Murari Singh[†] and Debaraj Sen
Department of Mathematics and Statistics
Concordia University, Montreal, Canada

Abstract

Coefficient of variation (CV) plays an important role in statistical practice and since an IG distribution may provide a good model for positive and positively skewed data, its distributional properties are of interest to practitioners. The variance stabilizing and symmetrizing transformations are often used for approximating the distribution in practice. In this paper we study the symmetrizing transformation of the square of the sample CV along the lines of Chaubey and Mudholkar (1983) that requires numerical techniques. The variance stabilizing transformation, on the other hand, is explicitly available, however its performance as an approximation to the distribution is extremely poor. An analysis of the symmetrizing transformation guided the authors to investigate the power transformation family which yielded an excellent approximation to the distribution function of the sample CV for sample sizes as small as 10 in the practical range of population CV. The resulting approximation is compared with others and its usefulness is illustrated in hypothesis testing example.

Keywords: Inverse Gaussian distribution; Coefficient of variation; Variance stabilizing transformation; Symmetrizing transformation.

[*]Currently visiting the Indian Statistical Institute-Delhi Center, New Delhi, India
[†]Currently at the International Center for Agricultural Research in the Dry Areas (ICARDA), Aleppo, Syria

1 Introduction

The coefficient of variation (CV) of a random variable (or that of the the corresponding population) is defined to be the ratio of the standard deviation to the mean of the corresponding population. It has been used in wide ranging applications in many areas of applied research including agro-biological, industrial, social and economic research (Johnson *et al.* 1994, Chapter 15). In these applications, the random variable of interest is assumed to follow a Gaussian distribution that is symmetric and has support on the whole real line [see Johnson *et al.* 1994; Laubscher 1960; Singh 1993; Chaubey *et al.* 2014]. However, in many of these applications the random variable may be more appropriately modeled by a distribution which is positively skewed and is supported on the positive half. To model such situations, inverse Gaussian (IG) distribution is often more justified compared to lognormal, gamma and Weibull distributions (see Chhikara and Folks 1977, 1989; Kumagai *et al.* 1996, Tagaki *et al.* 1997,). More recently, Mudholkar and Natarajan (2002) discussed comparisons of shape of the IG distribution with Gaussian distribution.

Since the distribution of the sample CV, in general is not easy to handle, various approximations, mostly centered around the Gaussian distribution have been discussed in the literature; see Banik and Kibria (2011) for a comprehensive review and comparison of various approximations. Recently Chaubey *et al.* (2013) have investigated approximately normalizing transformation of CV associated with a Gaussian population and contrasted its performance with the variance stabilizing transformation that is often employed in this context. The Likelihood ratio test for CV of an IG population has been investigated by Hsieh (1990) and more recently Chaubey *et al.* (2014) have demonstrated that this test is "best invariant" under the group of scale transformations.

The purpose of this paper is to investigate properties of variance stabilizing and skewness reducing transformations for CV in the context of the IG population. The organization of the paper is as follows. In Section 2, we list some basic properties of the IG distribution along with that of the corresponding sample CV. Note that Chaubey *et al.* (2013) consider the inverse of the sample CV in the context of the Gaussian distribution, as the sample mean in the denominator may present computational problems. This is not a problem in the IG case as the reciprocal of an IG random variable is well defined (see Folks and Chhikara 1978). Section 3 presents the general formulae for the variance stabilizing transformation (VST) and that for the symmetrizing transformation (ST) that is conjectured to provide a better approximation as compared to that given by the VST. An analysis of these transformations is carried also carried out in this with the aim of examining a) stability of variance of the symmetrizing transformation, b) symmetry of the distribution of variance stabilizing transformation, c) probability distribution of the sample CV based on these transformations. Section 4 presents an analysis of the symmetrizing transformation for small and large values of CV that motivates the examination of the power transformation family that has been carried out in Section 5. Section 6 presents a numerical investigation comparing various approximation for computing the distribution of the CV. The final section, Section 7 presents a numerical example illustrating the usefulness of the symmetrizing transformation in the context of hypothesis testing.

2 Inverse Gaussian Distribution and Estimation of *CV*

The probability density function *(pdf)* of the inverse Gaussian random variable X is given by

$$f(x|\mu,\lambda) = \left\{\frac{\lambda}{2\pi x^3}\right\}^{1/2} \exp\left\{-\frac{\lambda}{2\mu^2 x}(x-\mu)^2\right\}; \quad x>0, \;\; \mu>0, \;\; \lambda>0, \tag{2.1}$$

to be denoted by $IG(\mu,\lambda)$, where μ is the mean of the distribution and λ is the dispersion parameter. This distribution was studied in detail by Tweedie (1957a, 1957b) and was brought to limelight later by a seminal paper by Folks and Chhikara (1978). For a broad review and applications of the IG family and other related results, the reader may refer to the texts by Chhikara and Folks (1989) and Seshadri (1993, 1998). The variance of this density is given by μ^3/λ, hence the corresponding population CV is given by $\gamma = \sqrt{\mu/\lambda}$. For our purpose we will consider the parameter $\phi = \gamma^2 = \mu/\lambda$. Consider a random sample $X_1, X_2, ..., X_n$ from $IG(\mu,\lambda)$, then we have two standard results concerning the distributions of the sample mean $\bar{X} = n^{-1}\sum_{i=1}^n X_i$ and $U = (n-1)^{-1}\sum_{i=1}^n \left(\frac{1}{X_i} - \frac{1}{\bar{X}}\right)$, namely

$$(i)\; \bar{X} \sim IG(\mu, n\lambda) \quad \text{and} \quad (ii)\; (n-1)\lambda\, U \sim \chi^2_{n-1}. \tag{2.2}$$

Further, the random variables $\bar{X}$ and U are independent, thus we get an unbiased estimate of ϕ given by

$$\hat{\phi} = \hat{\mu}\, \widehat{\frac{1}{\lambda}} = \bar{X}\, U \tag{2.3}$$

Using the distributional properties of $\bar{X}$ and U we can write

$$\hat{\phi} \overset{\mathscr{D}}{=} \frac{ZY}{\nu} \tag{2.4}$$

where $Z \sim IG(\phi, n)$, $Y \sim \chi^2_\nu$, $Z \overset{\text{ind}}{\sim} Y$ and $\nu = n-1$. And thus using the moments of IG from Chhikara and Folks (1989) and those of the χ^2_ν random variable, the independence of $\bar{X}$ and U provides the following four raw moments of $\hat{\phi}$ that will be useful for later use:

$$E(\hat{\phi}) = \phi, \tag{2.5}$$

$$E(\hat{\phi}^2) = \phi^2 \left(1 + \frac{\phi}{n}\right)\left(1 + \frac{2}{\nu}\right), \tag{2.6}$$

$$E(\hat{\phi}^3) = \phi^3 \left(1 + \frac{3\phi}{n} + \left(\frac{3\phi}{n}\right)^2\right)\left(1 + \frac{2}{\nu}\right)\left(1 + \frac{4}{\nu}\right), \tag{2.7}$$

$$\text{and } E(\hat{\phi}^4) = \phi^4 \left(1 + \frac{6\phi}{n} + \left(\frac{15\phi}{n}\right)^2 + \left(\frac{15\phi}{n}\right)^3\right)\left(1 + \frac{2}{\nu}\right)\left(1 + \frac{4}{\nu}\right)\left(1 + \frac{6}{\nu}\right). \tag{2.8}$$

The central moments of $\hat{\phi}$ may therefore be deduced as

$$E(\hat{\phi} - \phi)^2 = \mu_2(\phi) = \phi^2 \left[\frac{2}{\nu} + \left(1 + \frac{2}{\nu}\right)\frac{\phi}{n}\right], \tag{2.9}$$

$$E(\hat{\phi} - \phi)^3 = \mu_3(\phi) = \phi^3 \left[\frac{8}{\nu^2} + \frac{12}{\nu}\left(1 + \frac{2}{\nu}\right)\frac{\phi}{n} + 3\left(1 + \frac{6}{\nu} + \frac{8}{\nu^2}\right)(\frac{\phi}{n})^2\right], \tag{2.10}$$

$$\begin{aligned} \text{and} \quad E(\hat{\phi} - \phi)^4 = \mu_4(\phi) = \phi^4 \Bigg[& \frac{12}{\nu^2}\left(1 + \frac{4}{\nu}\right) + \frac{12}{\nu}\left(1 + \frac{14}{\nu} + \frac{24}{\nu^2}\right)\left(\frac{\phi}{n}\right) \\ & + 3\left(1 + \frac{36}{\nu} + \frac{188}{\nu^2} + \frac{240}{\nu^3}\right)\left(\frac{\phi}{n}\right)^2 \\ & + 15\left(1 + \frac{12}{\nu} + \frac{44}{\nu^2} + \frac{48}{\nu^3}\right)\left(\frac{\phi}{n}\right)^3 \Bigg] \end{aligned} \tag{2.11}$$

3 Symmetrizing and Variance Stabilizing Transformations

The variance stabilizing transformation as first proposed by Bartlett (1947) is now widely available in standard texts (see *e.g* Rao 1973). The general formulation for a symmetrizing transformation following the same approach as of Bartlett (1947) has been put forward in Chaubey and Mudholkar (1981, 1983). Note that there have been attempts in proposing the symmetrizing transformations in a particular class. For example Hinkley (1975) considered symmetrizing transformation in the family of power transformations that has been further elaborated in Hinkley (1977) and Taylor (1985) (see also Hall 1992, and Yeo and Johnson 2000). The power transformations are easier to handle and the general symmetrizing transformation may be fretted upon due to numerical complexity. Here we follow Chaubey *et al.* (2013) that demonstrated the application of the general symmetrizing transformation on the inverse of the sample CV for a Gaussian population.

Let T_n denote a statistic based on a random sample of size n, constructed to estimate a parameter ϕ. Here we can take $T_n = \bar{X}U$, the unbiased estimator of the squared sample CV. Further, assume that $\sqrt{n}(T_n - \phi)$ tends to follow $N(0, \sigma^2(\phi))$ as $n \to \infty$. Denote the j^{th} central moment of T_n by

$$\mu_j(\phi) = E(T_n - \mu(\phi))^j, \; j = 1, 2, ... \tag{3.1}$$

where

$$\mu(\phi) = E(T_n).$$

We denote by $\xi_1(\phi) = \mu(\phi) - \phi$ as the bias of T_n and by $\mu_2(\phi) = \sigma^2(\phi) + \mu^2(\phi)$ as the MSE. Then for a smooth function $g(T_n)$, we approximately have for large n, the variance (μ_{2g} of $g(T_n)$ as (see Chaubey and Mudholkar 1983),

$$\mu_2(g(T_n)) = (g'(\phi))^2(1 + \xi_1(\phi)R)^2 \left[\mu_2(\phi) + R_1\mu_3(\phi) + \frac{1}{4}R_1^2(\mu_4(\phi) - {\mu_2}^2(\phi))\right] \tag{3.2}$$

where

$$R = \frac{g''(\phi)}{g'(\phi)} \text{ and } R_1 = \frac{R}{1 + \xi_1(\phi)R}. \tag{3.3}$$

And the third central moment μ_{3g} of T_n (up to order $O(1/n^2)$ is given by

$$\mu_3(g(T_n)) = (g'(\phi))^3(1 + \xi_1(\phi)R)^3 \left[\mu_3(\phi) + \frac{3}{2}R_1(\mu_4(\phi) - \mu_2^2(\phi))\right], \tag{3.4}$$

where we have omitted terms containing central moments of order higher than 4 (this assumes that the third and fourth central moments are of order $O(1/n^2)$ and the higher order moments are of lower order). In the present case $\xi_1 = 0$, hence $R_1 \equiv R$.

The *variance stabilizing transformation* (VST), may now be obtained using (3.2), ignoring the last two terms which are of $O(n^{-2})$, as

$$g'(\phi) = C\,\sigma(\phi)$$

where C is a constant. Hence

$$g(\phi) = C\int \frac{1}{\sigma(\phi)} d\phi. \tag{3.5}$$

The approximate *symmetrizing transformation* (ST), is obtained by equating the third moment of $g(X_n)$ given in (3.4) to be zero, as

$$g_s(\phi) = \int \mathrm{e}^{-a(\phi)} d\phi \tag{3.6}$$

where

$$a(\phi) = \frac{2}{3}\int \left\{\frac{f_1(\phi)}{f_2(\phi)}\right\} d\phi \tag{3.7}$$

with $f_1(.)$ and $f_2(.)$ being defined as

$$f_1(\phi) \;=\; \mu_3(\phi), \tag{3.8}$$
$$f_2(\phi) \;=\; \mu_4(\phi) - \mu_2^2(\phi). \tag{3.9}$$

In general the integrals may not be available in explicit forms. We will see that the VST is explicitly available, however, the ST is not. Chaubey *et al.* (2013) provided *R*-codes for solving the integral numerically that will be adopted here also for computations. However, we will develop explicit solutions for large ϕ and small ϕ and provide an approximate solution by considering an appropriate mixture of the two extreme cases. These are detailed in the next subsections.

3.1 Variance stabilizing transformation for $\hat{\phi}$.

The variance stabilizing transformation (VST), denoted by say $g_v(\hat{\phi})$ is obtained, from (3.4) by substituting

$$\sigma^2(\phi) = n\mathrm{Var}(\hat{\phi}) = n(a\phi^2 + b\phi^3), \tag{3.10}$$

where

$$a = \frac{2}{\nu},\; b = (1 + \frac{2}{\nu})/n.$$

Thus the (approximate) VST is given by

$$\begin{aligned} g_\nu(\phi) &= \int \frac{1}{\sqrt{nVar(\hat{\phi})}} d\phi \\ &= \frac{1}{\sqrt{n}} \int \frac{1}{\phi\sqrt{a+b\phi}} d\phi \end{aligned} \tag{3.11}$$

The integral may be obtained explicitly by substitution $\phi = (a/b)\tan^2\theta$ (see also Gradshteyn and Ryzhik 2007, formula 2.266; there is an additional constant of integration in our formula). This gives

$$\begin{aligned} \int \frac{d\phi}{\phi\sqrt{a+b\phi}} &= \frac{2}{\sqrt{a}} \int \mathrm{cosec}\theta d\theta \\ &= \frac{2}{\sqrt{a}} \ln\left(\frac{\sin\theta}{1+\cos\theta}\right) \\ &= \frac{2}{\sqrt{a}} \ln\left(\frac{\sqrt{b\phi}}{\sqrt{a}+\sqrt{a+b\phi}}\right). \end{aligned} \tag{3.12}$$

It is obvious that the above function is a decreasing function of ϕ, we consider the following (that differs by a multiplicative constant) as the

$$g_\nu(\phi) = \ln \frac{\sqrt{(2n/\nu)} + \sqrt{(2n/\nu) + (1+2/\nu)\phi}}{\sqrt{(1+2/\nu)\phi}}. \tag{3.13}$$

that can be written as

$$g_\nu(\phi) = \sinh^{-1}\left(\frac{B}{\sqrt{(\phi)}}\right) = \ln\left[\frac{B}{\sqrt{(\phi)}} + \sqrt{1+\frac{B^2}{\phi}}\right], \tag{3.14}$$

where

$$B = \sqrt{\frac{2n}{n+1}}$$

Remark: It may be noted that the above transformation is very similar to the one in the case of Gaussian distribution (see Singh 1993) given by

$$g\left(\frac{1}{\sqrt{\phi}}\right) = \sinh^{-1}\left(\frac{B_G}{\sqrt{\phi}}\right) = \ln\left[\frac{B_G}{\sqrt{\phi}} + \sqrt{1+\frac{B_G^2}{\phi}}\right],$$

where $B_G = (1+\frac{3}{4v})\sqrt{\frac{n}{2v}}$.

3.2 Symmetrizing transformation of $\hat{\phi}$

In order to obtain symmetrizing transformation of $\hat{\phi}$, we use the expression in (3.6) where $f_1(\phi)$ and $f_2(\phi)$ are computed using the values of $\mu_3(\phi)$ and $\mu_4(\phi)$ from equations (2.10) and (2.11) respectively. It is observed that the value of the ratio $f_1(\phi)/f_2(\phi)$ diverges for the values of ϕ near *zero* hence the algorithm

$$\int s(x)dx = S(x) = \int_0^x s(u)du + S(0).$$

as used by Chaubey *et al.* (2014) for the Gaussian case, is not convenient. Thus we use the formula

$$\begin{aligned}\int_1^x s(u)dx &= S(x) - S(1) \text{ for } x \geq 1\\ -\int_x^1 s(u)dx &= S(x) - S(1) \text{ for } x < 1\end{aligned}$$

which differs from $S(x)$ by an additive constant.

The `R-codes` (Ihaka and Gentleman (1996)] that generate the value of $c_0 g_s(\phi) + c_1$ for a given sample size and a given value of the parameter ϕ for some unknown constants c_0 and c_1 is given in Appendix A as the function `fsym.IG`. The codes are written in such way that the computations become are transparent.

Figure 1 is used to demonstrate the nature of the transformation for various sample sizes. We note that this function is an increasing function of ϕ.

Note that the logarithm involved in the variance stabilizing transformation acts on a value that is necessarily less than 1, hence the values of the transformation are negative; however such is not the case with the symmetrizing transformation, hence the two transformations will be qualitatively quite different. However, since the negative value of g_v will have the same variance as g_v we could effectively consider $-g_v$ as the variance stabilizing transformation. This is plotted in Figure 2 in order to assess the qualitative nature of VST in contrast to the ST.

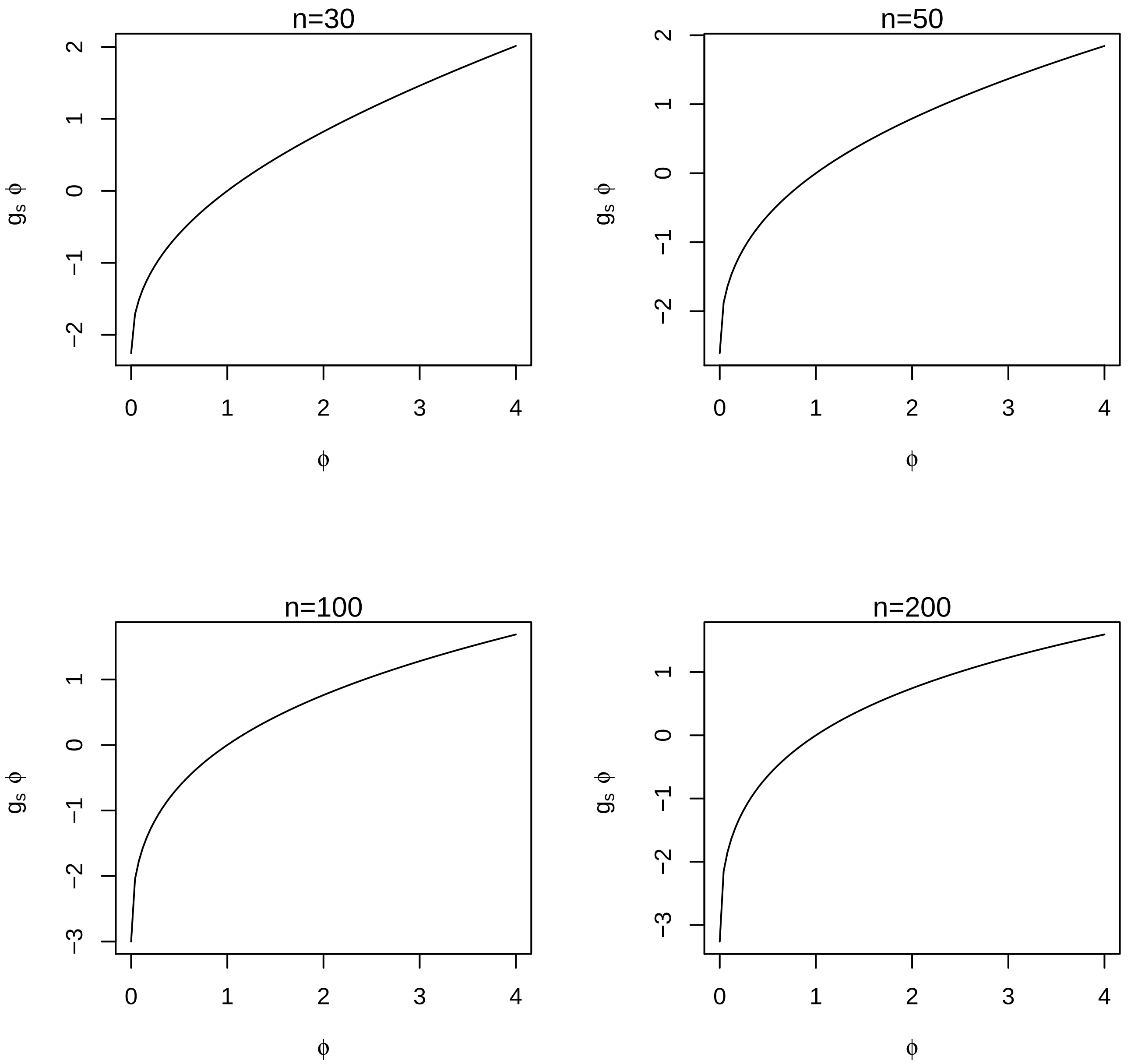

Figure 1: Symmetrizing transformation for CV for varying values of sample size

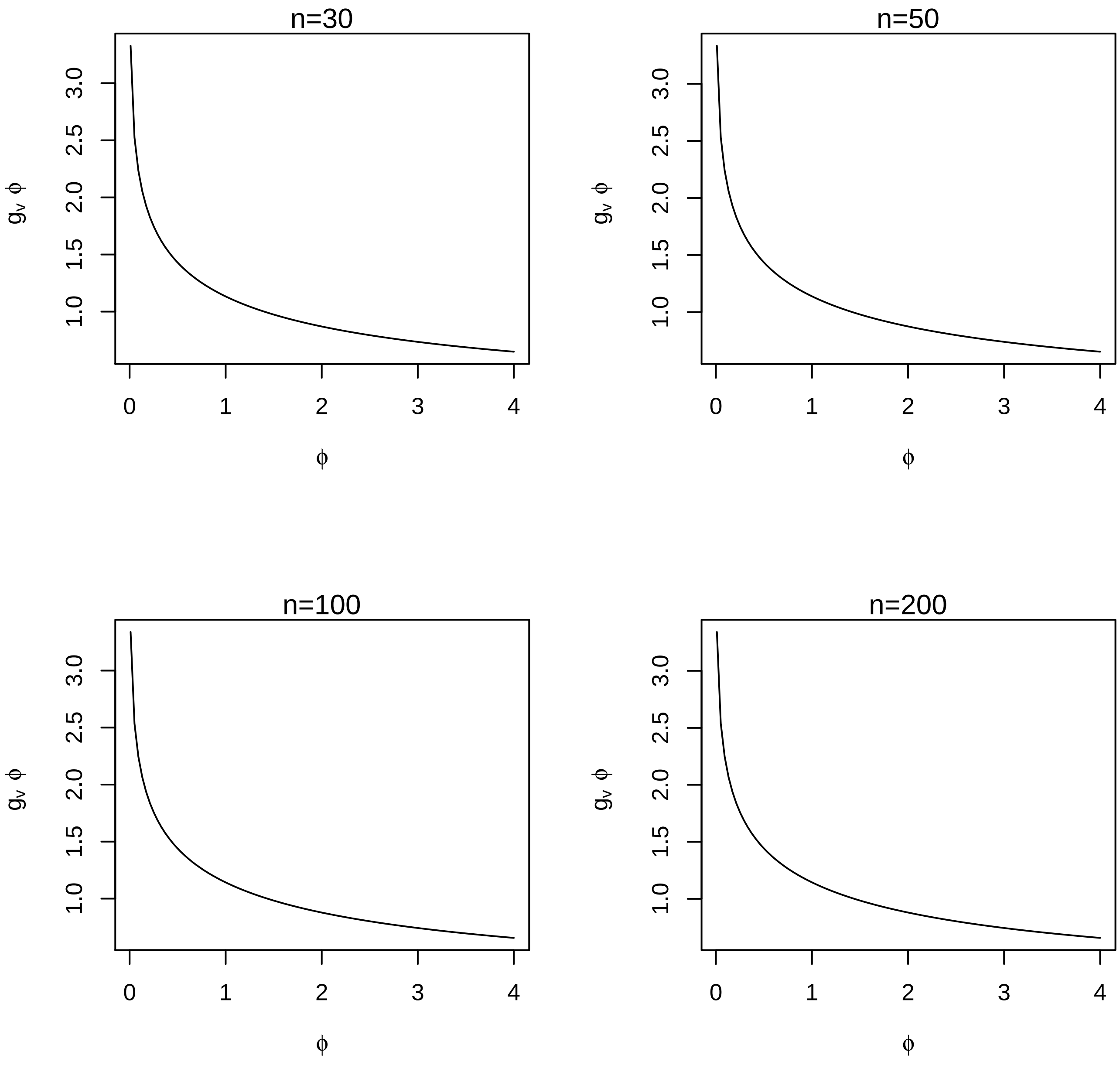

Figure 2: Variance stabilizing transformation for CV for varying values of sample size

3.3 Variance stabilizing and symmetrizing behavior of transformations

In this section, we investigate the following questions.

1. How far the variance stabilizing transformation (3.5) symmetrizes the distribution?

2. How far the symmetrizing transformation given in equation (3.6) stabilizes the variance?

To assess the degree of symmetry of VST (question 1) and untransformed statistic, we evaluate their skewness β_1 using equations (3.2) and (3.4), that is given by

$$\beta_1 = \frac{\mu_3(\phi) + \frac{3}{2}R(\mu_4(\phi) - \mu_2{}^2(\phi))}{[\mu_2(\phi) + R\mu_3(\phi) + \frac{1}{4}R^2(\mu_4(\phi) - \mu_2{}^2(\phi))]^{3/2}}, \tag{3.15}$$

where

$$R = -\frac{B\phi}{1+B\phi^2} \tag{3.16}$$

for VST, and it equals zero for untransformed case.

On the other hand to see how far the variance stability holds for the symmetrizing transformation (question 2) given in (3.6) explore the region on n and ϕ where the variance of $g(\hat{\phi})$ is nearly constant using equation (3.2).

Figure 6 gives a plot of the skewness of the VST and Figure 7 presents that of the untransformed statistic. These plots show that the VST reduces the skewness as compared to the untransformed one but the skewness is still significant even for sample sizes as large as 200. Figure 4 displays these values for various sample sizes and CV in the range of $[0, .3]$. It clearly demonstrates that the ST has poor variance stabilizing property.

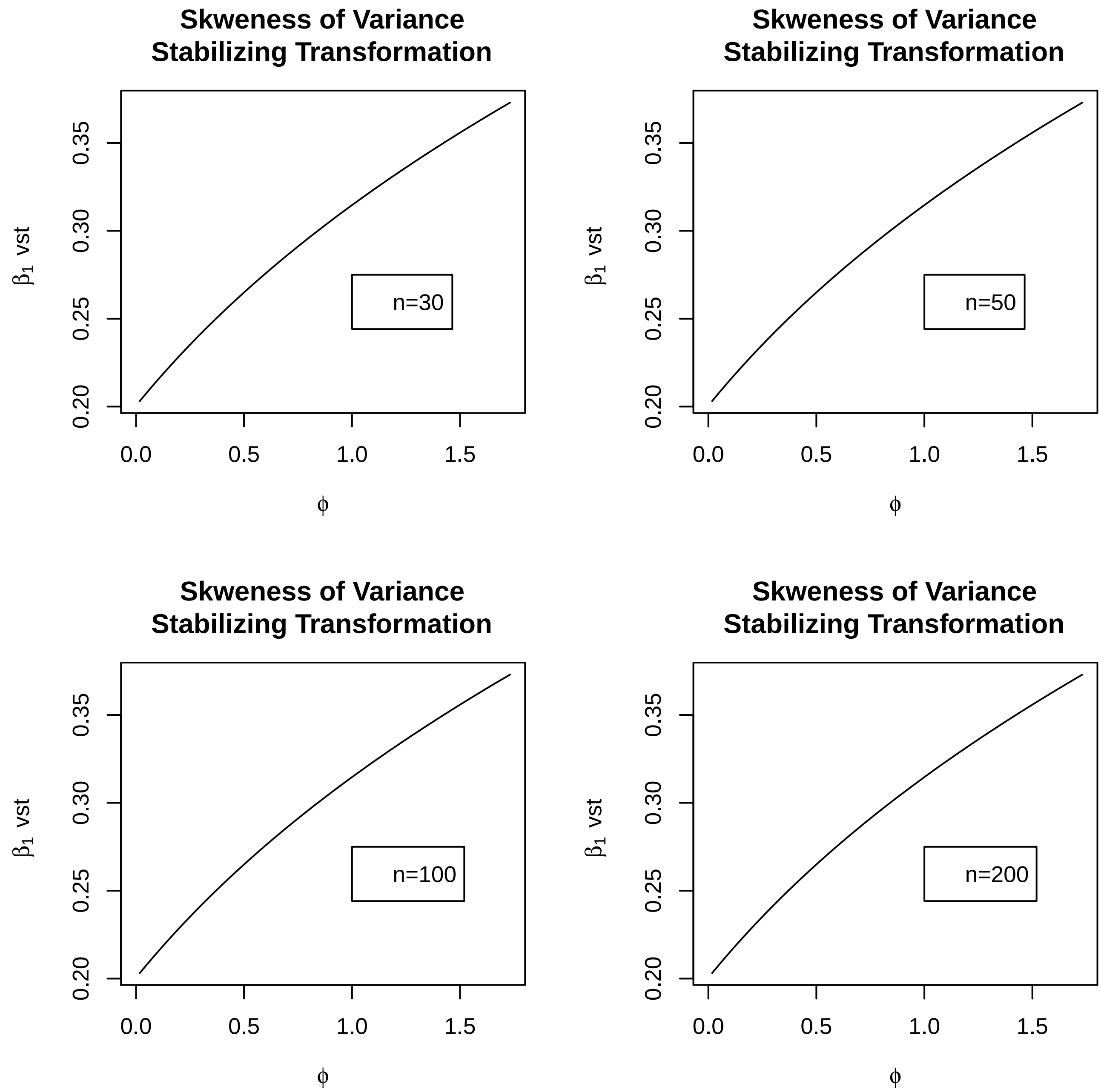

Figure 3: Skewness of the VST for varying values of sample size

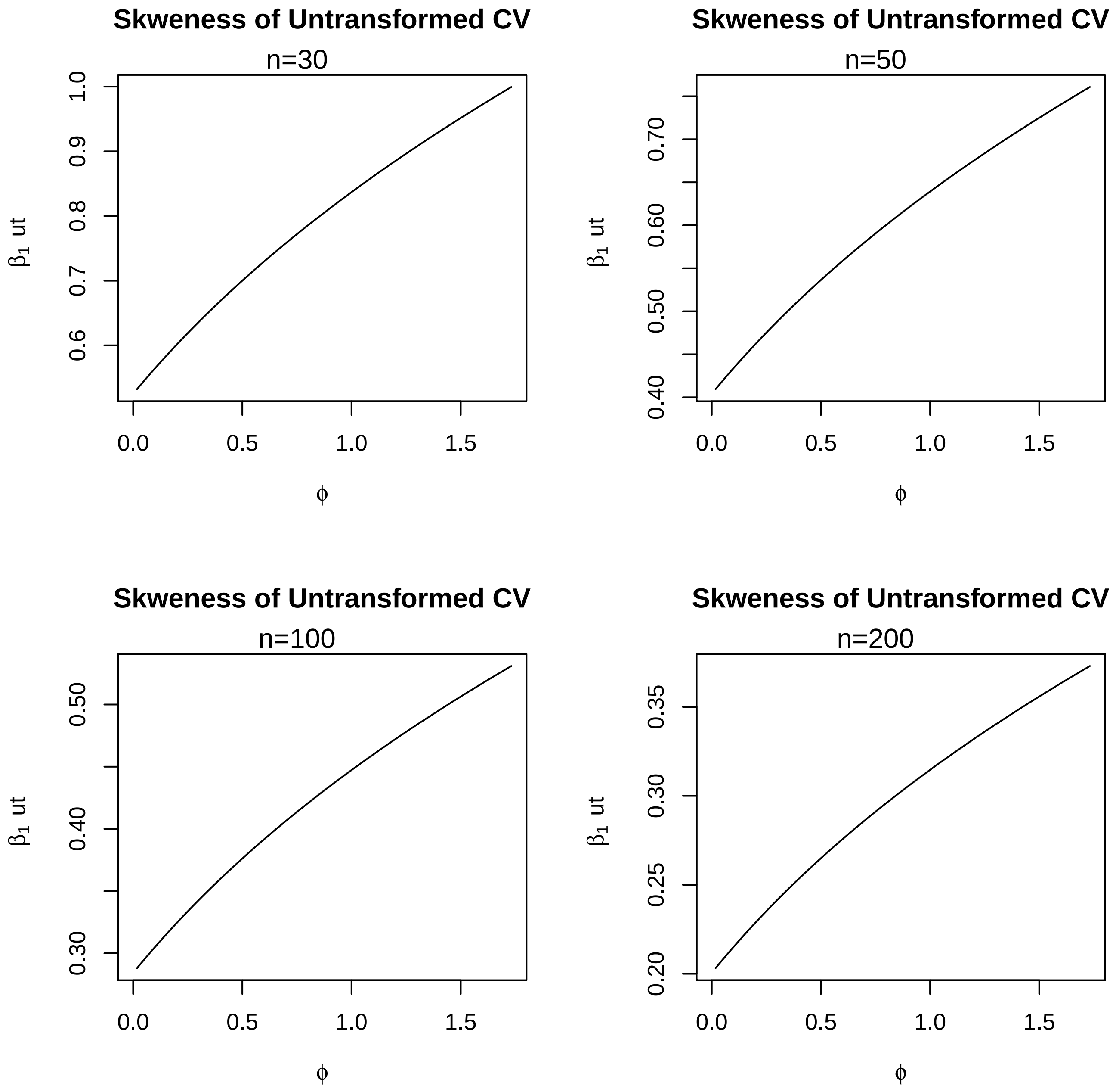

Figure 4: Skewness of the untransformed CV for varying sample sizes

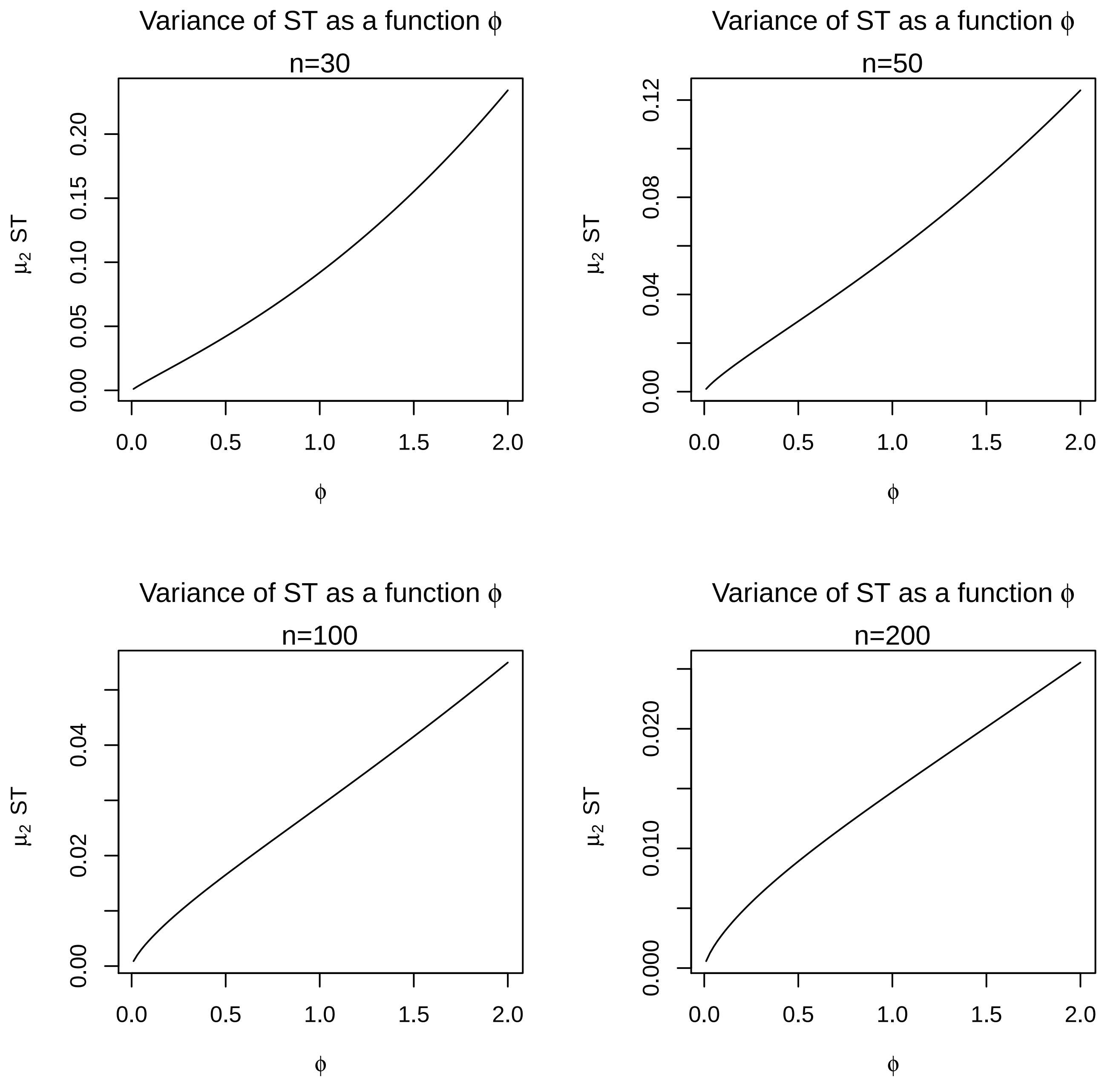

Figure 5: Variance of symmetrizing transformation of the coefficient of variation as a function of ϕ.

4 Nature of the symmetrizing transformation for large and small values of ϕ.

Small values of ϕ : For small values of ϕ we have approximately (for large values of n)

$$\begin{aligned}
\mu_2(\phi)^2 &\approx \frac{4\phi^4}{\nu^2}, \\
\mu_3(\phi) &\approx \frac{8\phi^3}{\nu^2}, \\
\text{and } \mu_4(\phi) &\approx \frac{12\phi^4}{\nu^2}.
\end{aligned}$$

Hence $f_1(\phi) = 8\phi^3/\nu^3$ and $f_2(\phi) = \mu_4(\phi) - \mu_2^2(\phi) = 8\phi^4/\nu^3$ that gives approximately

$$a(\phi) \approx \frac{2}{3}\int \frac{1}{\phi}d\phi = \frac{2}{3}\ln\phi, \tag{4.1}$$

and the corresponding expression for $g_s(\phi)$ therefore is given by

$$g_s(\phi) = \int \mathrm{e}^{-\frac{2}{3}\ln\phi}d\phi = \frac{1}{3}\phi^{1/3}. \tag{4.2}$$

This transformation heuristically makes sense as for small values of ϕ, Z is close μ and therefore $\hat{\phi}$ behaves like a χ^2 random variable for which the Wilson-Hilferty (1931) cube-root transformation is recognized as an excellent normalizing transformation. Now we consider the large values of ϕ.

Large values of ϕ : In this case retaining terms up to order $1/n^2$ and assuming $n \approx \nu$, we can write

$$\begin{aligned}
f_1(\phi) &\approx \phi^3[\frac{8}{\nu^2} + \frac{12}{\nu^2}\phi + \frac{3}{\nu^2}\phi^2] \\
&= \frac{\phi^3}{\nu^2}[8 + 12\phi + 3\phi^2].
\end{aligned}$$

and

$$
\begin{aligned}
f_2(\phi) &\approx \phi^4[\frac{8}{\nu^2}+\frac{8}{\nu^2}\phi+\frac{2}{\nu^2}\phi^2] \\
&= \frac{\phi^4}{\nu^2}[8+8\phi+2\phi^2].
\end{aligned}
$$

Thus we approximately have

$$
\begin{aligned}
\frac{f_1(\phi)}{f_2(\phi)} &= \frac{3\phi^2+12\phi+8}{\phi(8+8\phi+8\phi^2} \\
&= \frac{3\phi^2+12\phi+8}{2\phi(\phi+2)^2} \\
&= \frac{1}{\phi}+\frac{1}{2(\phi+2)}+\frac{1}{(\phi+2)^2}
\end{aligned}
$$

and then

$$
\begin{aligned}
a(\phi) &= \frac{2}{3}\int\frac{f_1(\phi)}{f_2(\phi)}d\phi \\
&= \frac{2}{3}[\ln\phi+\frac{1}{2}\ln(\phi+2)-\frac{1}{\phi+2}
\end{aligned}
$$

Using the approximation $\ln(1-\frac{2}{\phi+2})\approx-\frac{2}{\phi+2}$ the above can be simplified as

$$
a(\phi)=\frac{1}{3}[\ln\phi^2(\phi+2)\frac{\phi}{\phi+2}=\ln\phi. \tag{4.3}
$$

and therefore for large values of ϕ we have

$$
g_s(\phi)=\int \mathrm{e}^{-\ln\phi}\mathrm{d}\phi=\ln\phi \tag{4.4}
$$

and we find that the log-transformation is approximate symmetrizing transformation for large values of ϕ.

The above analysis shows that we might like to search for normalizing transformations in the family of power transformations $\phi\mapsto\phi^\lambda$ where $\lambda=0$ signifies logarithmic transformation. We can adopt the technique of Jensen and Solomon (1972) that was developed for seeking the best normalizing transformation for a quadratic form. This is explored in the next section.

5 Normalizing Transformation for CV in Power Transformation Family

The technique in Jensen and Solomon (1972) has been adopted to non-negative random variables by Mudholkar and Trivedi (1981) that we outline here. Let $\kappa_r,\ r=1,2,..$ denote the r^{th} cumulant of a non-negative random variable T and assume that $\psi_r=\kappa_r/\kappa_1,\ r=2,3,...$ are bounded as $\kappa_1\to\infty$. Then, using a Taylor series expansion, we can write the expectation of $(T/\kappa_1)^h$ as

$$\mu'_{1h}=1+\frac{h(h-1)\psi_2}{2\kappa_1}+\frac{h(h-1)(h-2)}{24\kappa_1^2}[4\psi_3+3(h-3)\psi_2^2]+O(k_1^{-3}). \tag{5.1}$$

The above expression may be used to obtain the r^{th} moment $\mu'_{rh}=E[(T/\kappa_1)^h]^r$ by a simple substitution of h by rh. This provides the following series expansions for the central moments $\mu_r(h)$ of $(T/\kappa_1)^h$, $r=2,3,4$ in terms of the powers of κ_1^{-1} :

$$\mu_{2h} = \frac{h^2\psi_2}{\kappa_1}+\frac{h^2(h-1)}{2\kappa_1^2}[2\psi_3+(3h-5)\psi_2^2]+O(\kappa_1^{-3}), \tag{5.2}$$

$$\mu_{3h} = \frac{h^3}{\kappa_1^2}[\psi_3+(3h-1)\psi_2^2]+O(\kappa_1^{-3}), \tag{5.3}$$

$$\mu_{4h} = \frac{3h^4\psi_2^2}{\kappa_1^2}+O(\kappa_1^{-3}). \tag{5.4}$$

$$\tag{5.5}$$

If T is asymptotically distributed as $\kappa_1\to\infty$ then as Mudholkar and Trivedi (1981) argue, so is T^h by Mann-Wald (1943) theorem. In order to accelerate the convergence to normality, we may choose h so that the leading term in $\mu_3(h)$ is zero. The resulting value of h denoted by h_0 that approximately symmetrizes $(T/\kappa_1)^h$ is thus given by

$$h_0=1-\frac{\kappa_1\kappa_3}{3\kappa_2^2}. \tag{5.6}$$

In order to use the above formulation for the CV, we take $T=\nu\hat{\phi}/\phi$. The cumulants of T needed for our purpose may be obtained from the central moments of $\hat{\phi}$ given in (2.9)-(2.10) that are given below:

$$\kappa_1 = \nu, \tag{5.7}$$

$$\kappa_2 = \nu[2+(\nu+2)\frac{\phi}{n}], \tag{5.8}$$

$$\kappa_3 = \nu[8+12(\nu+2)\frac{\phi}{n}+3(\nu^2+6\nu+8)(\frac{\phi}{n})^2]. \tag{5.9}$$

The asymptotic normality of T follows from that of $\hat{\phi}$ and obviously $\psi_r, r = 2,3,\ldots$ are bounded as $n \to \infty$. And hence we can approximate the distribution of $(\hat{\phi}/\phi)^{h_0}$ by the normal distribution with mean μ'_{1h_0} and variance $\sigma^2(h_0) = \mu_{2h_0}$ as given in (5.1) and (5.2), replacing h by h_0.

We plot the values of the powers h_0 for various sample sizes as a function of ϕ in Figure 3. It is interesting to note that for small values of ϕ the optimum power is close to $1/3$ and for large values of ϕ this is closer to zero. Owing to the analysis of the general symmetrizing transformation for small and large ϕ in the previous section, this implies that the power transformation family may be adequate in contrast to the general transformation g_s that can be only computed numerically.

6 A Comparison of the Approximations using the Transformed Statistics

The approximation afforded by the power transformation family provides an excellent approximation for computing the distribution of the CV as seen from the previous section. It is thus a natural question to ask if the numerical effort is worth in using the general transformation discussed in §3.2 over the simplicity of the explicit formulae using the power family of transformations. In order to investigate this issue we compare the two approximations, namely,

(i)

$$(\hat{\phi}/\phi)^{h_0} \sim N(\mu'_{1h_0}, \sigma^2(h_0))$$

with

(ii)

$$g_s(\hat{\phi}) \sim N(\mu_g(\phi), \sigma_g^2(\phi))$$

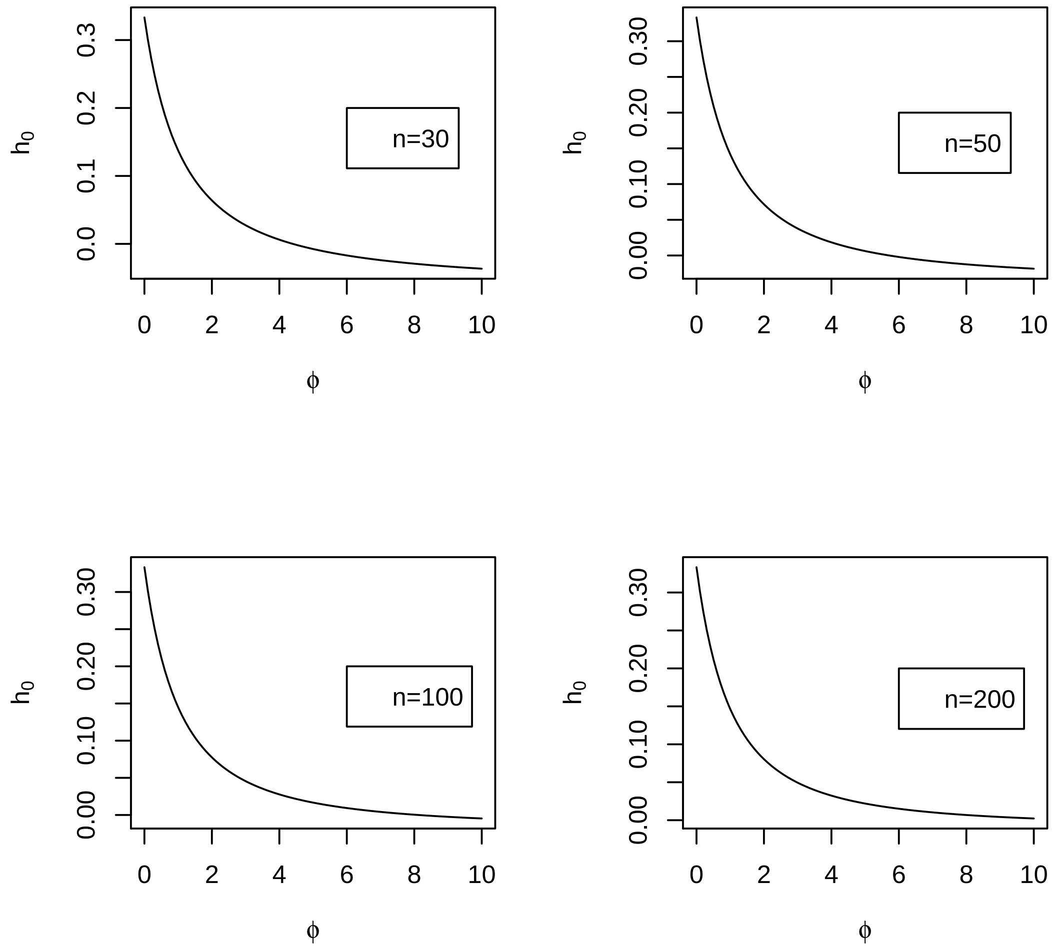

Figure 6: Optimum power for symmetrizing CV for varying values of sample size

where

$$\mu_g(\phi) = g_s(\phi) + \frac{1}{2} g_s''(\phi)\mu_2(\phi), \tag{6.1}$$

$$\sigma_g^2(\phi) = (g_s'(\phi))^2 \left[\mu_2(\phi) + R\mu_3(\phi) + \frac{1}{4}R_1^2(\mu_4(\phi) - \mu_2^2(\phi))\right] \tag{6.2}$$

where

$$R \equiv R(\phi) = -\frac{2}{3}\frac{\mu_3(\phi)}{\mu_4(\phi) - \mu_2^2(\phi)} d\phi$$

and

$$g_s'(\phi) = \int \mathrm{e}^{\mathrm{R}(\phi)} \mathrm{d}\phi.$$

Note that in the above approximation, we require g_s, g_s' and g_s'' that are numerically computed starting with g_s and g_s' and then computing g_s'' using the formula

$$g_s'' = g_s'(\phi) R(\phi).$$

Additionally we compare these in turn with approximating $\hat{\phi}$ and $\hat{\phi}_v$ with appropriate Gaussian distributions. These imply

(iii)

$$\hat{\phi} \sim N(\phi, \mu_2(\phi))$$

and

(iv)

$$g_v(\hat{\phi}) \sim N(\mu_{g_v}(\phi), \sigma^2(g_v))$$

where

$$\mu_{g_v}(\phi) = \sinh^{-1}(B/\sqrt{\phi}) + \frac{1}{2} g_v''(\phi)\mu_2(\phi) \tag{6.3}$$

and

$$\sigma^2(g_v) = g_v'(\phi)^2 \left[\mu_2(\phi) + R_v\mu_3(\phi) + \frac{1}{4}R_v^2(\mu_4(\phi) - {\mu_2}^2(\phi))\right], \tag{6.4}$$

where

$$\begin{aligned} R_v &= \frac{B(2B^2+3\phi)}{2\phi(B^2+\phi)}, \\ g_v'(\phi) &= -\frac{B}{2\phi\sqrt{B^2+\phi}}, \\ g_v''(\phi) &= \frac{B(2B^2+3\phi)}{4\phi^2(B^2+\phi)^{3/2}}. \end{aligned}$$

Figures 7 and 8 plot the distribution functions of $\hat{\phi}$ along with various approximations, where the exact values are computed using an integral formula outlined in Chaubey *et al.* (2014). These figures display the qualitative nature of the approximations and convey that the basic nature of the approximations are the same for the values of the parameters investigated. Hence we plot the errors as boxplots in the second column that clearly demonstrates that the normal approximations rendered by the untransformed statistic and the VST show the same performance whereas the symmetrizing transformation gives a significant improvement. However the power transformation even gives a better performance. One is tempted to ask 'why the general symmetrizing transformation derived numerically is poorer than the power transformation?' This may be due to accuracy lost during iterative computations of the integrals and approximation of the derivatives involved. Due to the simple nature of the power transformation and its accuracy in approximating the probabilities, we refrain from investigating the general symmetrizing transformation any further.

7 A Numerical Example

The following data from Chhikara and Folks (1977) presented below were found to fit the IG model well. These data show active repair times (in hours) of an airborne communication transceiver that are used for carrying out testing of hypothesis for coefficient of variation.

Such a test is carried out using the exact distribution in Chaubey *et al.* (2014). Let us test $H_0 : \phi \leq 2$ against $H_1 : \phi > 2$. Here $n = 46, \bar{X} = 3.6065, \sum_i((1/X_i) - (1/\bar{X})) = 27.73$. Hence,

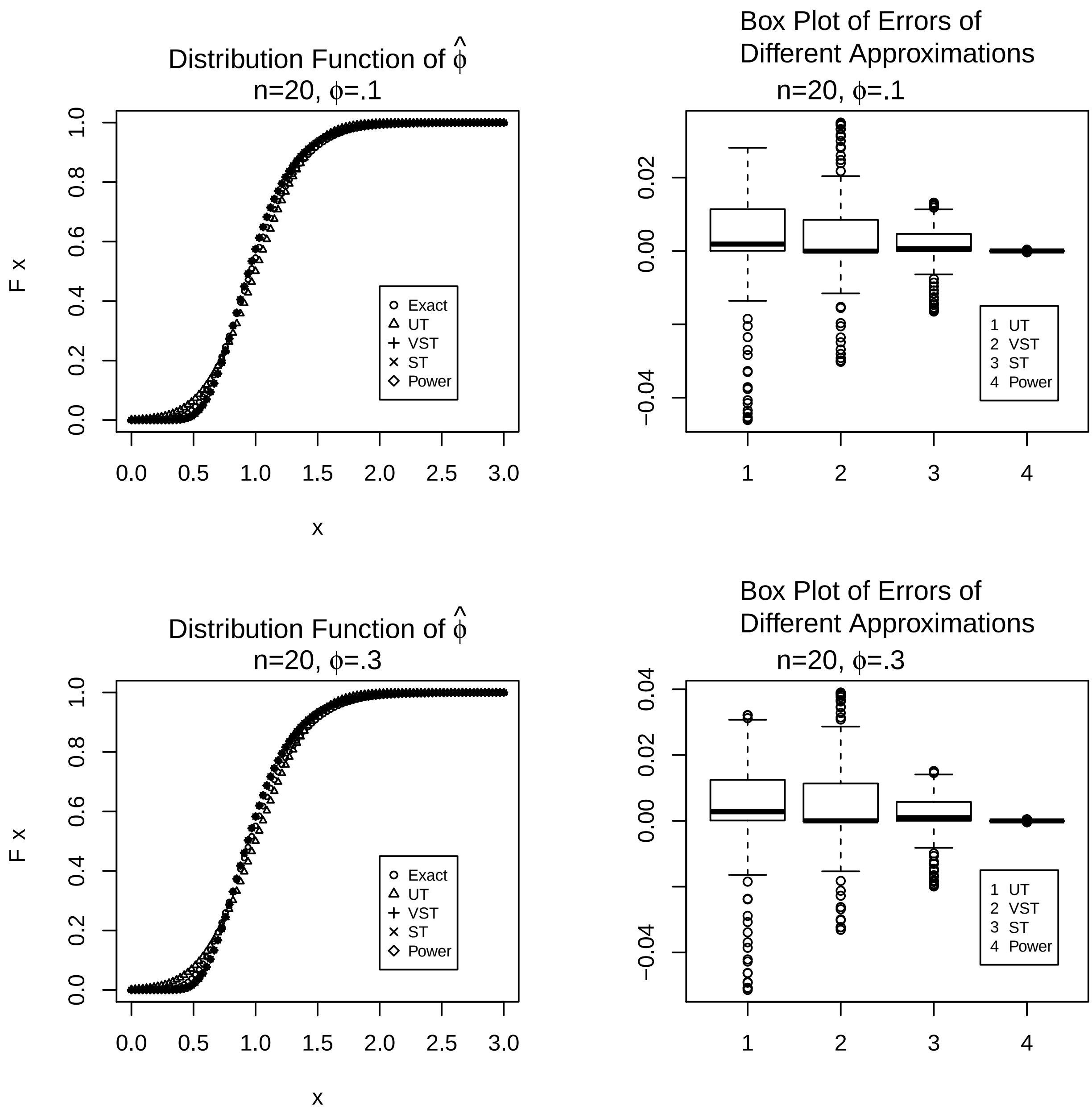

Figure 7: Normal Approximation for Various Transformations $n = 20$

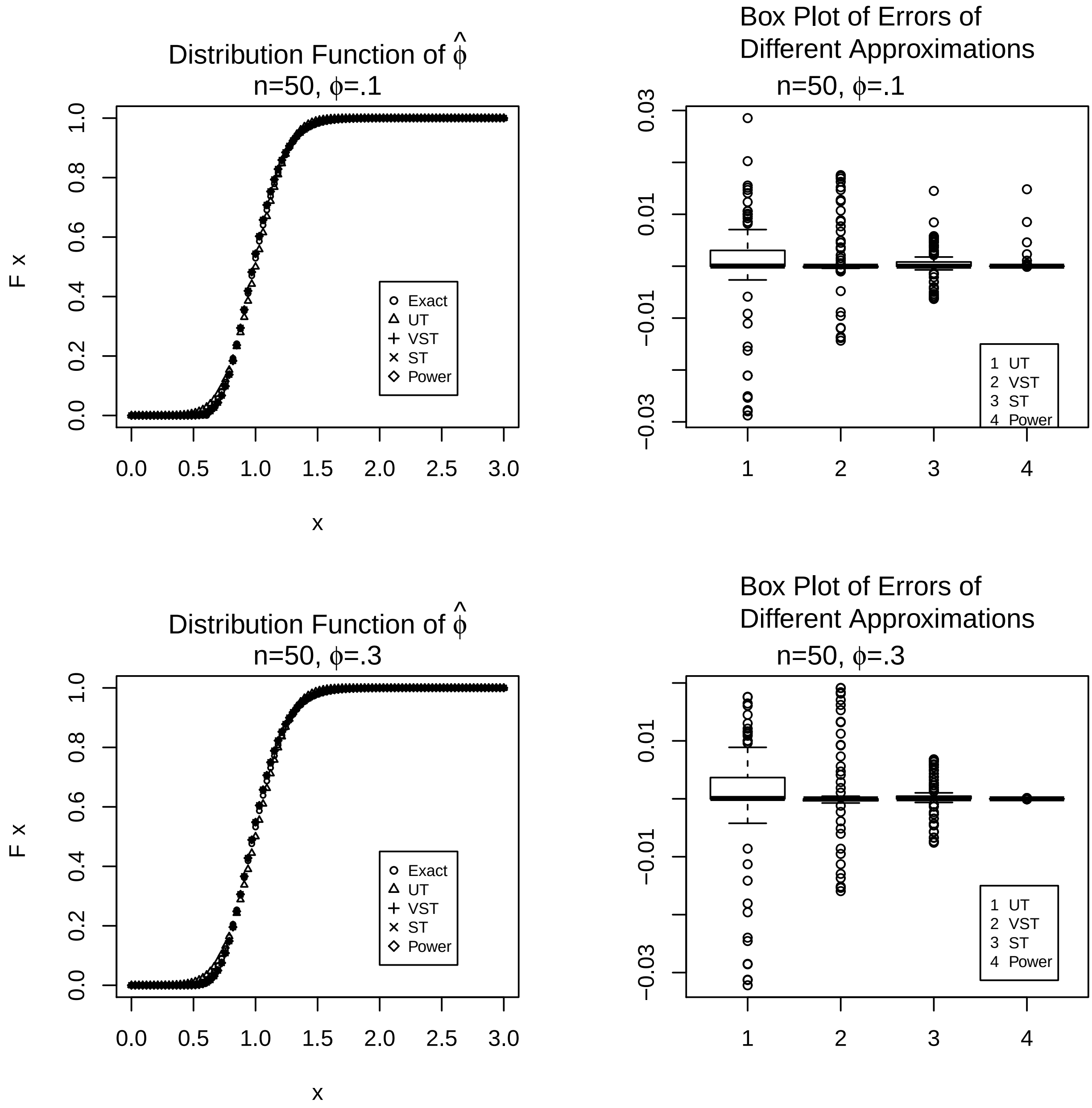

Figure 8: Normal Approximation for Various Transformations $n = 50$

Table 1: Active repair times (in hours) of an airborne communication.

0.2	0.3	0.5	0.5	0.5	0.5	0.6	0.6	0.7	0.7	0.7	0.8	0.8
1.0	1.0	1.0	1.0	1.1	1.3	1.5	1.5	1.5	1.5	2.0	2.0	2.2
2.5	2.7	3.0	3.0	3.3	3.3	4.0	4.0	4.5	4.7	5.0	5.4	5.4
7.0	7.5	8.8	9.0	10.3	22.0	24.5						

$\hat{\phi} = 3.6065 \times 27.73/45 = 2.2224$. We may use the test statistic based on the symmetrizing power transformation

$$Z = \frac{(\hat{\phi}/\phi_0)^{h_0} - \mu'_{1h_0}}{\sigma(h_0)};$$

here $\phi_0 = 2, h_0 = 0.0707, \mu'_{1h_0} = .99715, \sigma^2(h_0) = 0.0004347339$, and the observed value of Z is $Z_{obs.} = .4955$. As shown in Chaubey *et al.* (2014), the best invariant test under scale transformations rejects H_0 for larger values of $\hat{\phi}$, the test based on the Z statistic will reject the null hypothesis for $Z > Z_{obs}$ and the corresponding $P-$value $= 1 - .68988 = .31012$.

The exact P-value given in Chaubey *et al.* (2014) is .31087 that may be noted to be very close to the approximate value obtained by the symmetrizing transformation. This value is quite large for a 1% level of significance and therefore a squared CV of less than equal to 2 is accepted. This is not a surprising result for this data as the unbiased estimate of ϕ is $\hat{\phi}$ is just slightly larger than 2.

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Appendix A: R-Codes for the Symmetrizing Function

```
###Computing the symmetrization transformation as a function of phi
###Symmetrizing function
##Input:
##phi: Value of (mu/lambda)
```

```
##ss: Sample size n
##Output: g_s(phi)
###Computing the symmetrization transformation as a function of phi
################
### Symmetrizing unction
fsym.IG<-function(phi,ss){
if (phi>1) {xl<-1;xu<-phi}
else {xl<-phi;xu<-1}
fval<- integrate(f1f2Int.IG,xl,xu,subdivisions=1000,ss=ss)$value
if (phi<1) fval<--fval
fval}
######################
hfun.IG<-function(phi,ss=ss){
if (phi>0) {
nu<-ss-1;d<-phi/ss
c21<-(2/nu);c22<-(1+c21)
c31<-8/nu^2;c32<-12*c22/nu;c33<-3*(1+(6/nu)+(8/nu^2))
c41<-12*(1+(4/nu))/nu^2;c42<-12*(1+(14/nu)+(24/nu^2))/nu
c43<-3*(1+(36/nu)+(188/nu^2)+(240/nu^3))
c44<-15*(1+(12/nu)+(44/nu^2)+(48/nu^3))
mu2<-phi^2*(c21+c22*d)
mu3<-phi^3*(c31+c32*d+c33*d^2)
mu4<-phi^4*(c41+c42*d+c43*d^2+c44*d^3)
result<-mu3/(mu4-mu2^2)}
else result<-Inf
result}
##Vector version of hfun.IG
hfunInt.IG<-function(x,ss)sapply(x,hfun.IG,ss=ss)

f1f2.IG<-function(phi,ss){
if (phi==0) result<-Inf
else {
```

```
if (phi>1){xl<-1;xu<-phi}
else {xl<-phi;xu<-1}
 fval<- integrate(hfunInt.IG,xl,xu,subdivisions=1000,ss=ss)$value
if (phi<1) fval<--fval
result<- exp(-(2/3)*fval)}
result}


#Vectorised version of f1f2.IG
f1f2Int.IG<-function(x,ss)sapply(x,f1f2.IG,ss=ss)
```

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