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On the construction of asymmetric orthogonal arrays

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On the construction of asymmetric orthogonal arrays

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Abstract: A general method of construction of asymmetric orthogonal arrays was proposed by Suen, Das and Dey (2001), which led to several new families of orthogonal arrays of strength three and four. Using this method, we construct some more asymmetric orthogonal arrays of strength greater than two.

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1 Introduction and Preliminaries

Asymmetric orthogonal arrays introduced by Rao (1973) have received considerable attention in recent years. Such arrays are useful in experimental designs as universally optimal fractions of asymmetric factorials. Asymmetric orthogonal arrays have also been found very useful in industrial experimentation for quality improvement. Construction of asymmetric orthogonal arrays of strength *two* has been an area of intense research and one may refer to Hedayat, Sloane and Stufken (1999) for an excellent description of these. Relatively less is known on the construction of asymmetric orthogonal arrays of strength larger than two. Apart from the methods of construction of asymmetric orthogonal arrays of strength larger than two described in Dey and Mukerjee (1999) and Hedayat *et al.* (1999), further work on the construction of arrays of strength three or higher have been carried out e.g., by Suen *et al.* (2001), Suen and Dey (2003), Nguyen (2008) and Jiang and Yin (2013). In particular, Suen *et al.* (2001) proposed a general method to construct asymmetric orthogonal arrays of arbitrary strength. This method was then applied by them to obtain several families of asymmetric orthogonal arrays of strength three and four. Suen and Dey (2003) combined tools from finite projective geometry with the method of Suen *et al.* (2001) to construct some new families of asymmetric orthogonal arrays of strength three and four. In this paper, we apply the method of Suen *et al.* (2001) to obtain some more asymmetric orthogonal arrays of strength three. We also give an alternative method of construction of a family of asymmetric orthogonal arrays of strength four, which appears to be more direct than that of Suen and Dey (2003).

Recall that an orthogonal array $OA(N, n, s_1 \times \cdots \times s_n, g)$ of strength g, is an $N \times n$ matrix with symbols in the ith column from a finite set of $s_i(\geq 2)$ symbols, $1 \leq i \leq n$, such that in every $N \times g$ submatrix, all possible combinations of symbols appear equally often as a row. Orthogonal arrays with $s_1 = s_2 = \cdots = s_n = s$ (say) are called symmetric and are denoted by $OA(N, n, s, g)$; otherwise, the array is called *asymmetric* (or, with mixed levels).

Henceforth, the columns of an $OA(N, n, s_1 \times \cdots \times s_n, g)$ will be called *factors*, following the terminology in factorial experiments, and these factors will be denoted by $F_1, \ldots, F_n$. Throughout this paper, we take the integer $s \geq 2$ to be a prime or a prime power, i.e., $s = p^q$, where p is a prime and $q \geq 1$ is an integer. The Galois field of order s will be denoted by $GF(s)$, 0 and 1 being the identity elements of the field corresponding to the operations 'addition' and 'multiplication', respectively. Also, throughout a prime will denote transposition. We shall need the following results, the first of which is well known and the second one is due to Suen *et al.* (2001).

Lemma 1. *Let α and β be two elements of $GF(s)$ such that $\alpha^2 = \beta^2$. Then (i) $\alpha = \beta$ if s is even, (ii) either $\alpha = \beta$ or $\alpha = -\beta$, if s is odd.*

Lemma 2. *For a positive integer h, let D be a $(2h+1) \times s^h$ matrix with columns of the form $(\alpha_1^2, \ldots, \alpha_h^2, \alpha_1, \ldots, \alpha_h, 1)'$, where $(\alpha_1, \ldots, \alpha_h)$'s are all possible $h-$tuples with entries from $GF(s)$. Then any three distinct columns of D are linearly independent.*

If $\alpha_0, \alpha_1, \ldots, \alpha_{s-1}$ are the elements of $GF(s)$, then it follows from Lemma 1 that the set $S = \{\alpha_0^2, \alpha_1^2, \ldots, \alpha_{s-1}^2\}$ contains all the elements of $GF(s)$ if s is even. If s is odd, then one element of S is 0 and there are $(s-1)/2$ distinct non-zero elements of $GF(s)$, each appearing twice in S.

For the factor F_i $(1 \leq i \leq n)$, define the $m \times 1$ columns, $\boldsymbol{p}_{i1}, \ldots, \boldsymbol{p}_{iu_i}$ with elements from $GF(s)$. Then, for the n factors we have in all $\sum_{i=1}^n u_i$ columns. Also, let B be an $s^m \times m$ matrix whose rows are all possible m-tuples over $GF(s)$. Suen *et al.* (2001) proved the following result.

Theorem 1. *Consider an $m \times \sum_{i=1}^n u_i$ matrix $C = [A_1 \vdots A_2 \vdots \cdots \vdots A_n]$, $A_i = [\boldsymbol{p}_{i1}, ..., \boldsymbol{p}_{iu_i}]$, $1 \leq i \leq n$, such that for every choice of g matrices $A_{i_1}, ..., A_{i_g}$ from $A_1, ..., A_n$, the $m \times \sum_{j=1}^g u_{i_j}$ matrix $[A_{i_1}, ..., A_{i_g}]$ has full column rank over $GF(s)$. Then an $OA(s^m, n, (s^{u_1}) \times (s^{u_2}) \times ... \times (s^{u_n}), g)$ can be constructed.*

A little elaboration of the result in Theorem 1 seems to be in order to make the construction transparent. For a fixed choice of g indices $\{i_1, \ldots, i_g\} \in \{1, \ldots, n\}$, let $C_1 = [A_{i_1}, \ldots, A_{i_g}]$ and $r = \sum_{j=1}^g u_{i_j}$. By the rank condition of Theorem 1, it follows that in the product BC_1, each possible $1 \times r$ vector with entries from $GF(s)$ appears s^{m-r} times. Now, for each j, $1 \leq j \leq g$, replace the $s^{u_{i_j}}$ distinct combinations under A_{i_j} by $s^{u_{i_j}}$ distinct symbols using a 1–1 correspondence. In the resultant $s^m \times g$ matrix, (i) the i_jth column has $s^{u_{i_j}}$ symbols $(1 \leq j \leq g)$ and (ii) each of the $\prod_{j=1}^g s^{u_{i_j}}$ combinations of the symbols occurs equally often as a row. Hence, the desired orthogonal array with parameters as in Theorem 1 can be constructed.

2 Construction of orthogonal arrays of strength three

In this section, we construct two families of orthogonal arrays of strength three.

Theorem 2. *Let* $k \geq 2$ *be an integer and* t *denote the largest integer not exceeding* $k/2$.

(i) If s *is an odd prime or an odd prime power, then an orthogonal array* $OA(s^{2k+1}, s^k + (k-1)(s-1)^t + 2, (s^2) \times s^{s^k+(k-1)(s-1)^t+1}, 3)$ *can be constructed.*

(ii) If s *is a prime power of two, then an orthogonal array* $OA(s^{2k+1}, s^k + (k-1)s^t + 2, (s^2) \times s^{s^k+(k-1)s^t+1}, 3)$ *can be constructed.*

Proof. (i) Let s be an odd prime or an odd prime power. Let F_1 have s^2 symbols and the rest of the factors have s symbols each. The matrices A_i, $1 \leq i \leq n$, corresponding to the different factors are chosen as below, where $n = s^k + (k-1)(s-1)^t + 2$.

A_1 is chosen as $A_1 = \begin{bmatrix} I_2 & \mathbf{0}_{2,2k-2} & \boldsymbol{e} \end{bmatrix}'$, where I_u is the identity matrix of order u, $\mathbf{0}_{u,v}$ is a $u \times v$ null matrix, $\boldsymbol{e} = (0,1)'$. The matrix A_2 is chosen as $A_2 = [0, x, \mathbf{0}_{1,2k-2}, 1]'$, $x \in GF(s)$, $x \neq 0, 1$.

Suppose $\gamma_1, \ldots, \gamma_t$ are non-zero elements of $GF(s)$. For $3 \leq i \leq (s-1)^t + 2$, if k is even, then A_i is chosen to be of the form $A_i = [\gamma_1^2, , \ldots, \gamma_t^2, \gamma_t, \ldots, \gamma_1, 0, 1, \mathbf{0}_{1,k-1}]'$. For $(s-1)^t + 3 \leq i \leq 2(s-1)^t + 2$, let A_i be of the form

$$A_i = [\gamma_1^2, \ldots, \gamma_t^2, \gamma_t, \ldots, \gamma_1, 0, 0, 1, \mathbf{0}_{1,k-2}]',$$

i.e., 1 appears in the $(k+3)$th position. The other A_i matrices for this case are obtained by putting 1 in the $(k+4)$th, $(k+5)$th, …, $2k$th position, to get a total of $(k-1)(s-1)^t$ columns of such a form. If k is odd, A_i is of the form $A_i = [0, \gamma_1^2, \ldots, \gamma_t^2, \gamma_t, \ldots, \gamma_1, 0, 1, \mathbf{0}_{1,k-1}]', \ldots, [0, \gamma_1^2, \ldots, \gamma_t^2, \gamma_t, \ldots, \gamma_1, 0, \mathbf{0}_{1,k-2}, 1, 0]'$. Finally, the last s^k columns have the form $[\alpha_k^2, \ldots, \alpha_1^2, 1, \alpha_k, \ldots, \alpha_1]'$, where $\alpha_i \in GF(s)$.

(ii) Let s be an even prime power. In this case, the matrix A_2 is chosen as $A_2 = [\mathbf{0}_{1,2k}, 1]'$. $\gamma_1, \ldots, \gamma_t$ can be any element of $GF(s)$ and thus, each γ_i has s different choices. If k is even, the total number of columns of the types

$$[\gamma_t^2, \ldots, \gamma_1^2, \gamma_t, \ldots, \gamma_1, 0, 1, \mathbf{0_{1,k-1}}]', \ldots, [\gamma_t^2, \ldots, \gamma_1^2, \gamma_t, \ldots, \gamma_1, 0, \mathbf{0_{1,k-2}}, 1, 0]'$$

is $(k-1)s^t$. If k is odd, the total number of columns of the types

$$[0, \gamma_t^2, \ldots, \gamma_1^2, \gamma_t, \ldots, \gamma_1, 0, 1, \mathbf{0_{1,k-1}}]', \ldots, [0, \gamma_t^2, \ldots, \gamma_1^2, \gamma_t, \ldots, \gamma_1, 0, \mathbf{0_{1,k-2}}, 1, 0]'$$

is $(k-1)s^t$. The other columns of the matrices A_i are chosen as in case (i) above.

One can then verify that the rank condition of Theorem 1 is met with the above choices of the matrices A_i. We give below the proof for some of the non-trivial cases. To save space, we consider only the case when s is an odd prime or an odd prime power; the case when s is a power of two can be handled in a similar fashion.

For convenience, denote the columns A_1 by 1, A_2 by 2, those of A_i, $i \in \{3, ..., (k-1)(s-1)^t+2\}$ by a, and those of A_i, $i \in \{(k-1)(s-1)^t+3, ..., s^k+(k-1)(s-1)^t+2\}$ by b, if k is even. If k is odd, denote those of A_i, $i \in \{3, ..., (k-1)s^t+2\}$ by a, and those of A_i, $i \in \{(k-1)s^t+3, ..., s^k+(k-1)s^t+2\}$ by b. The proofs below are for the case when k is even. There are many different position of 1 in the form of $[0, \gamma_1^2, \ldots, \gamma_t^2, \gamma_t, \ldots, \gamma_1, 0, 1, \mathbf{0}_{1,k-1}]'$, $\ldots$,$[0, \gamma_1^2, \ldots, \gamma_t^2, \gamma_t, \ldots, \gamma_1, 0, \mathbf{0}_{1,k-2}, 1, 0]'$. For convenience, we can select any three of them to prove the result. Without loss of generality, we choose the last three of them.

$(1aa)_1$: Here, we have

$$[A_i, A_j, A_k] = \begin{bmatrix} 1 & \ldots & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 \\ 0 & \ldots & 0 & \ldots & 0 & 0 & \ldots & 0 & 1 \\ \beta_t^2 & \ldots & \beta_t & \ldots & \beta_1 & 0 & \ldots & 1 & 0 \\ \gamma_t^2 & \ldots & \gamma_t & \ldots & \gamma_1 & 0 & \ldots & 1 & 0 \end{bmatrix}'.$$

Since there exists at least one $\beta_i \neq \gamma_i$, for $i = 1, ..., t$, the determinant of the submatrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \beta_t^2 & \beta_i & 1 & 0 \\ \gamma_t^2 & \gamma_i & 1 & 0 \end{bmatrix}'$$

is $\beta_i - \gamma_i \neq 0$.

$(1aa)_2$: Here, we have

$$[A_i, A_j, A_k] = \begin{bmatrix} 1 & \ldots & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & \ldots & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 & 1 \\ \beta_t^2 & \ldots & \beta_t & \ldots & \beta_1 & 0 & \ldots & 0 & 1 & 0 \\ \gamma_t^2 & \ldots & \gamma_t & \ldots & \gamma_1 & 0 & \ldots & 1 & 0 & 0 \end{bmatrix}'.$$

The determinant of the submatrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \beta_t^2 & 0 & 1 & 0 \\ \gamma_t^2 & 1 & 0 & 0 \end{bmatrix}'$$

is 1.

$(2aa)_1$: Here

$$[A_i, A_j, A_k] = \begin{bmatrix} 0 & x & \ldots & 0 & \ldots & 0 & 0 & \ldots & 0 & 1 \\ \beta_t^2 & \beta_{t-1}^2 & \ldots & \beta_t & \ldots & \beta_1 & 0 & \ldots & 1 & 0 \\ \gamma_t^2 & \gamma_{t-1}^2 & \ldots & \gamma_t & \ldots & \gamma_1 & 0 & \ldots & 1 & 0 \end{bmatrix}'.$$

Since there exists at least one $\beta_i \neq \gamma_i$, for $i = 1, ..., t$, the determinant of the submatrix

$$\begin{bmatrix} 0 & 0 & 1 \\ \beta_i & 1 & 0 \\ \gamma_i & 1 & 0 \end{bmatrix}'$$

is $\beta_i - \gamma_i \neq 0$.

$(2aa)_2$: Here

$$[A_i, A_j, A_k] = \begin{bmatrix} 0 & x & \ldots & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 & 1 \\ \beta_t^2 & \beta_{t-1}^2 & \ldots & \beta_t & \ldots & \beta_1 & 0 & \ldots & 0 & 1 & 0 \\ \gamma_t^2 & \gamma_{t-1}^2 & \ldots & \gamma_t & \ldots & \gamma_1 & 0 & \ldots & 1 & 0 & 0 \end{bmatrix}'.$$

The determinant of the submatrix

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}'$$

equals 1.

$(12a)$: Here

$$[A_i, A_j, A_k] = \begin{bmatrix} 1 & 0 & \ldots & 0 & \ldots & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & \ldots & 0 & \ldots & 0 & 1 \\ 0 & x & \ldots & 0 & \ldots & 0 & \ldots & 0 & 1 \\ \gamma_t^2 & \gamma_{t-1}^2 & \ldots & \gamma_t & \ldots & \gamma_1 & \ldots & 1 & 0 \end{bmatrix}'.$$

The determinant of the submatrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & x & 0 & 1 \\ \gamma_t^2 & \gamma_{t-1}^2 & \gamma_i & 0 \end{bmatrix}'$$

is $x\gamma_i \neq 0$.

$(12b)$: Here

$$[A_i, A_j, A_k] = \begin{bmatrix} 1 & 0 & \ldots & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & \ldots & 1 \\ 0 & x & \ldots & 0 & \ldots & 1 \\ \alpha_k^2 & \alpha_{k-1}^2 & \ldots & 1 & \ldots & \alpha_1 \end{bmatrix}'.$$

The determinant of the submatrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & x & 0 & 1 \\ \alpha_k^2 & \alpha_{k-1}^2 & 1 & 0 \end{bmatrix}'$$

is $x \neq 0$.

$(1ab)$: Here

$$[A_i, A_j, A_k] = \begin{bmatrix} 1 & \ldots & 0 & \ldots & 0 & 0 \\ 0 & \ldots & 0 & \ldots & 0 & 1 \\ \gamma_t^2 & \ldots & 0 & \ldots & 1 & 0 \\ \alpha_k^2 & \ldots & 1 & \ldots & \alpha_2 & \alpha_1 \end{bmatrix}'.$$

The determinant of the submatrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \gamma_t^2 & 0 & 1 & 0 \\ \alpha_k^2 & 1 & \alpha_2 & \alpha_1 \end{bmatrix}'$$

is 1.

$(2ab)$: Here

$$[A_i, A_j, A_k] = \begin{bmatrix} 0 & x & \ldots & 0 & \ldots & 0 & 1 \\ \gamma_t^2 & \gamma_{t-1}^2 & \ldots & 0 & \ldots & 1 & 0 \\ \alpha_k^2 & \alpha_{k-1}^2 & \ldots & 1 & \ldots & \alpha_2 & \alpha_1 \end{bmatrix}'.$$

The determinant of the submatrix

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & \alpha_2 & \alpha_1 \end{bmatrix}'$$

is 1.

$(aab)_1$: Here

$$[A_i, A_j, A_k] = \begin{bmatrix} \beta_t^2 & \ldots & 0 & \ldots & 1 & 0 \\ \gamma_t^2 & \ldots & 0 & \ldots & 1 & 0 \\ \alpha_k^2 & \ldots & 1 & \ldots & \alpha_2 & \alpha_1 \end{bmatrix}'.$$

Since there exists at least one $\beta_i \neq \gamma_i$, for $i = 1, ..., t$, the determinant of the submatrix

$$\begin{bmatrix} \beta_i & 0 & 1 \\ \gamma_i & 0 & 1 \\ \alpha_j^2 & 1 & \alpha_2 \end{bmatrix}'$$

is $\gamma_i - \beta_i \neq 0$.

$(aab)_2$: Here

$$[A_i, A_j, A_k] = \begin{bmatrix} \beta_t^2 & \ldots & 0 & \ldots & 0 & 1 & 0 \\ \gamma_t^2 & \ldots & 0 & \ldots & 1 & 0 & 0 \\ \alpha_k^2 & \ldots & 1 & \ldots & \alpha_3 & \alpha_2 & \alpha_1 \end{bmatrix}'.$$

the determinant of the submatrix

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & \alpha_3 & \alpha_2 \end{bmatrix}'$$

is 1.

(abb): Here

$$[A_i, A_j, A_k] = \begin{bmatrix} \gamma_t^2 & \ldots & 0 & \ldots & 1 & 0 \\ \alpha_k^2 & \ldots & 1 & \ldots & \alpha_2 & \alpha_1 \\ \beta_k^2 & \ldots & 1 & \ldots & \beta_2 & \beta_1 \end{bmatrix}'.$$

If $\alpha_i \neq \beta_i$ for $i \neq 2$, the determinant of the submatrix

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & \alpha_i & \alpha_2 \\ 1 & \beta_i & \beta_2 \end{bmatrix}'$$

is $\beta_i - \alpha_i \neq 0$. If $\alpha_i = \beta_i$ for $i \neq 2$, then $\beta_2 \neq \gamma_2$, the determinant of the submatrix

$$\begin{bmatrix} \gamma_1 & 0 & 1 \\ \alpha_1^2 & 1 & \alpha_2 \\ \beta_1^2 & 1 & \beta_2 \end{bmatrix}'$$

is $\gamma_1(\beta_2 - \alpha_2) \neq 0$.

(bbb): Here

$$[A_i, A_j, A_k] = \begin{bmatrix} \alpha_k^2 & \ldots & 1 & \ldots & \alpha_1 \\ \beta_k^2 & \ldots & 1 & \ldots & \beta_1 \\ \gamma_k^2 & \ldots & 1 & \ldots & \gamma_1 \end{bmatrix}'.$$

by Lemma 2, the rank of the matrix is three.

$(1ab)$: Here

$$[A_i, A_j, A_k] = \begin{bmatrix} 1 & \ldots & 0 & \ldots & 0 & 0 \\ 0 & \ldots & 0 & \ldots & 0 & 1 \\ \gamma_t^2 & \ldots & 0 & \ldots & 1 & 0 \\ \alpha_k^2 & \ldots & 1 & \ldots & \alpha_2 & \alpha_1 \end{bmatrix}'.$$

The determinant of the submatrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \gamma_t^2 & 0 & 1 & 0 \\ \alpha_k^2 & 1 & \alpha_2 & \alpha_1 \end{bmatrix}'$$

is 1.

$(aaa)_1$: Here

$$[A_i, A_j, A_k] = \begin{bmatrix} \alpha_t^2 & \ldots & \alpha_1^2 & \alpha_t & \ldots & \alpha_1 & 0 & \ldots & 0 & 1 & 0 \\ \beta_t^2 & \ldots & \beta_1^2 & \beta_t & \ldots & \beta_1 & 0 & \ldots & 0 & 1 & 0 \\ \gamma_t^2 & \ldots & \gamma_1^2 & \gamma_t & \ldots & \gamma_1 & 0 & \ldots & 0 & 1 & 0 \end{bmatrix}'.$$

by Lemma 2, the rank of matrix is three.

$(aaa)_2$: Here

$$[A_i, A_j, A_k] = \begin{bmatrix} \alpha_t^2 & \ldots & \alpha_1^2 & \alpha_t & \ldots & \alpha_1 & 0 & \ldots & 0 & 1 & 0 \\ \beta_t^2 & \ldots & \beta_1^2 & \beta_t & \ldots & \beta_1 & 0 & \ldots & 1 & 0 & 0 \\ \gamma_t^2 & \ldots & \gamma_1^2 & \gamma_t & \ldots & \gamma_1 & 0 & \ldots & 1 & 0 & 0 \end{bmatrix}'.$$

Since $\beta_i \neq \gamma_i$ for at least one i, the determinant of the submatrix

$$\begin{bmatrix} \alpha_i & 0 & 1 \\ \beta_i & 1 & 0 \\ \gamma_i & 1 & 0 \end{bmatrix}'$$

is nonzero. $\square$

Remark. As before, let s be a prime or a prime power and let i, k be integers such that $1 \leq i \leq k$. Suen and Dey (2003) constructed the following families of orthogonal arrays:

(i) $OA(s^{2k+i}, s^k + 1, (s^k)^2 \times (s^i)^{s^k - 1}, 3)$, if s is odd and,

(ii) $OA(s^{2k+i}, s^k + 2, (s^k)^2 \times (s^i)^{s^k}, 3)$, if s is even.

Setting $i = 1$, one gets the arrays

(a) $OA(s^{2k+1}, s^k + 1, (s^k)^2 \times s^{s^k - 1}, 3)$, if s is odd and,

(b) $OA(s^{2k+1}, s^k + 2, (s^k)^2 \times s^{s^k}, 3)$, if s is even.

First, replace one of the s^k-symbol columns by k columns having s symbols each and then replace symbols in the other s^k-symbol column by all possible combinations arising out of one s^2-symbol column and $k - 2$ columns each having s symbols. By this process, one obtains the following arrays corresponding to the arrays in (a) and (b) respectively:

(c) $OA(s^{2k+1}, s^k + 2k - 2, (s^2) \times s^{s^k + 2k - 3}, 3)$, if s is odd, and,

(d) $OA(s^{2k+1}, s^k + 2k - 1, (s^2) \times s^{s^k+2k-2}, 3)$, if s is even.

It is easy to verify that the arrays given by Theorem 2 have more $s-$symbol columns than in the arrays (c) and (d) above. Thus, the arrays in Theorem 2 appear to be superior to the ones in (c) and (d) in terms of having more $s-$symbol columns.

In closing this paper, we make an observation about a family of asymmetric orthogonal arrays of strength *four*. Suen and Dey (2003) constructed a family of orthogonal arrays $OA(s^5, s + 3, (s^2) \times s^{s+2}, 4)$, where s is a power of two. We give below an alternative method of obtaining the same family, which appears to be more direct than that of Suen and Dey (2003).

Let F_1 have s^2 symbols and the rest of the factors have s symbols each. The matrices $A_i,\ 1 \le i \le s + 3$ are chosen as follows.

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}', \quad A_2 = [1\ 0\ 0\ 0\ 1]', \quad A_3 = [1\ 0\ 0\ 1\ 0]',$$

and for $4 \le j \le s+3$, $A_j = [0, \alpha_j^3, 1, \alpha_j, \alpha_j^2]'$, where $\alpha_4, \ldots, \alpha_{s+3}$ are distinct elements of $GF(s)$. It can be verified that with these choices of the A_i matrices, the rank condition of Theorem 1 is met. We omit the details.

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