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From particle counting to Gaussian tomography

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FROM PARTICLE COUNTING TO GAUSSIAN TOMOGRAPHY

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Abstract. The momentum and position observables in an n-mode boson Fock space $\Gamma(\mathbb{C}^n)$ have the whole real line $\mathbb{R}$ as their spectrum. But the total number operator N has a discrete spectrum $\mathbb{Z}_+ = \{0, 1, 2, \cdots\}$. An n-mode Gaussian state in $\Gamma(\mathbb{C}^n)$ is completely determined by the mean values of momentum and position observables and their covariance matrix which together constitute a family of $n(2n+3)$ real parameters. Starting with N and its unitary conjugates by the Weyl displacement operators and operators from a representation of the symplectic group $Sp(2n)$ in $\Gamma(\mathbb{C}^n)$ we construct $n(2n+3)$ observables with spectrum $\mathbb{Z}_+$ but whose expectation values in a Gaussian state determine all its mean and covariance parameters. Thus measurements of discrete-valued observables enable the tomography of the underlying Gaussian state and it can be done by using 5 one mode and 4 two mode Gaussian symplectic gates in single and pair mode wires of $\Gamma(\mathbb{C}^n) = \Gamma(\mathbb{C})^{\otimes n}$. Thus the tomography protocol admits a simple description in a language similar to circuits in quantum computation theory. Such a Gaussian tomography applied to outputs of a Gaussian channel with coherent input states permit a tomography of the channel parameters. However, in our procedure the number of counting measurements exceeds the number of channel parameters slightly. Presently, it is not clear whether a more efficient method exists for reducing this tomographic complexity.

As a byproduct of our approach an elementary derivation of the probability generating function of N in a Gaussian state is given. In many cases the distribution turns out to be infinitely divisible and its underlying Lévy measure can be obtained. However, we are unable to derive the exact distribution in all cases. Whether this property of infinite divisibility holds in general is left as an open problem.

Keywords. Gaussian state, Gaussian channel, Momentum and position observables, Weyl operators, Symplectic group, Tomography.

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1. Introduction

It is in the nature of quantum theory that properties of the state of quantum systems can be inferred from measurements of observables of physical significance taking values in a discrete set or a continuum. From an experimental point of view it is natural to seek as much information as possible from the discrete measurements. A typical measurement of the discrete type is counting the number of particles of a particular type. Suppose the unknown state of a system can be described in terms of some parameters which constitute a manifold of dimension k. Then it is natural to look for k discrete-valued observables from whose expectation values one can determine the values of these parameters. Of course, such observables should be physically meaningful and also experimentally measurable. This has been extensively studied in the book [PŘ04].

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In this article we explore this problem of determining the state when it is known that the state is Gaussian. Consider the Hilbert space $L^2(\mathbb{R}^n)$, or equivalently, the boson Fock space $\Gamma(\mathbb{C}^n)$ over the n-dimensional complex Hilbert space $\mathbb{C}^n$. A Gaussian state of n-modes in $L^2(\mathbb{R}^n)$ is completely described by its momentum and position mean values and a $2n \times 2n$ covariance matrix. Thus, an n-mode Gaussian state is determined by $n(2n+3)$ parameters. Here one has observables $a_j^\dagger a_j$, the number of particles in the j-th mode for each $j = 1, 2, \cdots, n$ and also their unitary equivalents in different frames which are obtained by Weyl (displacement) operators as well as unitary operators which implement the symplectic linear transformations in the position and momentum observables obeying the canonical commutation relations. Using these resources we shall construct $n(2n+3)$ number observables which have the discrete spectrum $\{0, 1, 2, \cdots\}$ and the property that all the means and covariances of the unknown Gaussian state can be easily determined from their expectation values. Measurements on these number observables on an ensemble of such a Gaussian state and the law of large numbers can be used to estimate the unknown parameters. In the process of such an investigation we shall determine the probability generating function of the distribution of the total number observable $\sum_{j=1}^n a_j^\dagger a_j$, its mean and variance in any fixed Gaussian state.

Following Heinosaari et al [HHW10] a Gaussian channel with n degrees of freedom is determined by a pair (A, B) where A is a $2n \times 2n$ real matrix, B is a $2n \times 2n$ real positive semidefinite matrix satisfying the matrix inequality

$$B + \imath(A^T J_{2n} A - J_{2n}) \geq 0,$$

where J_{2n} is defined in equation (2.12). Thus such a channel is determined by $4n^2 + n(2n+1)$ real parameters. Such a channel yields an output Gaussian state for any given input Gaussian state. By choosing a few appropriate input coherent states and performing a Gaussian tomography on the output Gaussian states, we show how the matrices A and B of the quasifree channel can be estimated. We use coherent states as they constitute an important class of mathematical objects, which are easy to realize experimentally. Further, creation of different coherent states with different amplitudes and phases can be possible from the same experimental set-up. We note that, a similar approach of using coherent states for tomography (although in a different scheme) has also been taken by Lobino et al [LKK+08], and was further developed in [RKSM+11, WYH+13].

Bosonic particle counting is an important tool in quantum optics, both for theory and for experiments [SW87b, SW87a]. Gaussian states, channels and their applications have been used extensively in quantum information theory. These concepts have been studied in detail in the book of Holevo [Hol12] and in the survey article by Weedbrook et al [WPGP+12]. We also refer to the books by de Gosson [dG06] and Parthasarathy [Par92] for the connection between symplectic geometry and quantum stochastic calculus. We refer to the survey article by Lvovsky and Raymer [LR09] and the references therein, for experimental processes of continuous variable tomography.

We organise the paper as follows. To increase the readability of the article, in §2, we write a short introduction to notions like exponential vector, Weyl operator, Gaussian state, Fourier transform, Gaussian channels and other necessary concepts which will be used in subsequent sections. In this venture, we mostly follow the approach taken in the papers [ADMS95, Par10, Par14a]. In §3 we derive the basic formulae for expectation and variance of the total number operator which will be used in §4 to estimate an unknown Gaussian state. In §5 we use the tomographic method derived for Gaussian states in §4 to estimate the unknown parameters of a Gaussian channel.

2. Notation and preliminaries

In this section we give a short survey on Gaussian state and other necessary concepts. Though these have been extensively studied in various references, we follow the method and notation adopted in the book [Par92] and the papers [Par10, Par13, Par14b, Par14a].

2.1. **Exponential vector.** Let $\mathcal{H}$ be a finite dimensional complex Hilbert space. When $\dim \mathcal{H} = n$ and $\mathcal{H}$ is identified with $\mathbb{C}^n$ we express its elements as column vectors $\mathbf{z} = (z_1, z_2, \cdots, z_n)^T$ with z_j being complex scalars and the scalar product between two elements $\mathbf{z}$ and $\mathbf{z}'$ as

$$\langle \mathbf{z} | \mathbf{z}' \rangle = \sum_{j=1}^{n} \bar{z}_j z'_j.$$

Define the *boson Fock space* $\Gamma(\mathcal{H})$ over $\mathcal{H}$ by

$$\Gamma(\mathcal{H}) = \mathbb{C} \oplus \mathcal{H} \oplus \mathcal{H}^{\circledS^2} \oplus \cdots \oplus \mathcal{H}^{\circledS^r} \oplus \cdots \tag{2.1}$$

where $\circledS^r$ denotes r-fold symmetric tensor product. Elements of the subspace $\mathcal{H}^{\circledS^r}$ in $\Gamma(\mathcal{H})$ are called r-particle vectors and elements of the form

$$u_0 \oplus u_1 \oplus \cdots \oplus u_r \oplus \cdots$$

where all but a finite number of u_r's are null, are called *finite particle vectors*. Finite particle vectors constitute a dense linear manifold $\mathcal{F}$ in $\Gamma(\mathcal{H})$. For any $\mathbf{u} \in \mathcal{H}$ we associate the *exponential vector* $\mathbf{e}(\mathbf{u})$ in $\Gamma(\mathcal{H})$ defined by

$$\mathbf{e}(\mathbf{u}) = \mathbf{1} \oplus \mathbf{u} \oplus \frac{\mathbf{u}^{\otimes 2}}{\sqrt{2!}} \oplus \cdots \oplus \frac{\mathbf{u}^{\otimes r}}{\sqrt{r!}} \oplus \cdots . \tag{2.2}$$

Then

$$\langle \mathbf{e}(\mathbf{u}) | \mathbf{e}(\mathbf{v}) \rangle = \exp \langle \mathbf{u} | \mathbf{v} \rangle \quad \forall \mathbf{u}, \mathbf{v} \in \mathcal{H}. \tag{2.3}$$

The linear manifold $\mathcal{E}$ generated by all the exponential vectors is called *exponential domain*. The two dense linear manifolds $\mathcal{E}$ and $\mathcal{F}$ are useful domains for constructing several operators of physical significance.

2.2. **Weyl operator.** For any $\mathbf{u} \in \mathcal{H}$ we associate the *Weyl displacement operator* $W(\mathbf{u})$ by putting

$$W(\mathbf{u})\mathbf{e}(\mathbf{v}) = e^{-\frac{1}{2}\|u\|^2 - \langle \mathbf{u} | \mathbf{v} \rangle} \mathbf{e}(\mathbf{u} + \mathbf{v}) \tag{2.4}$$

for all $\mathbf{v} \in \mathcal{H}$, observing that $W(\mathbf{u})$ is scalar product preserving on $\mathcal{E}$ and therefore extends naturally to $\Gamma(\mathcal{H})$. The Weyl operators obey the multiplication property

$$W(\mathbf{u})W(\mathbf{v}) = e^{-\imath \mathrm{Im} \langle \mathbf{u} | \mathbf{v} \rangle} W(\mathbf{u} + \mathbf{v}) \tag{2.5}$$

for all $\mathbf{u}$, $\mathbf{v} \in \mathcal{H}$ and yield a strongly continuous, irreducible, factorizable and projective unitary representation of the additive group $\mathcal{H}$. By 'factorizable' we mean the property that under the isomorphism between $\Gamma(\mathcal{H}_1 \oplus \mathcal{H}_2)$ and $\Gamma(\mathcal{H}_1) \otimes \Gamma(\mathcal{H}_2)$ through the identification $\mathcal{I}\mathbf{e}(\mathbf{u_1} \oplus \mathbf{u_2}) = \mathbf{e}(\mathbf{u_1}) \otimes \mathbf{e}(\mathbf{u_2})$ one has

$$\mathcal{I}W(\mathbf{u_1} \oplus \mathbf{u_2})\mathcal{I}^{-1} = W(\mathbf{u_1}) \otimes W(\mathbf{u_2}).$$

If $\mathbf{u} \to \tilde{W}(\mathbf{u})$ is another strongly continuous map from $\mathcal{H}$ into the unitary group of a Hilbert space $\mathcal{K}$ such that equation (2.5) holds with W replaced by $\tilde{W}$ and $\tilde{W}(\cdot)$ is irreducible, then there exists a unitary isomorphism $V : \Gamma(\mathcal{H}) \to \mathcal{K}$ such that

$$VW(\mathbf{u})V^{-1} = \tilde{W}(\mathbf{u}) \quad \forall \mathbf{u} \in \mathcal{H}.$$

Thus the Weyl operators constitute a unique multiplicative family up to unitary equivalence but with the presence of the factor $\exp -\imath \mathrm{Im}\langle \mathbf{u}|\mathbf{v}\rangle$ in equation (2.5). We call $\mathbf{u} \to W(\mathbf{u})$ the *Weyl representation* of $\mathcal{H}$ and we shall exploit its properties to define a natural quantum Fourier transform for states in $\Gamma(\mathcal{H})$. For now, we shall introduce some basic observations arising from the Weyl operators.

For any fixed $\mathbf{u} \in \mathcal{H}$, the map $t \mapsto W(t\mathbf{u})$ yields a strongly continuous one parameter unitary group as t varies in $\mathbb{R}$. By Stone's theorem [Par92] there exists a self adjoint operator $p(\mathbf{u})$ such that

$$W(t\mathbf{u}) = e^{-\imath t p(\mathbf{u})}, \quad t \in \mathbb{R}. \tag{2.6}$$

Define the operators

$$\begin{aligned} q(\mathbf{u}) &= -p(\imath\mathbf{u}), \\ a(\mathbf{u}) &= \frac{1}{2}(q(\mathbf{u}) + \imath p(\mathbf{u})), \\ a^\dagger(\mathbf{u}) &= \frac{1}{2}(q(\mathbf{u}) - \imath p(\mathbf{u})). \end{aligned}$$

All these operators have domains including the exponential domain $\mathcal{E}$ and the domain $\mathcal{F}$ of finite particle vectors. Indeed, any finite linear combination of these operators have the same property and we denote their respective closures by the same symbols. With this convention one has

$$W(\mathbf{u}) = e^{-\imath p(\mathbf{u})} = e^{a^\dagger(\mathbf{u}) - a(\mathbf{u})}.$$

When $\mathcal{H} = \mathbb{C}^n = \mathbb{R}^n + \imath\mathbb{R}^n$ and $\mathbf{u} = \mathbf{x} + \imath\mathbf{y}$ with $\mathbf{x} = \mathrm{Re}\mathbf{u},\ \mathbf{y} = \mathrm{Im}\mathbf{u}$ we also have

$$W(\mathbf{u}) = W(\mathbf{x} + \imath\mathbf{y}) = e^{-\imath(p(\mathbf{x}) - q(\mathbf{y}))}.$$

Furthermore, one has the following commutation relations.

$$\begin{aligned} [p(\mathbf{u}), p(\mathbf{v})] &= 2\imath\, \mathrm{Im}\langle \mathbf{u}|\mathbf{v}\rangle, \\ [q(\mathbf{u}), q(\mathbf{v})] &= 2\imath\, \mathrm{Im}\langle \mathbf{u}|\mathbf{v}\rangle, \\ [q(\mathbf{u}), p(\mathbf{v})] &= 2\imath\, \mathrm{Re}\langle \mathbf{u}|\mathbf{v}\rangle, \\ [a(\mathbf{u}), a(\mathbf{v})] &= 0, \\ \left[a^\dagger(\mathbf{u}), a^\dagger(\mathbf{v})\right] &= 0, \\ \left[a(\mathbf{u}), a^\dagger(\mathbf{v})\right] &= \langle \mathbf{u}|\mathbf{v}\rangle \end{aligned}$$

on the domains $\mathcal{E}$ and $\mathcal{F}$ for all $\mathbf{u}$, $\mathbf{v}$ in $\mathcal{H}$.

Choose and fix an orthonormal basis $\{\mathbf{e_j}\},\ j=1,2,\cdots$ in $\mathcal{H}$, and define

$$p_j = \frac{1}{\sqrt{2}}p(\mathbf{e_j}), \qquad q_j = -\frac{1}{\sqrt{2}}p(\imath\mathbf{e_j}),$$
$$a_j = \frac{1}{\sqrt{2}}(q_j+\imath p_j), \qquad a_j^\dagger = \frac{1}{\sqrt{2}}(q_j-\imath p_j).$$

Then one has the canonical commutation relations in the form

$$[p_r,p_s]=[q_r,q_s]=0, \quad [q_r,p_s]=\imath\delta_{rs},$$

or equivalently,

$$[a_r,a_s]=[a_r^\dagger,a_s^\dagger]=0, \quad [a_r,a_s^\dagger]=\delta_{rs},$$

in the domains $\mathcal{E}$ and $\mathcal{F}$. The observables $p_1,p_2,\cdots$ are called momentum operators and $q_1,q_2,\cdots$ are called position operators in the basis $\{\mathbf{e_j},\ j=1,2,\cdots\}$.

2.3. **Quantum Fourier transform.** For any trace class operator ρ in $\Gamma(\mathcal{H})$ its *quantum Fourier transform* or simply *Fourier transform* $\hat{\rho}$ on $\mathcal{H}$ is defined by

$$\hat{\rho}(\mathbf{u}) = \mathrm{Tr}\rho W(\mathbf{u}), \quad \mathbf{u}\in\mathcal{H}.$$

Then $\hat{\rho}$ is a bounded continuous function of $\mathbf{u}$ satisfying $\hat{\rho}(\mathbf{0})=\mathrm{Tr}\rho$. If ρ is positive then $\hat{\rho}$ obeys the Bochner property: for any finite set $\{c_r,\ r=1,2,\cdots,k\}$ of scalars and elements $\{\mathbf{u_r},\ r=1,2,\cdots,k\}$ in $\mathcal{H}$ one has the inequality

$$\sum_{r,s}\bar{c}_r c_s \exp(\imath \mathrm{Im}\langle \mathrm{u_r}|\mathrm{u_s}\rangle)\hat{\rho}(\mathbf{u_s}-\mathbf{u_r})\geq 0.$$

Conversely, if φ is a continuous function with $\varphi(\mathbf{0})=1$ and φ satisfies the Bochner property above then there exists a unique state ρ such that $\hat{\rho}=\varphi$. When $\mathcal{H}=\mathbb{C}^n$ one has the Fourier inversion formula:

$$\rho = \frac{1}{\pi^n}\int \overline{\hat{\rho}(\mathbf{u})}W(\mathbf{u})\,\mathrm{d}\mathbf{u}$$

where $\mathrm{d}\mathbf{u}$ denotes integration with respect to the $2n$ dimensional Lebesgue measure in $\mathbb{R}^{2n}$ with $\mathrm{d}\mathbf{u}=\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y},\ \mathbf{u}=\mathbf{x}+\imath\mathbf{y},\ \mathbf{x}=\mathrm{Re}(\mathbf{u}),\ \mathbf{y}=\mathrm{Im}(\mathbf{u})$.

With the help of Fourier transform we shall now construct a natural Hilbert space isomorphism between the Hilbert space of Hilbert-Schmidt operators in $\Gamma(\mathcal{H})$ and the Hilbert space $L^2(\mathbb{R}^{2n})$.

Proposition 2.1. *Let $\mathcal{H}=\mathbb{C}^n$. Then*

$$\frac{1}{\pi^n}\int \exp\left[-\|\mathbf{w}\|^2+\langle\mathbf{u}|\mathbf{w}\rangle+\langle\mathbf{w}|\mathbf{v}\rangle\right]\mathrm{d}\mathbf{w} = \exp\langle\mathbf{u}|\mathbf{v}\rangle.$$

Proof. Immediate from standard formulae for Gaussian integrals. □

Proposition 2.2. *Denote by $L^2(\mathcal{H})$, the Hilbert space of square integrable functions on $\mathcal{H}$ with the scalar product*

$$\langle f|g\rangle = \int \overline{f(\mathbf{u})}g(\mathbf{u})\frac{\mathrm{d}\mathbf{u}}{\pi^n}$$

and by $\mathcal{B}_2(\Gamma(\mathcal{H}))$ the Hilbert space of all Hilbert-Schmidt operators on $\Gamma(\mathcal{H})$ with the scalar product

$$\langle \rho_1 | \rho_2 \rangle = \mathrm{Tr} \rho_1^\dagger \rho_2.$$

Then there exists a unique Hilbert space isomorphism $\mathbb{F} : \mathcal{B}_2(\Gamma(\mathcal{H})) \to L^2(\mathcal{H})$ such that for any $\mathbf{u}, \mathbf{v} \in \mathcal{H}$,

$$(\mathbb{F}(|\mathbf{e}(\mathbf{u})\rangle\langle \mathbf{e}(\mathbf{v})|))(\mathbf{w}) = \mathrm{Tr}|\mathbf{e}(\mathbf{u})\rangle\langle \mathbf{e}(\mathbf{v})| W(\mathbf{w}) \quad \forall \mathbf{w} \in \mathcal{H}. \tag{2.7}$$

Proof. The right hand side of equation (2.7) is equal to

$$\begin{aligned} & \langle \mathbf{e}(\mathbf{v}) | W(\mathbf{w}) | \mathbf{e}(\mathbf{u}) \rangle \\ = \; & e^{-\frac{1}{2}\|\mathbf{w}\|^2} \langle \mathbf{e}(\mathbf{v}) | e^{-\langle \mathbf{w}|\mathbf{u}\rangle} | \mathbf{e}(\mathbf{u}+\mathbf{w}) \rangle \\ = \; & \exp\left[-\frac{1}{2}\|\mathbf{w}\|^2 + \langle \mathbf{v}|\mathbf{w}\rangle - \langle \mathbf{w}|\mathbf{u}\rangle + \langle \mathbf{v}|\mathbf{u}\rangle \right] \end{aligned} \tag{2.8}$$

If we put $\rho_j = |\mathbf{e}(\mathbf{u_j})\rangle\langle \mathbf{e}(\mathbf{v_j})|, \; j = 1, 2$ then

$$\mathrm{Tr} \rho_1^\dagger \rho_2 = \exp\left[\langle \mathbf{u_1} | \mathbf{u_2} \rangle + \langle \mathbf{v_2} | \mathbf{v_1} \rangle \right]. \tag{2.9}$$

On the other hand the scalar product between $\mathrm{Tr}|\mathbf{e}(\mathbf{u_j})\rangle\langle \mathbf{e}(\mathbf{v_j})| W(\mathbf{w}), \; j = 1, 2$ reduces by Proposition 2.1 and equation (2.8) to the right hand side of (2.9). Now we observe that rank one operators of the form $|\mathbf{e}(\mathbf{u})\rangle\langle \mathbf{e}(\mathbf{v})|, \; \mathbf{u}, \mathbf{v} \in \mathcal{H}$ constitute a total set in $\mathcal{B}_2(\Gamma(\mathcal{H}))$. Thus $\mathbb{F}$ defined by (2.7) extends uniquely to the whole of $\mathcal{B}_2(\Gamma(\mathcal{H}))$. On the other hand functions of $\mathbf{w}$ of the form on the right hand side of (2.8) constitute a total set in $L^2(\mathcal{H})$. Thus $\mathbb{F}$ extends to a Hilbert space isomorphism between $\mathcal{B}_2(\Gamma(\mathcal{H}))$ and $L^2(\mathcal{H})$. □

2.4. **Gaussian state.**

Definition 2.1. *A state ρ in $\Gamma(\mathcal{H})$ with $\mathcal{H} = \mathbb{C}^n$ is called an n-mode Gaussian state if its Fourier transform $\hat{\rho}$ is given by*

$$\hat{\rho}(\mathbf{x} + \imath \mathbf{y}) = \exp\left[-\imath\sqrt{2}(\mathbf{l}^T \mathbf{x} - \mathbf{m}^T \mathbf{y}) - \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}^T S \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} \right]. \tag{2.10}$$

for all $\mathbf{x}, \; \mathbf{y} \in \mathbb{R}^n$ where $\mathbf{l}, \; \mathbf{m}$ are elements of $\mathbb{R}^n$ and S is a real $2n \times 2n$ symmetric matrix satisfying the matrix inequality

$$2S + \imath J_{2n} \geq 0 \tag{2.11}$$

with

$$J_{2n} = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}, \tag{2.12}$$

I_n being the identity matrix of order n.

Remark 2.1. Equations (2.10)–(2.12) have been written keeping in mind the orders of the canonical momentum and position observables as $p_1, p_2, \cdots, p_n, q_1, q_2, \cdots, q_n$. Sometimes it is more convenient to distinguish the different modes of a Gaussian state by using the order $p_1, q_1, p_2, q_2, \cdots, p_m, q_n$. This is usually achieved by employing the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & 2n-1 & 2n \\ 1 & n+1 & 2 & n+2 & \cdots & n & n+n \end{pmatrix}.$$

Then the right hand sides of (2.10)–(2.12) are obtained by changing $(\mathbf{x}^T, \mathbf{y}^T)$, $(\mathbf{l}^T, \mathbf{m}^T)$ and S respectively to $(x_1, y_1, x_2, y_2, \cdots, x_n, y_n)$, $(l_1, m_1, l_2, m_2, \cdots, l_n, m_n)$, and $\sigma S \sigma^{-1}$ with J_{2n} replaced by

$$\tilde{J_{2n}} = \begin{bmatrix} 0 & -1 & & & & & & \\ 1 & 0 & & & & & & \\ & & 0 & -1 & & & & \\ & & 1 & 0 & & & & \\ & & & & \ddots & & & \\ & & & & & & 0 & -1 \\ & & & & & & 1 & 0 \end{bmatrix},$$

By abuse of notation we may denote both J_{2n} and $\tilde{J_{2n}}$ by the same symbol J_{2n}.

We choose the canonical orthonormal basis $\mathbf{e_j} = (0, 0, \cdots, 1, 0, \cdots, 0)^T$ with 1 in the j-th position for $j = 1, 2, \cdots$, then the momentum and position operators satisfy the relations

$$\operatorname{Tr} p_j \rho = l_j, \quad \operatorname{Tr} q_j \rho = m_j$$

and S is the covariance matrix of $(p_1, p_2, \cdots, p_n, -q_1, -q_2, \cdots, -q_n)$ in the state ρ satisfying (2.10-2.12). Whenever 2.10 is satisfied we write

$$\rho = \rho_g(\mathbf{l}, \mathbf{m}; S).$$

Thus ρ is completely described by $n(2n+3)$ parameters.

Proposition 2.3. *Let $\rho_g(\mathbf{l_j}, \mathbf{m_j}; S_j)$, $j = 1, 2$ be two n-mode Gaussian states. Then*

$$\operatorname{Tr}\ \rho_g(\mathbf{l_1}, \mathbf{m_1}; S)\rho_g(\mathbf{l_2}, \mathbf{m_2}; T) = \frac{\exp\left[-\frac{1}{2}\begin{bmatrix} \mathbf{l_1} - \mathbf{l_2} \\ -(\mathbf{m_1} - \mathbf{m_2}) \end{bmatrix}^T (S+T)^{-1} \begin{bmatrix} \mathbf{l_1} - \mathbf{l_2} \\ -(\mathbf{m_1} - \mathbf{m_2}) \end{bmatrix}\right]}{\sqrt{\det(S+T)}}. \tag{2.13}$$

Proof. Any state in $\Gamma(\mathbb{C}^n)$ is a positive operator of unit trace and hence a Hilbert-Schmidt operator. Thus by Proposition 2.2 we have

$$\begin{aligned} &\operatorname{Tr}\ \rho_g(\mathbf{l_1}, \mathbf{m_1}; S)\rho_g(\mathbf{l_2}, \mathbf{m_2}; T) \\ = &\ \frac{1}{\pi^n} \int \exp\left[\imath\sqrt{2}((\mathbf{l_1} - \mathbf{l_2})^T \mathbf{x} - (\mathbf{m_1} - \mathbf{m_2})^T \mathbf{y}) - \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}^T (S+T) \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}\right] \mathrm{d}\mathbf{x}\, \mathrm{d}\mathbf{y}. \end{aligned}$$

The rest follows from the standard formula for the characteristic function of a multivariate normal density function in statistics. ☐

Proposition 2.4. *For any $\mathbf{u} \in \mathbb{C}^n$ with $\mathbf{x} = \operatorname{Re}(\mathbf{u})$, $\mathbf{y} = \operatorname{Im}(\mathbf{u})$*

$$W(\mathbf{u})\rho_g(\mathbf{l}, \mathbf{m}; S)W(\mathbf{u})^\dagger = \rho_g(\mathbf{l}', \mathbf{m}'; S)$$

where

$$\mathbf{l}' = \mathbf{l} + \sqrt{2}\mathbf{y}, \quad \mathbf{m}' = \mathbf{m} + \sqrt{2}\mathbf{x}.$$

Proof. Immediate from Corollary 3.3 in [Par10]. ☐

We denote by $Sp(2n)$ the symplectic group $Sp(2n, \mathbb{R})$ of all real $2n \times 2n$ matrices L satisfying the relation

$$L^T J_{2n} L = J_{2n}.$$

Let $\Gamma(L)$ be the unitary operator in $\Gamma(\mathbb{C}^n)$ which is unique upto a scalar multiple of modulus unity and satisfies the relation

$$\Gamma(L) W(\mathbf{x} + \imath\, \mathbf{y}) \Gamma(L)^{-1} = W(\mathbf{x}' + \imath\, \mathbf{y}'), \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n,$$

where

$$L \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} \mathbf{x}' \\ \mathbf{y}' \end{pmatrix}.$$

If U is any $n \times n$ unitary matrix and $U = A + \imath\, B$ where $A = \mathrm{Re}(U)$ and $B = \mathrm{Im}(U)$ then the $2n \times 2n$ matrix

$$L = \begin{bmatrix} A & -B \\ B & A \end{bmatrix}$$

is an orthogonal matrix which is also an element of $Sp(2n)$. We denote the corresponding $\Gamma(L)$ by $\Gamma(U)$ and call it the second quantization of U. We can realize $\Gamma(U)$ as the unique unitary operator satisfying

$$\Gamma(U)\mathbf{e}(\mathbf{u}) = \mathbf{e}(U\mathbf{u}) \quad \forall \mathbf{u} \in \mathbb{C}^n.$$

With these notations we have

Proposition 2.5. *For any $L \in Sp(2n)$*

$$\Gamma(L)\rho_g(\mathbf{l}, \mathbf{m}; S)\Gamma(L)^\dagger = \rho_g(\mathbf{l}', \mathbf{m}'; S')$$

where

$$\begin{aligned} \begin{pmatrix} \mathbf{l}' \\ -\mathbf{m}' \end{pmatrix} &= (L^{-1})^T \begin{pmatrix} \mathbf{l}' \\ -\mathbf{m}' \end{pmatrix} \\ S' &= (L^{-1})^T S L^{-1}. \end{aligned}$$

Proof. This is Corollary 3.5 of [Par10]. ☐

3. Particle counts and their statistics in a Gaussian state

Let $\rho_g(\mathbf{l}, \mathbf{m}; S)$ be an n-mode Gaussian state in $\Gamma(\mathbb{C}^n)$ and whose Fourier transform is given by equation (2.10). Define the observables

$$\begin{aligned} N_j &= a_j^\dagger a_j = \frac{1}{2}(p_j^2 + q_j^2 - 1), \quad 1 \leq j \leq n. \\ N &= \sum_{j=1}^n N_j. \end{aligned}$$

Both N_j and N are observables with spectrum $\{0, 1, 2, \cdots\}$. N_j is called the number operator or observable which counts the number of particles (photons) in the j-th mode. Since the N_j's commute with each other they have a joint distribution in the state $\rho_g(\mathbf{l}, \mathbf{m}; S)$ with support in $\{0, 1, 2, \cdots\}^n$. Using Proposition 2.3 we shall derive a formula for the probability generating function of this joint distribution and arrive at some natural corollaries. To this end we introduce the Gaussian state

$$\rho(\mathbf{0},\mathbf{0};T)=\prod_{j=1}^{n}(1-e^{-t_j})e^{-\sum_{j=1}^{n}t_ja_j^\dagger a_j};$$

where

$$\begin{aligned} T &= \begin{bmatrix} \frac{1}{2}\left(\frac{1+e^{-t_1}}{1-e^{-t_1}}\right)I_2 & & \\ & \ddots & \\ & & \frac{1}{2}\left(\frac{1+e^{-t_n}}{1-e^{-t_n}}\right)I_2 \end{bmatrix} \\ &= D\left(\frac{1}{2}\left(\frac{1+e^{-t_j}}{1-e^{-t_j}}\right)I_2,\ 1\le j\le n\right); \qquad t_j>0 \quad \forall j, \end{aligned}$$

with D indicating the diagonal block matrix with blocks of order 2×2 and the diagonal entries being enumerated within ( ). These are the well-known thermal states. It follows immediately from Proposition 2.3, equation(2.13) that

$$\mathrm{Tr}\rho_g(\mathbf{l},\mathbf{m};S)e^{-\sum_{j=1}^{n}t_ja_j^\dagger a_j}=\frac{\exp\left[-\frac{1}{2}\begin{pmatrix}\mathbf{l}\\ -\mathbf{m}\end{pmatrix}^T\left(S+D\left(\frac{1}{2}\left(\frac{1+e^{-t_j}}{1-e^{-t_j}}\right)I_2,\ 1\le j\le n\right)\right)^{-1}\begin{pmatrix}\mathbf{l}\\ -\mathbf{m}\end{pmatrix}\right]}{\prod_{j=1}^{n}(1-e^{-t_j})\sqrt{\det\left[S+D\left(\frac{1}{2}\left(\frac{1+e^{-t_j}}{1-e^{-t_j}}\right)I_2,\ 1\le j\le n\right)\right]}}.$$

Substituting $N_j=a_j^\dagger a_j$, $x_j=e^{-t_j}$ we get
(3.1)

$$\mathrm{Tr}\rho_g(\mathbf{l},\mathbf{m};S)x_1^{N_1}\cdots x_n^{N_n}=\frac{\exp\left[-\frac{1}{2}\begin{pmatrix}\mathbf{l}\\ -\mathbf{m}\end{pmatrix}^T\left(S+D\left(\frac{1}{2}\left(\frac{1+x_j}{1-x_j}\right)I_2,\ 1\le j\le n\right)\right)^{-1}\begin{pmatrix}\mathbf{l}\\ -\mathbf{m}\end{pmatrix}\right]}{\prod_{j=1}^{n}(1-x_j)\sqrt{\det\left[S+D\left(\frac{1}{2}\left(\frac{1+x_j}{1-x_j}\right)I_2,\ 1\le j\le n\right)\right]}}$$

for $0<x_j<1\ \forall j$. The right hand side of this equation is nothing but the probability generating function of the joint distribution of the observables N_j, $1\le j\le n$.

Theorem 3.1. *Let $\rho_g(\mathbf{l},\mathbf{m};S)$ be an n-mode Gaussian state in $\Gamma(\mathbb{C}^n)$ whose covariance matrix S has eigenvalues $\lambda_1\ge\lambda_2\ge\cdots\ge\lambda_{2n}$ with respective real eigenvectors $\mathbf{b_1},\mathbf{b_2},\cdots,\mathbf{b_{2n}}$ and momentum position mean $\begin{pmatrix}\mathbf{l}\\ \mathbf{m}\end{pmatrix}$ satisfying*

$$\begin{pmatrix}\mathbf{l}\\ -\mathbf{m}\end{pmatrix}=\sum_{j=1}^{2n}\tau_j\mathbf{b_j}.$$

Suppose

$$\alpha_j=\frac{\lambda_j-\frac{1}{2}}{\lambda_j+\frac{1}{2}},\qquad 1\le j\le n.$$

Then the probability generating function $G_N(x)$ of the distribution of the total number operator N in the state $\rho_g(\mathbf{l}, \mathbf{m}; S)$ is given by

$$\begin{aligned} G_N(x) &= \mathrm{Tr}\ \rho_g(\mathbf{l}, \mathbf{m}; S)\, x^N \\ &= \prod_{j=1}^{2n} \sqrt{\frac{1-\alpha_j}{1-\alpha_j x}} \exp\left[-\frac{1}{2}\tau_j^2 \frac{(1-x)(1-\alpha_j)}{(1-\alpha_j x)}\right], \qquad 0 \leq x < 1. \end{aligned} \tag{3.2}$$

Proof. Putting $x_j = x$ for all j in (3.1) and making use of the eigenbasis and eigenvalues of S we see that (3.1) becomes

$$G_N(x) = \prod_{j=1}^{2n} \frac{\exp\left[-\frac{1}{2}\tau_j^2 \left(\lambda_j + \frac{1}{2}\frac{1+x}{1-x}\right)^{-1}\right]}{\sqrt{(1-x)\lambda_j + \frac{1}{2}(1+x)}}.$$

The rest is elementary algebra using the definitions of α_j in terms of λ_j for every j. □

Corollary 3.1. *The probability distribution of N in the state $\rho_g(\mathbf{l}, \mathbf{m}; S)$ satisfies the following:*

$$(i) \qquad \Pr(N = 0) = \prod_{j=1}^{2n} (1-\alpha_j) \exp\left[-\frac{1}{2}\tau_j^2 (1-\alpha_j)\right], \tag{3.3}$$

$$(ii) \qquad \langle N \rangle = \frac{1}{2}\left[\mathrm{Tr}\left(S - \frac{1}{2}\right) + \left\|\begin{pmatrix}\mathbf{l} \\ -\mathbf{m}\end{pmatrix}\right\|^2\right], \tag{3.4}$$

$$(iii) \qquad \mathit{Variance}(N) = \frac{1}{2}\mathrm{Tr}\left(S - \frac{1}{2}\right)\left(S + \frac{1}{2}\right) + \begin{pmatrix}\mathbf{l} \\ -\mathbf{m}\end{pmatrix}^T S \begin{pmatrix}\mathbf{l} \\ -\mathbf{m}\end{pmatrix}, \tag{3.5}$$

Proof. Property (i) follows by putting $x = 0$ in (3.2). Properties (ii) and (iii) are obtained from (3.2) by taking the logarithm of G_N, differentiating twice and taking $\lim_{x\to 1} \frac{\mathrm{d}}{\mathrm{d}x}(\log G(x))$ and $\lim_{x\to 1} \frac{\mathrm{d}^2}{\mathrm{d}^2 x}(\log G(x))$. □

Remark 3.1. Equations (3.3)-(3.5) show that the probability of presence of a particle and the expectation and variance of the total number of particles get enhanced when the Gaussian state has a nonzero momentum position mean vector. Indeed, the mean and variance of N tend to infinity as the length of the momentum position mean vector increases to infinity. Equations (3.3) and (3.5) indicate the possibility of estimating the mean and covariance parameters of a Gaussian state by measuring the number operator under different displacements. We shall discuss this approach to the tomography of a Gaussian state in great detail in the next section §4.

It may be noted that the parameters α_j in Theorem 3.1 satisfy the inequality $|\alpha_j| < 1$ for every j. If $\mathbf{l}$ and $\mathbf{m}$ are null-vectors and $\alpha_j \geq 0$ for every j then $G_N(x)$ assumes the form

$$G_N(x) = \prod_{j=1}^{2n} \left(\frac{1-\alpha_j}{1-\alpha_j x}\right)^{\frac{1}{2}}$$

and the corresponding distribution of N is a convolution of $2n$ negative binomial distributions of index $\frac{1}{2}$, some of which may be degenerate at 0. In particular, it is infinitely divisible. In this case

$$\text{Variance}(N) - \langle N \rangle = \frac{1}{2}\mathrm{Tr}\left(S - \frac{1}{2}\right)^2 \geq 0$$

and the distribution exhibits a super Poissonian property. When $S = \frac{1}{2}I_{2n}$ the state becomes the vacuum state and N has the degenerate distribution, degenerate at 0.

Now we shall analyse the distribution of N in a pure Gaussian state. In this case $S = \frac{1}{2}L^T L$ for some element L of the group $Sp(2n)$ and its eigenvalues can be expressed as

$$\left(\frac{c_1}{2}, \frac{1}{2c_1}, \frac{c_2}{2}, \frac{1}{2c_2}, \cdots, \frac{c_k}{2}, \frac{1}{2c_k}, \frac{1}{2}, \cdots, \frac{1}{2}\right) \tag{3.6}$$

where $c_j > 1$ for $1 \leq j \leq k$. Then

$$\begin{array}{lcll} \alpha_{2j-1} & = & \frac{c_j-1}{c_j+1} > 0 & \text{for} \quad 1 \leq j \leq k, \\ \alpha_{2j} & = & \frac{1-c_j}{1+c_j} < 0 & \text{for} \quad 1 \leq j \leq k, \\ \alpha_r & = & 0 & \text{for} \quad 2k+1 \leq r \leq 2n. \end{array}$$

Write $\beta_j = \alpha_{2j-1}$, $1 \leq j \leq k$ so that $\alpha_{2j} = -\beta_j$, $1 \leq j \leq k$. Then the probability generating function of the total number operator N in (3.2) assumes the form

$$G_N(x) = G_1(x)G_2(x)G_3(x) \tag{3.7}$$

where

$$G_1(x) = \prod_{j=1}^{k} \sqrt{\frac{1-\beta_j^2}{1-\beta_j^2 x^2}} \tag{3.8}$$

$$G_2(x) = \exp \frac{1}{2}\sum_{j=1}^{k}\left[\tau_{2j-1}^2 \frac{(x-1)(1-\beta_j)}{1-\beta_j x} + \tau_{2j}^2 \frac{(x-1)(1+\beta_j)}{1+\beta_j x}\right] \tag{3.9}$$

$$G_3(x) = \exp\left[\frac{1}{2}\sum_{j=2k+1}^{2n} \tau_j^2 (x-1)\right] \tag{3.10}$$

where $0 < \beta_j < 1$ for $1 \leq j \leq k$.

Writing

$$\gamma_j = \tau_{2j-1}^2(1-\beta_j), \tag{3.11}$$

$$\delta_j = \tau_{2j}^2(1+\beta_j), \quad 1 \leq j \leq k, \tag{3.12}$$

one can express $G_2(x)$ in (3.9) as

$$G_2(x) = \exp \frac{1}{2}\sum_{j=1}^{k} \frac{\beta_j(\gamma_j - \delta_j)(x^2-1) + \left[\gamma_j(1-\beta_j) + \delta_j(1+\beta_j)\right](x-1)}{1-\beta_j^2 x^2} \tag{3.13}$$

where $0 < \beta_j < 1$, $\gamma_j \geq 0$, $\delta_j \geq 0$ for $1 \leq j \leq k$.

From (3.7) it follows that $G_1(x)$ is the probability generating function of a convolution of probability distributions μ_j, $1 \leq j \leq k$ where the probability generating function of μ_j is equal to

$$(1-\beta_j^2)^{\frac{1}{2}} \sum_{r=0}^{\infty} \frac{1.3.5.\cdots(2r+1)}{r!} \beta_j^{2r} x^{2r}, \quad 1 \leq j \leq k.$$

In particular μ_j is an infinitely divisible distribution with support in $\{0, 2, 4, \cdots\}$. Equation (3.10) shows that $G_3(x)$ is the probability generating function of a Poisson distribution with mean value $\frac{1}{2}\sum_{r=2k+1}^{2n} \tau_r^2$.

If $\gamma_j \geq \delta_j$ for every $j = 1, 2, \cdots, k$ then $G_2(x)$ is clearly the probability generating function of an infinitely divisible distribution with support in $\{0, 1, 2, \cdots\}$. Its Lévy measure can be easily read off from (3.13). Under this assumption, i.e. $\gamma_j \geq \delta_j$ for each $1 \leq j \leq k$ it follows that N has an infinitely divisible distribution in the pure Gaussian state we started with.

However, we do not know the answer to the question whether the distribution of the total number operator N in every Gaussian state is infinitely divisible and hence of a mixed Poisson type.

4. From particle counting to the tomography of a Gaussian state

A Gaussian state with n modes can be constructed if its momentum and position means $\mathbf{l}$, $\mathbf{m}$ and its covariance matrix S are known. Our aim is to express these parameters in terms of the expectation values of conjugates of the total number operator by a few elementary gates in the Hilbert space $\Gamma(\mathbb{C}^n)$. To this end we shall make use of the Weyl displacement operators $W(\mathbf{u}),\ u \in \mathbb{C}^n$ and the Gaussian symmetries $\Gamma(L),\ L \in Sp(2n)$ described in Section §3. We start with an elementary but basic result for achieving this tomography of a Gaussian state. For any observable X denote by $\langle X \rangle$ its mean value in the state relevant to the context.

Theorem 4.1. *Let $\rho_g(\mathbf{l}, \mathbf{m}; S)$ be a Gaussian state and let N be the number operator in the n-mode Hilbert space $\Gamma(\mathbb{C}^n)$. Then the following hold:*

(i) For all $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^n$

$$\left\langle W(\mathbf{x}+\imath\mathbf{y})^\dagger N W(\mathbf{x}+\imath\mathbf{y}) \right\rangle - \langle N \rangle = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 + \sqrt{2}(\mathbf{y}^T\mathbf{l} + \mathbf{x}^T\mathbf{m}) \tag{4.1}$$

(ii) For any $L \in Sp(2n)$

(4.2)

$$\left\langle \Gamma(L)^\dagger N \Gamma(L) \right\rangle - \langle N \rangle = \frac{1}{2}\left[\operatorname{Tr} S\left(L^{-1}L^{-1^T} - I_{2n}\right) + \begin{pmatrix} \mathbf{l} \\ -\mathbf{m} \end{pmatrix}^T \left(L^{-1}L^{-1^T} - I_{2n}\right) \begin{pmatrix} \mathbf{l} \\ -\mathbf{m} \end{pmatrix} \right].$$

Proof. We have from Proposition 2.4 and equation (3.4)

$$\begin{aligned}
\left\langle W(\mathbf{x}+\imath\,\mathbf{y})^\dagger N W(\mathbf{x}+\imath\,\mathbf{y}) \right\rangle &= \operatorname{Tr}\ \rho_g(\mathbf{l}, \mathbf{m}; S) W(\mathbf{x}+\imath\,\mathbf{y})^\dagger N W(\mathbf{x}+\imath\,\mathbf{y}) \\
&= \operatorname{Tr}\ W(\mathbf{x}+\imath\,\mathbf{y}) \rho_g(\mathbf{l}, \mathbf{m}; S) W(\mathbf{x}+\imath\,\mathbf{y})^\dagger N \\
&= \frac{1}{2}\left[\operatorname{Tr}\left(S - \frac{1}{2}\right) + \|\mathbf{l} + \sqrt{2}\mathbf{y}\|^2 + \|\mathbf{m} + \sqrt{2}\mathbf{x}\|^2 \right].
\end{aligned}$$

Now subtracting the value of $\langle N \rangle$ given by (3.4) we obtain equation (4.1).

Similarly , we have from Proposition 2.5 and equation (3.4)

$$\begin{aligned} \mathrm{Tr}\ \rho_g(\mathbf{l},\mathbf{m};S)\Gamma(L)^\dagger N\Gamma(L) &= \mathrm{Tr}\ \Gamma(L)\rho_g(\mathbf{l},\mathbf{m};S)\Gamma(L)^\dagger N \\ &= \frac{1}{2}\left[\mathrm{Tr}\ \left(L^{-1^T}SL^{-1}-\frac{1}{2}\right)+\begin{pmatrix}\mathbf{l}\\-\mathbf{m}\end{pmatrix}^T L^{-1^T}L^{-1}\begin{pmatrix}\mathbf{l}\\-\mathbf{m}\end{pmatrix}\right]. \end{aligned}$$

Now, subtracting the value of $\langle N\rangle$ given by (3.4) get get (4.2). ☐

For any unitary operator U in the 1-mode Hilbert space $\Gamma(\mathbb{C})$ we say that the operator $I\otimes\cdots\otimes I\otimes U\otimes I\otimes\cdots\otimes I$ with U in j-th position, acting in the n-mode Hilbert space $\Gamma(\mathbb{C}^n)=\underbrace{\Gamma(\mathbb{C})\otimes\Gamma(\mathbb{C})\otimes\cdots\otimes\Gamma(\mathbb{C})}_{n\text{ times}}$ is the gate U applied on the j-th mode and denote it by $U^{(j)}$. In Figure 1 we represent $U^{(j)}$ by following the notion of circuit diagrams in quantum computation.

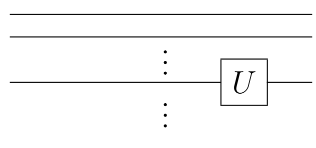
$U^{(j)} =$ 

FIGURE 1.

Here each wire stands for a $\Gamma(\mathbb{C})$ and in the j-th wire the unitary operator U is applied.

Similarly, if V is a unitary operator in the 2-mode Hilbert space $\Gamma(\mathbb{C}^2)=\Gamma(\mathbb{C})\otimes\Gamma(\mathbb{C})$ we construct the unitary operator $V^{(i,j)}$ in the 2 modes representing the i-th and the j-th wire. For example $V^{(1,2)}$ is represented in Figure 2.

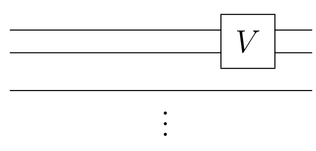
$V^{(1,2)} =$ 

FIGURE 2.

When $i<j$ are not successive we can apply a permutation to make them successive, apply V and follow by the reverse permutation. One may use Figure 3. For achieving the tomography, we shall use only one and two mode gates.

To begin with we consider the two 1-mode Weyl displacement operators $W(2^{-\frac{1}{2}}\imath)$ and $W(2^{-\frac{1}{2}})$ where $\imath=\sqrt{-1}$. Put

$$G_p = W(2^{-\frac{1}{2}}\imath) \tag{4.3}$$

$$G_q = W(2^{-\frac{1}{2}}) \tag{4.4}$$

If $\mathbf{e}_j=(0,\cdots,0,1,0\cdots,0)^T$ with 1 in the j-th position then applying G_p and G_q in the j-th mode is equivalent to using the displacement operator $W(2^{-\frac{1}{2}}\imath\mathbf{e}_j)$ and $W(2^{-\frac{1}{2}}\mathbf{e}_j)$ respectively.

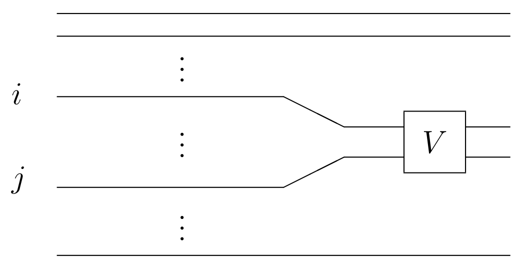


FIGURE 3.

Then equation (4.1) in Theorem 4.1 reduces to

$$l_j = \left\langle G_p^{(j)\dagger} N G_p^{(j)} \right\rangle - \langle N \rangle - 1, \tag{4.5}$$

$$m_j = \left\langle G_q^{(j)\dagger} N G_q^{(j)} \right\rangle - \langle N \rangle - 1. \tag{4.6}$$

Furthermore (3.4) implies

$$\mathrm{Tr}\ S = 2\langle N \rangle - \|\mathbf{l}\|^2 - \|\mathbf{m}\|^2 + n. \tag{4.7}$$

In other words, the measurement of counting observables $N,\ G_p^{(j)\dagger} N G_p^{(j)},\ G_q^{(j)\dagger} N G_q^{(j)},\ 1 \leq j \leq n$ which constitute a set of cardinality $2n+1$ yields the $2n+1$ parameters $l_j,\ m_j\ (1 \leq j \leq n)$ and Tr S concerning the Gaussian state $\rho_g(\mathbf{l}, \mathbf{m}; S)$.

Till now, the only resources we have used are identity (or zero mode) gate and the one mode gates G_p and G_q. Now we shall pass on to 1-mode Gaussian symmetry gates.

We consider the element $L(x, \alpha) \in Sp(2)$ defined by

$$L(x,\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} x & 0 \\ 0 & \frac{1}{x} \end{bmatrix} \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}, \quad x > 1, \quad 0 \leq \alpha < 2\pi \tag{4.8}$$

and the unitary operator

$$G_{sp}(x, \alpha) = \Gamma(\tau(L(x, \alpha))) \tag{4.9}$$

where $\tau(L) = (L^{-1})^T$ in any symplectic group. We now view $G_{sp}(x, \alpha)$ as a 1-mode gate and apply it in different modes. We express it by Figure 4 where the box is in the j-th wire.

$G^j_{sp}(x, \alpha) =$

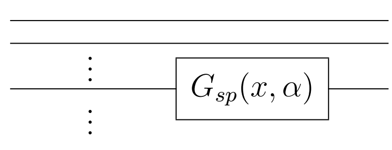


FIGURE 4.

Expressing the covariance matrix S of the n-mode Gaussian state as a block matrix

$$S = [[S_{i,j}]], \qquad i, j \in \{1, 2, \cdots, n\}$$

where each $S_{i,j}$ is a 2×2 matrix we observe that S_{jj} is the covariance matrix of the j-th marginal Gaussian state and

$$\begin{bmatrix} S_{ii} & S_{ij} \\ S_{ij}^T & S_{jj} \end{bmatrix}, \quad i<j \tag{4.10}$$

is the covariance matrix of the $i\ j$-marginal Gaussian state of $\rho_g(\mathbf{l},\mathbf{m};S)$.

Let

$$S_{jj} = \begin{bmatrix} \sigma_{pp} & \sigma_{pq} \\ \sigma_{qp} & \sigma_{qq} \end{bmatrix} \tag{4.11}$$

with $\sigma_{qp}=\sigma_{pq}$. Thus S_{jj} has three parameters. Applying the gate $G_{sp}^{(j)}(x,\alpha)$ defined by (4.9) and Figure 4, and using part (ii) of Theorem 4.1 we get

$$\begin{aligned}
&(x^2\cos^2\alpha + x^{-2}\sin^2\alpha - 1)\sigma_{pp} + (x^2\sin^2\alpha + x^{-2}\cos^2\alpha - 1)\sigma_{qq} + 2(x^2-x^{-2})\sin\alpha\cos\alpha \\
&\quad = 2\left(\left\langle G_{sp}^{(j)}(x,\alpha)^\dagger N G_{sp}^{(j)}(x,\alpha)\right\rangle - \langle N\rangle\right) + [1-(x^2\cos^2\alpha + x^{-2}\sin^2\alpha)]l_j^2 \\
&\qquad + [1-(x^2\sin^2\alpha + x^{-2}\cos^2\alpha)]m_j^2 + 2l_jm_j(x^2-x^{-2})\sin\alpha\cos\alpha
\end{aligned} \tag{4.12}$$

Choosing $\alpha=0$, (4.12) becomes

$$\begin{aligned}
&(x^2-1)\sigma_{pp} + (x^{-2}-1)\sigma_{qq} \\
&= 2\left(\left\langle G_{sp}^{(j)}(x,0)^\dagger N G_{sp}^{(j)}(x,0)\right\rangle - \langle N\rangle\right) + (1-x^2)l_j^2 + (1-x^{-2})m_j^2.
\end{aligned} \tag{4.13}$$

Choosing $x=\sqrt{2}$ and $x=\sqrt{3}$ successively (4.13) yields two linearly independent equations for determining σ_{pp} and σ_{qq} in terms of $l_j,\ m_j,\ \langle N\rangle$ and $\left\langle G_{sp}^{(j)}(x,0)^\dagger N G_{sp}^{(j)}(x,0)\right\rangle$.

Now we go back to the equation (4.12) and choose $x=\sqrt{2}$, $\alpha=\frac{\pi}{4}$. Then we get

$$\begin{aligned}
\frac{1}{4}(\sigma_{pp}+\sigma_{qq}) + \frac{3}{2}\sigma_{pq} &= 2\left(\left\langle G_{sp}^{(j)}\left(\sqrt{2},\frac{\pi}{4}\right)^\dagger N G_{sp}^{(j)}\left(\sqrt{2},\frac{\pi}{4}\right)\right\rangle - \langle N\rangle\right) \\
&\quad -\frac{1}{4}(l_j^2+m_j^2) + \frac{3}{2}l_jm_j.
\end{aligned} \tag{4.14}$$

Since $\sigma_{pp},\ \sigma_{qq}$ and $l_j,\ m_j$ have already been determined, (4.14) determines σ_{pq} by using values of $\langle N\rangle$ and $\left\langle G_{sp}^{(j)}\left(\sqrt{2},\frac{\pi}{4}\right)^\dagger N G_{sp}^{(j)}\left(\sqrt{2},\frac{\pi}{4}\right)\right\rangle$. Thus S_{jj} can be completely determined by measurements of N using the gates $G_{sp}(\sqrt{2},0),\ G_{sp}(\sqrt{3},0)$ and $G_{sp}(\sqrt{2},\frac{\pi}{4})$ in different modes after knowing the vectors $\mathbf{l}$ and $\mathbf{m}$. However, after determining S_{jj} for $1\le j\le n-1$, in order to determine S_{nn} it is enough to use only the two gates $G_{sp}(\sqrt{2},0)$ and $G_{sp}(\sqrt{2},\frac{\pi}{4})$ in the n-th mode because we already know $\operatorname{Tr} S$ form (4.7). Thus we need only $(3n-1)$ new measurements to determine the $3n$ parameters occurring in the block diagonals $S_{jj},\ 1\le j\le n$.

Now it remains to determine for any $i<j$ the off-diagonal block S_{ij}. To achieve this goal we shall use a 2-mode gate of the form $\Gamma(L),\ L\in Sp(4)$. We start with a unitary matrix of order 2 of the form

$$U = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}, \quad |\alpha|^2+|\beta|^2=1$$

where $\alpha = \alpha_1 + \imath\,\alpha_2$, $\beta = \beta_1 + \imath\,\beta_2$ with α_j, β_j being real. If we view U as a real linear transformation of $\mathbb{R}^4$ we get an element of $Sp(4)$ of the form

$$\mathcal{O} = \begin{bmatrix} \alpha_1 & -\alpha_2 & \beta_1 & -\beta_2 \\ \alpha_2 & \alpha_1 & \beta_2 & \beta_1 \\ -\beta_1 & -\beta_2 & \alpha_1 & \alpha_2 \\ \beta_2 & -\beta_1 & -\alpha_2 & \alpha_1 \end{bmatrix} \tag{4.15}$$

where $\mathcal{O}$ is a real orthogonal matrix. Define

$$L(U, x_1, x_2) = \mathcal{O} \begin{bmatrix} x_1 & & & \\ & x_1^{-1} & & \\ & & x_2 & \\ & & & x_2^{-1} \end{bmatrix} \mathcal{O}^T, \quad x_1 > 1,\ x_2 > 1. \tag{4.16}$$

We write

$$L(U, x_1, x_2)^T L(U, x_1, x_2) = \begin{bmatrix} A & B^T \\ B & C \end{bmatrix} \tag{4.17}$$

and note that A, C are 2×2 positive definite matrices and B is given by

$$B = \begin{bmatrix} -\beta_1\alpha_1(x_1 - x_2) + \beta_2\alpha_2(x_1^{-1} - x_2^{-1}) & -\beta_1\alpha_2(x_1 - x_2^{-1}) - \beta_2\alpha_1(x_1^{-1} - x_2) \\ \beta_2\alpha_1(x_1 - x_2^{-1}) + \beta_1\alpha_2(x_1^{-1} - x_2) & \beta_2\alpha_2(x_1 - x_2) - \beta_1\alpha_1(x_1^{-1} - x_2^{-1}) \end{bmatrix}. \tag{4.18}$$

Using (4.10) and (4.17) we observe that

$$\begin{aligned} \mathrm{Tr}\ \begin{bmatrix} S_{ii} & S_{ij} \\ S_{ji} & S_{jj} \end{bmatrix} L(U, x_1, x_2)^T L(U, x_1, x_2) &= \mathrm{Tr}\ \begin{bmatrix} S_{ii} & S_{ij} \\ S_{ji} & S_{jj} \end{bmatrix} \begin{bmatrix} A & B^T \\ B & C \end{bmatrix} \\ &= \mathrm{Tr}(S_{ii}A + S_{jj}C) + 2\mathrm{Tr}\, S_{ij}B \end{aligned} \tag{4.19}$$

where the first sum on the right hand side depends only on S_{ii} and S_{jj} which have already been determined in terms of the expectations of N and its conjugates by chosen one mode gates. In order to determine S_{ij} we use the 2-mode gate

$$G_{sp}(U, x_1, x_2) = \Gamma(\tau(L(U, x_1, x_2))) \tag{4.20}$$

in the (i, j)-modes and use part (ii) of Theorem 4.1 to obtain the relation

$$\begin{aligned} &\left\langle G_{sp}^{i,j}(U, x_1, x_2)^\dagger N G_{sp}^{i,j}(U, x_1, x_2) \right\rangle - \langle N \rangle \\ = \ & \frac{1}{2} \left[\mathrm{Tr} \begin{bmatrix} S_{ii} & S_{ij} \\ S_{ji} & S_{jj} \end{bmatrix} \begin{bmatrix} A - I_2 & B^T \\ B & C - I_2 \end{bmatrix} + \begin{pmatrix} l_1 \\ -m_1 \\ l_2 \\ -m_2 \end{pmatrix}^T \begin{bmatrix} A - I_2 & B^T \\ B & C - I_2 \end{bmatrix} \begin{pmatrix} l_1 \\ -m_1 \\ l_2 \\ -m_2 \end{pmatrix} \right]. \end{aligned}$$

Using (4.19) this reduces to

$$\begin{aligned} \mathrm{Tr}\ S_{ij}B &= \left\langle G_{sp}^{i,j}(U, x_1, x_2)^\dagger N G_{sp}^{i,j}(U, x_1, x_2) \right\rangle - \langle N \rangle \\ &\quad - \frac{1}{2}\mathrm{Tr}\ \left[S_{ii}(A - I_2) + S_{jj}(C - I_2) \right] \\ &= f(U, x_1, x_2) \text{ say.} \end{aligned} \tag{4.21}$$

When $l_i,\ m_i,\ l_j,\ m_j,\ S_{ii}$ and S_{jj} are already determined, the term $f(U, x_1, x_2)$ depends only on $U,\ x_1,\ x_2$. Let

$$S_{ij} = \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix}.$$

We now make four special cases for (U, x_1, x_2).

(i) $U = H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix},\ x_1 = 1,\ x_2 = 2$. Put $r_1 = f(H, 1, 2)$. Then (4.21) becomes

$$\frac{1}{2}\gamma_{11} - \frac{1}{4}\gamma_{22} = r_1. \tag{4.22}$$

(ii) $U = H,\ x_1 = 1,\ x_2 = 3$. Put $r_2 = f(H, 1, 3)$. Then (4.21) becomes

$$\gamma_{11} - \frac{1}{3}\gamma_{22} = r_2. \tag{4.23}$$

(iii) $U = K = \frac{1}{\sqrt{2}} \begin{bmatrix} \imath & 1 \\ -1 & -\imath \end{bmatrix},\ x_1 = 1,\ x_2 = 2$. Put $r_3 = f(K, 1, 2)$. Then (4.21) becomes

$$-\frac{1}{4}(\gamma_{21} + 2\gamma_{12}) = r_3. \tag{4.24}$$

(iv) $U = K,\ x_1 = 1,\ x_2 = 3$. Put $r_4 = f(K, 1, 3)$. Then (4.21) becomes

$$-\frac{1}{3}(\gamma_{21} + 3\gamma_{12}) = r_4. \tag{4.25}$$

The four equations (4.22)–(4.25) in the unknowns $\gamma_{11},\ \gamma_{22},\ \gamma_{12},\ \gamma_{21}$ are linear and linearly independent. Thus they determine the matrix S_{ij} for any fixed $i,\ j$. For this purpose we have used exactly four measurements described by the four 2-mode gates for four parameters.

In all we have used exactly $(2n + 1) + (3n - 1) + 4\frac{n(n-1)}{2} = n(2n + 3)$ measurements to determine the $n(2n + 3)$ parameters of the Gaussian state $\rho_g(\mathbf{l}, \mathbf{m}; S)$.

5. Tomography of Gaussian channels

An n-mode Gaussian channel $\mathcal{K}(A, B)$ is described by a pair of real $2n \times 2n$ matrices (A, B) where B is positive semidefinite and the following matrix inequality holds:

$$B + \imath(A^T J_{2n} A - J_{2n}) \geq 0$$

with J_{2n} as in equation (2.12). Thus $\mathcal{K}(A, B)$ is determined by $6n^2 + n$ real parameters. Such a channel has the property that for any Gaussian input state $\rho_g(\mathbf{l}, \mathbf{m}; S)$, the corresponding output is again Gaussian and has the form $\rho_g(\mathbf{l}', \mathbf{m}'; S')$ where

$$\begin{pmatrix} \mathbf{l}' \\ -\mathbf{m}' \end{pmatrix} = A^T \begin{pmatrix} \mathbf{l} \\ -\mathbf{m} \end{pmatrix} \tag{5.1}$$

$$S' = A^T S A + \frac{1}{2}B. \tag{5.2}$$

our aim is to determine A and B by performing tomography on the output states for a small number of coherent input statea $|\psi(\mathbf{u})\rangle$ where

$$|\psi(\mathbf{u})\rangle\langle\psi(\mathbf{u})| = \rho_g\left(\sqrt{2}\mathbf{y}, \sqrt{2}\mathbf{x}; \frac{1}{2}I_{2n}\right)$$

where $\mathbf{x} = \mathrm{Re}(\mathbf{u}),\ \mathbf{y} = \mathrm{Im}(\mathbf{u})$. By (5.1) and (5.2) the output state is

$$\rho_g\left(\mathbf{y}', \mathbf{x}'; \frac{1}{2}(A^TA + B)\right) \tag{5.3}$$

where

$$\begin{pmatrix} \mathbf{y}' \\ -\mathbf{x}' \end{pmatrix} = A^T \begin{pmatrix} \sqrt{2}\mathbf{y} \\ -\sqrt{2}\mathbf{x} \end{pmatrix}. \tag{5.4}$$

Now we specialize the values of $\mathbf{u}$ and select the $2n$ input states

$$\rho_j = |\psi(2^{-\frac{1}{2}}\imath\mathbf{e_j})\rangle\langle\psi(2^{-\frac{1}{2}}\imath\mathbf{e_j})|, \quad 1 \le j \le n, \tag{5.5}$$

$$\rho_j' = |\psi(2^{-\frac{1}{2}}\mathbf{e_j})\rangle\langle\psi(2^{-\frac{1}{2}}\mathbf{e_j})|, \quad 1 \le j \le n. \tag{5.6}$$

Then the corresponding output states are

$$\tilde{\rho_g}\left(\tilde{\mathbf{l}}_\mathbf{j}, \tilde{\mathbf{m}}_\mathbf{j}, \frac{1}{2}(A^TA + B)\right) \tag{5.7}$$

with

$$\begin{pmatrix} \tilde{\mathbf{l}}_\mathbf{j} \\ -\tilde{\mathbf{m}}_\mathbf{j} \end{pmatrix} = A^T \begin{pmatrix} \mathbf{e_j} \\ \mathbf{0} \end{pmatrix}, \quad 1 \le j \le n \tag{5.8}$$

and

$$\tilde{\rho_g}'\left(\tilde{\mathbf{l}}_\mathbf{j}', \tilde{\mathbf{m}}_\mathbf{j}', \frac{1}{2}(A^TA + B)\right) \tag{5.9}$$

with

$$\begin{pmatrix} \tilde{\mathbf{l}}_\mathbf{j}' \\ -\tilde{\mathbf{m}}_\mathbf{j}' \end{pmatrix} = A^T \begin{pmatrix} \mathbf{0} \\ \mathbf{e_j} \end{pmatrix}, \quad 1 \le j \le n. \tag{5.10}$$

A full tomography on (5.7) with $j = 1$ as outlined in Section §4 yields the first row of A and the matrix A^TA+B by using $n(2n+3)$ measurements. A similar but partial tomography of the remaining $(n-1)$ states in (5.7) and all the states in (5.9) but only for the mean values yields the remaining $(2n-1)$ rows of A. This needs an additional set of $(2n-1)(2n+1) = 4n^2 - 1$ measurements. In all, our approach requires $6n^2 + 3n - 1$ measurements for getting the $6n^2 + n$ parameters. It will be interesting to know whether one can determine A and B with less measurements. If this is not possible our problem will carry an intrinsic tomographic complexity.

6. Conclusions

All the $2n^2+3n$ mean and covariance parameters of an n-mode Gaussian state can be recovered from the expectation values of the same number of conjugates of the total number operator by Gaussian symmetries. Such symmetries can be realised by five one mode and four two mode gates. The complete tomography of a Gaussian state can be expressed by circuit diagrams and measurements akin to those in quantum computation theory. An application of this tomography to the output of an n-mode Gaussian channel corresponding to appropriate coherent inputs determines all the $6n^2+n$ parameters by $6n^2+3n-1$ measurements. Improvement in this channel tomography and finding the probability distribution of the number operator from its explicitly computable probability generating function in a general n-mode Gaussian state seem to be interesting problems arising from our investigations.

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