isid/ms/2015/06
June 23, 2015
http://www.isid.ac.in/~statmath/index.php?module=Preprint

Exchangeable, stationary and entangled chains of Gaussian states

K. R. Parthasarathy and Ritabrata Sengupta

Indian Statistical Institute, Delhi Centre
7, SJSS Marg, New Delhi–110 016, India

EXCHANGEABLE, STATIONARY AND ENTANGLED CHAINS OF GAUSSIAN STATES

K. R. PARTHASARATHY AND RITABRATA SENGUPTA

Abstract. We explore conditions on the covariance matrices of a consistent chain of mean zero finite mode Gaussian states in order that the chain may be exchangeable or stationary. For an exchangeable chain our conditions are necessary and sufficient. Every stationary Gaussian chain admits an asymptotic entropy rate. Whereas an exchangeable chain admits a simple expression for its entropy rate, in our examples of stationary chains the same admits an integral formula based on the asymptotic eigenvalue distribution for Toeplitz matrices. An example of a stationary entangled Gaussian chain is given.

Keywords. Gaussian state, exchangeable, stationary and entangled Gaussian chains, entropy rate.

Mathematics Subject Classification (2010): 81P45, 94A15, 54C70.

1. Introduction

The importance of finite mode Gaussian states and their covariance matrices in general quantum theory as well as quantum information has been highlighted extensively in the literature. A comprehensive survey of Gaussian states and their properties can be found in the book of Holevo [Hol11]. For their applications to quantum information theory the reader is referred to the survey article by Weedbrook et al [WPGP+12] as well as Holevo's book [Hol12]. For our reference we use [ADMS95, Par10, Par13] for Gaussian states and for notations in the following sections we use [PS15]. While working on this paper, Mozrzymas, Rutkowski, and Studziński had posted in arXiv [MRS15] an article similar spirit for finite dimensional Hilbert spaces.

In the present paper our concern is with a chain of finite mode Gaussian states constituting a consistent sequence exhibiting properties like exchangeability, stationarity, and entanglement. All these notions are easily translated into properties of infinite covariance matrices.

If ρ is a state of a quantum system and X_i, $i = 1, 2$ are two real valued observables, or equivalently, self-adjoint operators with finite second moments in a state ρ then the covariance between X_1 and X_2 in the state ρ is the scalar quantity $\operatorname{Tr}\left(\frac{X_1X_2+X_2X_1}{2}\right)\rho - \operatorname{Tr}X_1\rho \cdot \operatorname{Tr}X_2\rho$, which is denoted by $\operatorname{Cov}_\rho(X_1, X_2)$. Suppose $q_1, p_1; q_2, p_2; \cdots ; q_n, p_n$ are the position - momentum pairs of observables of a quantum system with n degrees of freedom obeying the canonical commutation relations. Then we express

$$(X_1, X_2, \cdots , X_{2n}) = (q_1, p_1, q_2, p_2, \cdots , q_n, p_n).$$

If ρ is a state in which all the X_j's have finite second moments we write

$$S_\rho = [[\operatorname{Cov}_\rho(X_i, X_j)]], \quad i, j \in \{1, 2, \cdots , n\}. \tag{1.1}$$

RS acknowledges financial support from the National Board for Higher Mathematics, Govt. of India.

We call S_ρ the covariance matrix of the position momentum observables. If we write

$$J_{2n} = \begin{bmatrix} 0 & 1 & & & & & \\ -1 & 0 & & & & & \\ & & 0 & 1 & & & \\ & & -1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & 1 \\ & & & & & -1 & 0 \end{bmatrix} \tag{1.2}$$

or equivalently $\bigoplus_1^n \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ for the $2n \times 2n$ block diagonal matrix, the complete Heisenberg uncertainty principle for all the position and momentum observables assumes the form of the following matrix inequality

$$S_\rho + \frac{\imath}{2} J_{2n} \geq 0. \tag{1.3}$$

Conversely, if S is any real $2n \times 2n$ symmetric matrix obeying the inequality $S_\rho + \frac{\imath}{2} J_{2n} \geq 0$, then there exists a state ρ such that S is the covariance matrix S_ρ of the observables $q_1, p_1; q_2, p_2; \cdots; q_n, p_n$. In such a case ρ can be chosen to be a Gaussian state with mean zero. In view of this theorem we make a formal definition.

Definition 1.1. *A $2n \times 2n$ real symmetric positive matrix S is said to be a* G*-matrix if it satisfies the inequality*

$$S + \frac{\imath}{2} J_{2n} \geq 0, \tag{1.4}$$

where $J_{2n} = \bigoplus_{n\text{-}copy} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Suppose

$$\Sigma = [[A_{ij}]], \quad i,\, j \in \{1, 2, \cdots\} \tag{1.5}$$

is an infinite matrix where each A_{ij} is a $2k \times 2k$ real matrix and $A_{ij}^T = A_{ji}$ for all i, j. For any finite subset $I = \{i_1 < i_2 < \cdots < i_n\} \subset \{1, 2, \cdots\}$, let

$$\Sigma(I) = [[A_{i_r i_s}]], \quad r,\, s \in \{1, 2, \cdots\} \tag{1.6}$$

be the $2kn \times 2kn$ matrix obtained from Σ by restriction to its rows and columns numbered $i_1 < i_2 < \cdots < i_n$.

Definition 1.2. *We say that Σ is a* G-chain of order k *if $\Sigma(I)$ is the covariance matrix of a kn-mode zero mean Gaussian state $\rho(I)$ in the boson Fock space $\Gamma(\mathbb{C}^{kn}) = \mathcal{H}_{i_1} \otimes \mathcal{H}_{i_2} \otimes \cdots \otimes \mathcal{H}_{i_n}$ where $\mathcal{H}_j$ denotes the j-th copy of the Hilbert space $\mathcal{H} = \Gamma(\mathbb{C}^k)$, $j = 1, 2, \cdots$.*

If $I = \{i_1 < i_2 < \cdots < i_n\}$ and $I' = \{i_1 < i_2 < \cdots < i_{n+m}\}$ then clearly $\rho(I)$ is the I-marginal of the state $\rho(I')$. Thus Σ describes a consistent family of zero mean Gaussian states with finite number of modes which are multiples of k. It is a standard result of Gaussian states that Σ is a G-chain of order k if and only if the following matrix inequalities hold:

$$\Sigma(\{1,2,\cdots,n\}) + \frac{\imath}{2}J_{2kn} \geq 0, \quad n = 1,2,\cdots \tag{1.7}$$

Definition 1.3. *We say that Σ in equation (1.5) is an* exchangeable G-chain *if it is a G-chain and there exist two $2k \times 2k$ matrices $A,\ B$ such that*

$$A_{ij} = \begin{cases} B & \text{if } j > i, \\ A & \text{if } j = i, \\ B^T & \text{if } j < i. \end{cases} \tag{1.8}$$

In such a case we write

$$\begin{aligned} \Sigma &= \Sigma(A,B), \\ \Sigma(I) &= \Sigma(I;A,B). \end{aligned} \tag{1.9}$$

It is clear that for any two finite subsets I and I' of $\{1,2,\cdots\}$ with the same cardinality an exchangeable G-chain Σ satisfies

$$\Sigma(I;A,B) = \Sigma(I';A,B)$$

and therefore the corresponding Gaussian states $\rho(I)$ and $\rho(I')$ are same, i.e. the quantum version of de Finitti exchangeability property [de 29] holds.

In this paper we shall show that two $2k \times 2k$ matrices A and B determine a G-chain $\Sigma(A,B)$ if and only if $B = B^T,\ B > 0$ and $A - B$ is the covariance matrix of a k-mode Gaussian state.

For any finite mode Gaussian state ρ with covariance matrix A denote its von Neumann entropy $S(\rho)$ by $S(A)$. we shall prove that for any exchangeable G-chain $\Sigma(A,B)$, the sequence $\left\{\frac{1}{n}S(\Sigma(\{1,2,\cdots,n\},A,B))\right\}$ decreases monotonically to the limit $S(A-B)$.

Definition 1.4. *We say that a G-chain Σ given by (1.5) is* stationary *if there exist $2k \times 2k$ matrices $A,\ B_1,\ B_2,\cdots$ such that*

$$A_{ij} = \begin{cases} A & \text{if } i = j, \\ B_{j-i} & \text{if } j > i, \\ B_{i-j}^T & \text{if } j < i \end{cases} \tag{1.10}$$

for all $i,\ j$.

For such a stationary G-chain the Gaussian states $\rho(I)$ and $\rho(I+1)$ are same for any finite subset $I \subset \{1,2,\cdots\}$. This translation invariance property shows that our definition of stationarity is similar to such a notion in classical theory of the stochastic processes [Doo53]. It is an interesting problem to find necessary and sufficient conditions on the matrices $A,\ B_1,\ B_2,\cdots$ so that (1.10) yields a stationary G-chain. Even though we do not have an answer to this question we shall construct a large class of examples of such stationary G-chains.

For any stationary G-chain Σ, consider the von Neumann entropy of the Gaussian state $\rho(\{1,2,\cdots,n\},A,B)$, namely, $S(\Sigma(\{1,2,\cdots,n\}:A,B))$. The strong sub-additivity property of entropy [LR73] implies that the sequence $\left\{\frac{1}{n}S(\Sigma(\{1,2,\cdots,n\}:A,B))\right\}$ decreases monotonically to a limit $\bar{S}(\Sigma)$, which may be called the *entropy rate* of the stationary chain. Using the Kac-Murdock-Szegö theory for asymptotic eigenvalue distributions of Toeplitz matrices

[KMS53, Tro84] we shall get an integral formula for the entropy rate of an interesting example of a stationary G-chain of order k.

When $k = 1$, we shall construct a stationary G-chain in which states of the form $\rho(\{1, 2\})$ and hence $\rho(\{1, 2, \cdots, n\})$, $n = 2, 3, 4, \cdots$ are entangled. However it would be more interesting to find examples of stationary G-chains in which, for some fixed finite nonempty subset I of $\{1, 2, \cdots\}$, states of the form $\rho(I \cup I + n)$ with marginals $\rho(I)$ remain entangled for arbitrarily large n. That would ensure preservation of entanglement after an arbitrarily large lapse of time.

2. Exchangeable G-chain

Our first main result gives necessary and sufficient conditions on a pair (A, B) of real $2k \times 2k$ matrices so that the infinite matrix $\Sigma(A, B)$ in Definition 1.3 is an exchangeable G-chain.

Theorem 2.1. *Let (A, B) be a pair of real $2k \times 2k$ matrices. Then $\Sigma(A, B)$ is a G-chain if and only if A and B are nonnegative definite and $A - B$ is a G-matrix.*

Proof. Choose and fix a positive integer n and put $I = \{1, 2, \cdots, n\}$. Write

$$\Sigma_n(A, B) = \Sigma(I : A, B).$$

Denote by I_n the identity matrix of order n and N_n the upper triangular nilpotent matrix with all its upper triangular entries equal to 1 and the rest equal to zero. Let

$$|\psi_n\rangle = \frac{1}{\sqrt{n}}[1, 1, \cdots, 1]^T,$$

a unit column vector of length n. Then

$$I_n + N_n + N_n^T = n\, |\psi_n\rangle\langle\psi_n|\, .$$

Then we have

$$\Sigma_n(A, B) = A \otimes I_n + B \otimes N_n + B^T \otimes N_n^T.$$

Noting that $J_{2kn} = J_{2k} \otimes I_n$ we can now write

$$\begin{aligned} \Sigma_n(A, B) + \frac{\imath}{2} J_{2kn} \;\; &= \;\; \left(A + \frac{\imath}{2} J_{2k} - \frac{1}{2}(B + B^T)\right) \otimes (I_n - |\psi_n\rangle\langle\psi_n|) \\ &\quad + \left(A + \frac{\imath}{2} J_{2k} + \frac{1}{2}(n-1)(B + B^T)\right) \otimes |\psi_n\rangle\langle\psi_n| \\ &\quad + \frac{1}{2}(B - B^T) \otimes (N_n - N_n^T). \end{aligned} \tag{2.1}$$

In order that $\Sigma(A, B)$ may be a G-chain it is necessary and sufficient that the left hand side of (2.1) is nonnegative definite for every n. To prove necessity we multiply first both sides of (2.1) by $I_n \otimes |\psi_n\rangle\langle\psi_n|$ and take relative trace over the second component. Noting that for any real vector $|\psi\rangle$ in the second Hilbert space, $\langle\psi|N - N^T|\psi\rangle = 0$ we get the inequality

$$A + \frac{\imath}{2} J_{2k} + \frac{1}{2}(n-1)(B + B^T) \geq 0 \quad \forall n. \tag{2.2}$$

Dividing by $n - 1$ and letting $n \to \infty$ we get

$$\frac{1}{2}(B + B^T) \geq 0. \tag{2.3}$$

Choosing an arbitrary real unit vector $|\psi\rangle$ in the range of $I_n - |\psi_n\rangle\langle\psi_n|$, multiplying both sides of (2.1) by $I \otimes |\psi\rangle\langle\psi|$ and tracing out over the second Hilbert space we get

$$A + \frac{\imath}{2}J_{2k} - \frac{1}{2}(B + B^T) \geq 0. \tag{2.4}$$

In other words $A - \frac{1}{2}(B + B^T)$ is a G-matrix. Now we consider the complex unitary vector

$$|\phi_n\rangle = \frac{1}{\sqrt{n}}[1, \omega, \omega^2, \cdots, \omega^{n-1}]^T$$

where $\omega = e^{\frac{2\pi\imath}{n}}$, an n-th root of unity. Simple algebra shows that

$$\langle\phi_n|N_n|\phi_n\rangle = \frac{1}{\bar{\omega} - 1}$$

and therefore

$$\langle\phi_n|N_n - N_n^T|\phi_n\rangle = 2\imath Im\frac{1}{\bar{\omega} - 1} = \imath \cot\frac{\pi}{n}.$$

Now, multiplying both sides of (2.1) by $I_n \otimes |\phi_n\rangle\langle\phi_n|$ and tracing over the second Hilbert space we get the inequality

$$A + \frac{\imath}{2}J_{2k} - \frac{1}{2}(B + B^T) + \imath\frac{B - B^T}{2}\cot\frac{\pi}{n} \geq 0 \tag{2.5}$$

for $n = 1, 2, \cdots$. Multiplying by $\tan\frac{\pi}{n}$ (for $n \geq 3$) and letting $n \to \infty$ we get the inequality

$$\frac{\imath}{2}(B - B^T) \geq 0.$$

Since the left hand side is a Hermitian matrix with trace zero it follows that $B = B^T$ and (2.4) implies that $A - B$ is a G-matrix. Together with (2.3) the proof of necessity is complete.

To prove sufficiency, observe that $A - B$ being a G-matrix and B being positive implies that $A - B + nB$ is a G-matrix for every $n = 1, 2, \cdots$. Now the identity (2.1) implies that

$$\Sigma_n(A, B) + \frac{\imath}{2}J_{2kn} = \left[(A - B) + \frac{\imath}{2}J_{2k}\right] \otimes (I_n - |\psi_n\rangle\langle\psi_n|) + \left[A + nB + \frac{\imath}{2}J_{2k}\right] \otimes |\psi_n\rangle\langle\psi_n| \geq 0$$

for every n and therefore $\Sigma_n(A, B)$ is a G-matrix for every n. In other words $\Sigma(A, B)$ is a G-chain. □

Corollary 2.1. *In any exchangeable G-chain $\Sigma(A, B)$ of order k, for every finite set $I \subset \{1, 2, \cdots\}$, the underlying Gaussian state $\rho(I)$ is separable.*

Proof. Without loss of generality we may assume that $I = \{1, 2, \cdots, n\}$ for some n. Then the covariance matrix of $\rho(I)$ is equal to the $n \times n$ block matrix

$$\begin{bmatrix} A & B & \cdots & B \\ B & A & \cdots & B \\ \vdots & \vdots & \ddots & \vdots \\ B & B & \cdots & A \end{bmatrix} = (A - B) \otimes I_n + B \otimes \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}.$$

By Theorem 2.1, $(A - B) \otimes I_n$ is the covariance matrix of an n-fold product Gaussian state and the second summand on the right hand side of the equation above is a nonnegative definite matrix. Hence by Werner and Wolf's theorem [WW01] $\rho(I)$ is separable. □

3. Examples of stationary G-chain

Let A, B be real $2k \times 2k$ symmetric matrices. For any fixed $j = 1, 2, \cdots$, denote by $\Delta^j(A, B)$ the infinite block matrix all of whose diagonal blocks are equal to A, $(n, n+j)$-th and $(n+j, n)$-th blocks are equal to B for every n and all the remaining blocks are zero matrices of order $2k \times 2k$. For example,

$$\Delta^1(A, B) = \begin{bmatrix} A & B & 0 & 0 & 0 & \cdots \\ B & A & B & 0 & 0 & \cdots \\ 0 & B & A & B & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Denote by $\Delta^j_n(A, B)$ the $2kn \times 2kn$ matrix obtained by $\Delta^j(A, B)$ by restriction to the first n rows and column blocks. For example,

$$\Delta^1_2(A, B) = \begin{bmatrix} A & B \\ B & A \end{bmatrix}$$

$$\Delta^2_3(A, B) = \begin{bmatrix} A & 0 & B \\ 0 & A & 0 \\ B & 0 & A \end{bmatrix}$$

and so on.

Our first result gives a necessary and sufficient condition for $\Delta^j(A, B)$ to be a G-chain.

Theorem 3.1. *Let A, B be a pair of $2k \times 2k$ real symmetric matrices. In order that $\Delta^j(A, B)$ may be a G-chain of order k it is necessary and sufficient that $A + tB$ is a G-matrix for every $t \in [-2, 2]$.*

Proof. Denote by L^j_n the upper triangular matrix whose $(j+1)$-th upper diagonal entries are all equal to 1 and all the remaining entries are zero. Thus L^j_n is defined for $1 \leq j \leq n-1$. Then

$$\Delta^j_n(A, B) = A \otimes I_n + B \otimes (L^j_n + (L^j_n)^T). \tag{3.1}$$

Consider the spectral decomposition of the $n \times n$ symmetric matrix $L^j_n + (L^j_n)^T$:

$$L^j_n + (L^j_n)^T = \sum_{r=1}^{n} \lambda_{nr} \, |\psi_{nr}\rangle\langle\psi_{nr}| \tag{3.2}$$

where $\{\lambda_{nr} : r = 1, 2, \cdots, n\}$ are the eigenvalues and $\{|\psi_{nr}\rangle : r = 1, 2, \cdots, n\}$ are the corresponding orthonormal basis of eigenvectors for $L^j_n + (L^j_n)^T$. Since each L^j_n is a matrix with operator norm equal to unity and therefore $L^j_n + (L^j_n)^T$ has operator norm not exceeding 2 it is clear that

$$|\lambda_{nr}| \leq 2, \quad 1 \leq r \leq n, \; n = 1, 2, \cdots. \tag{3.3}$$

Equations (3.1)–(3.2) imply

$$\Delta^j_n(A, B) + \frac{\imath}{2} J_{2kn} = \sum_{r=1}^{n} \left(A + \lambda_{nr} B + \frac{\imath}{2} J_{2k} \right) \otimes |\psi_{nr}\rangle\langle\psi_{nr}|. \tag{3.4}$$

Thus $\Delta_n^j(A,B)$ is a G-matrix if and only if $A+\lambda_{nr}B$ is a G-matrix for each $r=1,2,\cdots,n$. This together with (3.3) already proves the sufficiency part of the theorem.

To prove necessity, we appeal to the theorem of Kac, Murdock and Szegö [KMS53]. Consider the probability distribution

$$\mu_n = \frac{1}{n}\sum_{r=1}^{n}\delta_{\lambda_{nr}}$$

where λ_{nr}, $r=1,2,\cdots,n$ are as in (3.2). The left hand side of (3.2) is a Toeplitz matrix of order n for each n. Thus Kac, Murdock, Szegö theorem implies that the sequence $\{\mu_n\}$ converges weakly as $n\to\infty$ to the probability measure Lh^{-1} where L is the Lebesgue measure in the unit interval and $h(t)=2\cos 2\pi jt$, $t\in[0,1]$. This, in particular, implies that $\{\lambda_{nr}: r=1,2,\cdots,n,\ n=1,2,\cdots\}$ is dense in the interval $[-2,2]$. The proof of necessity is now complete. □

Corollary 3.1. *Let A, B_1, B_2, $\cdots$ be real $2k\times 2k$ symmetric matrices satisfying the condition that $A+tB_j$ is a G-matrix for every $j=1,2,\cdots$ and $t\in[-2,2]$. Suppose $p_1,p_2,\cdots,$ is a probability distribution on the set $\{1,2,\cdots\}$. Then the block Toeplitz matrix*

$$\Sigma(A;p_1B_1,p_2B_2,\cdots)=\begin{bmatrix} A & p_1B_1 & P_2B_2 & \cdots & \cdots\\ p_1B_1 & A & p_1B_1 & p_2B_2 & \cdots\\ p_2B_2 & p_1B_1 & A & p_1B_1 & \cdots\\ \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}$$

is a stationary G-chain

Proof. This is immediate from the theorem because

$$\Sigma(A;p_1B_1,p_2B_2,\cdots)=\sum_{j=1}^{\infty}p_j\Delta^j(A,B_j)$$

and each $\Delta^j(A,B_j)$ is a G-chain. □

4. Entropy rate of the stationary G-chain

Suppose $\Sigma=\Sigma(A,B_1,B_2,\cdots)$ is a stationary G-chain. For any G-matrix C denote by $S(C)$ the von Neumann entropy of a Gaussian state ρ with covariance matrix C. Let

$$\begin{aligned}\Sigma_n &= \Sigma(\{1,2,\cdots,n\}),\\ S_n &= S(\Sigma_n).\end{aligned}$$

Proposition 4.1. *The sequences $\{S_n-S_{n-1}\}$, $\left\{\frac{1}{n}S_n\right\}$ monotonically decrease to the same limit $\bar{S}\geq 0$ as $n\to\infty$.*

Proof. Consider three Gaussian quantum systems P, Q, R so that PQR is in the mean zero Gaussian state $\rho(\{1,2,\cdots,n+1\})$, Q in $\rho(\{2,\cdots,n\})$, PQ in $\rho(\{1,2,\cdots,n\})$ and QR in

$\rho(\{2, \cdots, n+1\})$. Then using stationarity we have

$$\begin{aligned} S(\rho(PQR)) &= S_{n+1} \\ S(\rho(PQ)) &= S_n \\ S(\rho(QR)) &= S_n \\ S(\rho(Q)) &= S_{n-1}. \end{aligned}$$

By the strong subadditivity property of entropy [LR73] we have

$$S_{n+1} + S_{n-1} \leq 2S_n$$

or

$$S_{n+1} - S_n \leq S_n - S_{n-1}.$$

Since $S_{n-1} - S_n \geq 0$ it follows that $S_{n-1} - S_n$ decreases monotonically to a limit $\bar{S} \geq 0$. Since

$$\frac{S_n}{n} = \frac{(S_n - S_{n-1}) + (S_{n-1} - S_{n-2}) + \cdots + (S_1 - S_0)}{n}$$

with S_0 defined as 0, it follows that $\frac{S_n}{n}$ also monotonically decreases to the same limit $\bar{S}$ and $\bar{S} \geq 0$. □

Definition 4.1. *We denote the limit $\bar{S}$ in Proposition 4.1 by $\bar{S}(\Sigma)$ and call it the* entropy rate *of the stationary G-chain Σ.*

Theorem 4.1. *Let $\Sigma = \Sigma(A, B)$ be an exchangeable G-chain. Then*

$$\bar{S}(\Sigma) = S(A - B).$$

Proof. If C, D are two G-matrices so is $C \oplus D$ and $S(C \oplus D) = S(C) + S(D)$. By Theorem 2.1 and equation (2.1) we have the identity

$$\Sigma_n(A, B) = (A - B) \otimes (I_n - |\psi_n\rangle\langle\psi_n|) + (A + \overline{n-1}B) \otimes |\psi_n\rangle\langle\psi_n|$$

where we have adopted the notations in the proof of Theorem 2.1. Thus

$$S_n = S(\Sigma_n(A, B)) = (n-1)S(A - B) + S(A + \overline{n-1}B). \tag{4.1}$$

Denote by ρ^A the zero mean Gaussian state with covariance matrix A for any G-matrix A. Thus we have

$$\rho^{A+\overline{n-1}B} = \int_{\mathbb{R}^{2k}} W(\boldsymbol{\xi})\rho^A W(\boldsymbol{\xi})^\dagger \phi(\boldsymbol{\xi})\, \mathrm{d}\boldsymbol{\xi}$$

where $\boldsymbol{\xi} = \boldsymbol{\xi}_1 \oplus \boldsymbol{\xi}_2$, $W(\boldsymbol{\xi})$ is Weyl or displacement operator at $\boldsymbol{\xi}_1 + \imath\boldsymbol{\xi}_2$ and $\phi(\boldsymbol{\xi})$ is the Gaussian density function with mean zero and covariance matrix $(n-1)B$. By Proposition 6.2 of [OP93] we have

$$\begin{aligned} S(A + \overline{n-1}B) &= S(\rho^{A+\overline{n-1}B}) \\ &\leq \int S(A)\phi(\boldsymbol{\xi})\, \mathrm{d}\boldsymbol{\xi} + H(\phi) \end{aligned} \tag{4.2}$$

where $H(\phi)$ is the Shannon differential entropy of the density function ϕ. Since (by [CT06])

$$H(\phi) = k \log 2\pi e + \frac{1}{2} \log \det(\overline{n-1}B) \tag{4.3}$$

it follows from (4.1)–(4.3) that

$$\begin{aligned}\left|\frac{S_n}{n}-\frac{n-1}{n}S(A-B)\right| &\leq \frac{S(A)}{n}+\frac{k}{n}\log 2\pi e+\frac{1}{2n}\log(n-1)^{2k}\det B\\ &\leq \frac{1}{n}\left[S(A)+k\log 2\pi e+\frac{1}{2}\log\det B\right]+\frac{k}{n}\log(n-1).\end{aligned}$$

Letting $n\to\infty$ we get

$$\bar{S}(\Sigma)=S(A-B).$$

□

Theorem 4.2. *Let $p_1, p_2, \cdots$ be a probability distribution over $\{1,2,3,\cdots\}$, and let A and B be $2k\times 2k$ symmetric real matrices satisfying the condition that $A+tB$ is a G-matrix for every $t\in[-2,2]$. Let Σ be the stationary G-chain defined by the infinite block Toeplitz matrix*

$$\Sigma=\begin{bmatrix} A & p_1B & p_2B & \cdots & \cdots\\ p_1B & A & p_1B & p_2B & \cdots\\ p_2B & p_1B & A & p_1B & \cdots\\ \vdots & \vdots & \vdots & \ddots & \ddots\end{bmatrix}.$$

Then the entropy rate of Σ is given by

$$\bar{S}(\Sigma)=\int_0^1 S(A+h(s)B)\,\mathrm{d}s$$

where

$$h(t)=2\sum_{j=1}^{\infty}p_j\cos 2\pi js,\quad s\in[0,1].$$

Proof. We can express Σ_n as

$$\Sigma_n=A\otimes I_n+B\otimes T_n(\boldsymbol{p})$$

where Σ_n is Σ restricted to its first n row and column blocks and $T_n(\boldsymbol{p})$ is the Toeplitz matrix given by

$$T_n(\boldsymbol{p})=\begin{bmatrix} 0 & p_1 & p_2 & \cdots & p_{n-1}\\ p_1 & 0 & p_1 & \cdots & p_{n-2}\\ p_2 & p_1 & 0 & \cdots & p_{n-3}\\ \vdots & \cdots & \vdots & \ddots & \vdots\\ p_{n-1} & p_{n-2} & p_{n-3} & \cdots & 0\end{bmatrix}.$$

Let $\lambda_{n1},\lambda_{n2},\cdots,\lambda_{nn}$ be the eigenvalues of $T_n(\boldsymbol{p})$ and let $|\psi_{n1}\rangle,|\psi_{n2}\rangle,\cdots,|\psi_{nn}\rangle$ the corresponding eigenvectors constituting an orthonormal basis for $\mathbb{R}^n$ so that

$$\Sigma_n=\sum_{j=1}^{n}(A+\lambda_{nj}B)\otimes|\psi_{nj}\rangle\langle\psi_{nj}|.$$

This shows that

$$\begin{aligned}\frac{1}{n}S(\Sigma_n) &= \frac{1}{n}\sum_{j=1}^{n} S(A+\lambda_{nj}B)\\ &= \int S(A+sB)\,\mathrm{d}\mu_n(s),\end{aligned}$$

where μ_n is the probability measure defined by

$$\mu_n = \frac{1}{n}\sum_{j=1}^{n}\delta_{\lambda_{nj}}.$$

By Kac-Murdock-Szegö theorem μ_n converges weakly as $n \to \infty$ to the distribution Lh^{-1} where L denotes the Lebesgue measure in $[0,1]$ and

$$h(s) = 2\sum_{j=1}^{\infty} p_j \cos 2\pi j s.$$

Note that $\|T_n(\boldsymbol{p})\| \leq 2$ and the eigenvalues λ_{nj} lie in the interval $[-2,2]$. Furthermore, the symplectic spectrum of $A+sB$ is a continuous function of s and hence the entropy $S(A+sB)$ is a continuous function of s in $[-2,2]$. Thus

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}S(\Sigma_n) &= \int_{-2}^{2} S(A+sB)Lh^{-1}(\mathrm{d}s)\\ &= \int_{0}^{1} S(A+h(s)B)\,\mathrm{d}s.\end{aligned}$$

□

5. Entanglement in a stationary G-chain

We have already noted in Corollary 2.1 that in any exchangeable G-chain $\Sigma(A,B)$ of order k all the underlying Gaussian states $\rho(I)$ with $\#I \geq 2$ are separable. In the class of examples of stationary G-chains in Corollary 3.1 it is natural to examine the presence of entanglement. It would be particularly interesting to construct an example of a stationary G-chain for which the underlying Gaussian state $\rho(I \cup I+n)$ with marginals $\rho(I)$ and $\rho(I+n) = \rho(I)$ is entangled for arbitrarily large n. That would imply the preservation of entanglement after an arbitrarily large lapse of time. For such problems we do not have any answer. However, we shall examine the special case of a stationary G-chain of order 1 where the matrices A, B_1, B_2, $\cdots$ in Corollary 3.1 are given by

$$\begin{aligned}A &= \lambda I_2\\ B_j &= B = b\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} \quad j=1,2,\cdots,\end{aligned}$$

where λ and b are positive scalars with $\lambda^2 > \frac{1}{4}$. We start with two elementary lemmas. Let

$$\Sigma = \begin{bmatrix} \lambda I_2 & p_1 B & p_2 B & \cdots & \cdots \\ p_1 B & \lambda I_2 & p_1 B & p_2 B & \cdots \\ p_2 B & p_1 B & \lambda I_2 & p_1 B & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix} \tag{5.1}$$

where $p_1, p_2, \cdots$ is a probability distribution over $\{1, 2, \cdots\}$ and λ, B as above.

Lemma 5.1. *The infinite block matrix Σ in (5.1) is a stationary G-chain of order one if $b < \frac{1}{2}\left(\lambda^2 - \frac{1}{4}\right)^{\frac{1}{2}}$.*

Proof. By Corollary 3.1, Σ is a stationary G-chain of order one if $\lambda I_2 + tB$ is a G-matrix for all $t \in [-2, 2]$. This is fulfilled if the matrix inequalities

$$\begin{bmatrix} \lambda + tb & \frac{\imath}{2} \\ -\frac{\imath}{2} & \lambda - tb \end{bmatrix} > 0, \quad t \in [-2, 2]$$

hold. Clearly this is satisfied if $b < \frac{1}{2}\left(\lambda^2 - \frac{1}{4}\right)^{\frac{1}{2}}$. □

Lemma 5.2. *Let $\lambda > \frac{1}{2}$, $c > 0$. Then the matrix*

$$\Gamma = \begin{bmatrix} \lambda & 0 & c & 0 \\ 0 & \lambda & 0 & -c \\ c & 0 & \lambda & 0 \\ 0 & -c & 0 & \lambda \end{bmatrix}$$

is the covariance matrix of an entangled 2-mode Gaussian state if

$$\lambda - \frac{1}{2} < c < \left(\lambda^2 - \frac{1}{4}\right)^{\frac{1}{2}}.$$

Proof. By Simon's criterion [Sim00], Γ has the required property if the following two matrix inequalities hold:

$$\begin{bmatrix} \lambda & \frac{\imath}{2} \\ -\frac{\imath}{2} & \lambda \end{bmatrix} - c^2 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \lambda & \frac{\imath}{2} \\ -\frac{\imath}{2} & \lambda \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad > \quad 0, \tag{5.2}$$

$$\begin{bmatrix} \lambda & \frac{\imath}{2} \\ -\frac{\imath}{2} & \lambda \end{bmatrix} - c^2 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \lambda & -\frac{\imath}{2} \\ \frac{\imath}{2} & \lambda \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \ngeq \quad 0. \tag{5.3}$$

Simple algebra shows that (5.2) holds whenever $c^2 < \lambda^2 - \frac{1}{4}$. Inequality (5.3) reduces to

$$\begin{bmatrix} (1 - d^2)\lambda & \frac{\imath}{2}(1 + d^2) \\ -\frac{\imath}{2}(1 + d^2) & (1 - d^2)\lambda \end{bmatrix} \ngeq 0 \tag{5.4}$$

where

$$d = \frac{c}{\sqrt{\lambda^2 - \frac{1}{4}}}. \tag{5.5}$$

W now choose $d < 1$ such that the determinant of the matrix on the left hand side of (5.4) is strictly negative. This finally leads to the inequality $c > \lambda - \frac{1}{2}$. Thus the inequality $\lambda - \frac{1}{2} < c < \sqrt{\lambda^2 - \frac{1}{4}}$ is sufficient to ensure entanglement. □

Proposition 5.1. *Let $\frac{1}{2} < \lambda < \frac{5}{6}$, $\lambda - \frac{1}{2} < b < \frac{1}{2}\sqrt{\lambda^2 - \frac{1}{4}}$. Suppose $p_j b > \lambda - \frac{1}{2}$ for some j. Then the 2-mode Gaussian state $\rho(\{1, j\})$ determined by the stationary G-chain Σ defined by (5.1) is entangled.*

Proof. Since $\frac{1}{2} < \lambda < \frac{5}{6}$, the interval $\left(\lambda - \frac{1}{2}, \frac{1}{2}\left(\lambda^2 - \frac{1}{4}\right)^{\frac{1}{2}}\right)$ is open and nonempty. Thus it is possible to choose b in this interval. By Lemma 5.1 the block matrix Σ in (5.1) is a stationary G-chain of order one. The covariance matrix of the 2-mode state $\rho(\{1, j\})$ defined by the G-chain Σ has covariance matrix

$$\begin{bmatrix} \lambda & 0 & p_j b & 0 \\ 0 & \lambda & 0 & -p_j b \\ p_j b & 0 & \lambda & 0 \\ 0 & -p_j b & 0 & \lambda \end{bmatrix}$$

where by hypothesis

$$\lambda - \frac{1}{2} < p_j b < \left(\lambda^2 - \frac{1}{4}\right)^{\frac{1}{2}}.$$

By Lemma 5.2 it follows that $\rho(\{1, j\})$ is entangled. □

Remark 5.1. It follows from Proposition 5.1 that for $\frac{1}{2} < \lambda < \frac{5}{6}$, $\lambda - \frac{1}{2} < b < \frac{1}{2}\left(\lambda^2 - \frac{1}{4}\right)^{\frac{1}{2}}$ the matrix

$$\Sigma = \begin{bmatrix} \lambda & 0 & b & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdots\cdots \\ 0 & \lambda & 0 & -b & 0 & 0 & \cdot & \cdot & \cdot & \cdots\cdots \\ b & 0 & \lambda & 0 & b & 0 & 0 & 0 & \cdot & \cdots\cdots \\ 0 & -b & 0 & \lambda & 0 & -b & 0 & 0 & \cdot & \cdots\cdots \\ 0 & 0 & b & 0 & \lambda & 0 & b & 0 & \cdot & \cdots\cdots \\ 0 & 0 & 0 & -b & 0 & \lambda & 0 & -b & \cdot & \cdots\cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \cdots\cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots\cdots \end{bmatrix}$$

(which is 2×2 block tridiagonal) is a stationary G-chain in which $\rho(\{1, 2\}) = \rho(\{n, n+1\})$, $n = 2, 3, \cdots$ is entangled. In this example $\rho(\{n, n + j\})$, $j \geq 2$ is a product state with covariance matrix

$$\begin{bmatrix} \lambda I_2 & 0 \\ 0 & \lambda I_2 \end{bmatrix}.$$

6. Conclusion

An exchangeable chain of mean zero finite mode Gaussian states is completely determined by two matrices A, B such that $A - B$ is the covariance matrix of a Gaussian state and B is a nonnegative definite matrix. Its asymptotic entropy rate is equal to $S(\rho^{A-B})$, the von Neumann entropy of the mean zero Gaussian state ρ^{A-B} with covariance matrix $A - B$.

A class of examples of stationary Gaussian chains is constructed and their asymptotic entropy rates are evaluated. An example of a stationary entangled chain is presented.

References

[ADMS95] Arvind, B Dutta, N Mukunda, and R Simon. The real symplectic groups in quantum mechanics and optics. *Pramana*, 45:471–497, 1995.

[CT06] Thomas M. Cover and Joy A. Thomas. *Elements of information theory*. Wiley-Interscience [John Wiley & Sons], Hoboken, NJ, second edition, 2006.

[de 29] B. de Finetti. Sulle funzioni a incremento aleatorio. *Atti Accad. Naz. Lincei, Rend., VI. Ser.*, 10:163–168, 1929.

[Doo53] J. L. Doob. *Stochastic processes*. John Wiley & Sons, Inc., New York; Chapman & Hall, Limited, London, 1953.

[Hol11] Alexander Holevo. *Probabilistic and statistical aspects of quantum theory*, volume 1 of *Quaderni. Monographs*. Edizioni della Normale, Pisa, second edition, 2011. With a foreword from the second Russian edition by K. A. Valiev.

[Hol12] Alexander S. Holevo. *Quantum systems, channels, information. A mathematical introduction.* Berlin: de Gruyter, 2012.

[KMS53] M. Kac, W. L. Murdock, and G. Szegö. On the eigenvalues of certain Hermitian forms. *J. Rational Mech. Anal.*, 2:767–800, 1953.

[LR73] Elliott H. Lieb and Mary Beth Ruskai. Proof of the strong subadditivity of quantum-mechanical entropy. *J. Mathematical Phys.*, 14:1938–1941, 1973. With an appendix by B. Simon.

[MRS15] Marek Mozrzymas, Adam Rutkowski, and Michał Studziński. Separable Decomposition and Quantum Correlations in Toeplitz Matrices. *ArXiv e-prints*, May 2015, 1505.06422.

[OP93] Masanori Ohya and Dénes Petz. *Quantum entropy and its use*. Texts and Monographs in Physics. Springer-Verlag, Berlin, 1993.

[Par10] K. R. Parthasarathy. What is a Gaussian state? *Commun. Stoch. Anal.*, 4(2):143–160, 2010.

[Par13] Kalyanapuram R. Parthasarathy. The symmetry group of Gaussian states in $L^2(\mathbb{R}^n)$. In Albert N. Shiryaev, S. R. S. Varadhan, and Ernst L. Presman, editors, *Prokhorov and contemporary probability theory. In honour of Yuri V. Prokhorov on the occasion of his 80th birthday*, volume 33 of *Springer Proceedings in Mathematics & Statistics*, pages 349–369. Berlin: Springer, 2013.

[PS15] K. R. Parthasarathy and Ritabrata Sengupta. From particle counting to Gaussian tomography. *ArXiv e-prints*, April 2015, 1504.07054.

[Sim00] R. Simon. Peres-Horodecki separability criterion for continuous variable systems. *Phys. Rev. Lett.*, 84:2726–2729, Mar 2000.

[Tro84] Hale F. Trotter. Eigenvalue distributions of large Hermitian matrices; Wigner's semicircle law and a theorem of Kac, Murdock, and Szegő. *Adv. in Math.*, 54(1):67–82, 1984.

[WPGP+12] Christian Weedbrook, Stefano Pirandola, Raúl García-Patrón, Nicolas J. Cerf, Timothy C. Ralph, Jeffrey H. Shapiro, and Seth Lloyd. Gaussian quantum information. *Rev. Mod. Phys.*, 84:621–669, May 2012.

[WW01] R. F. Werner and M. M. Wolf. Bound entangled gaussian states. *Phys. Rev. Lett.*, 86:3658–3661, Apr 2001.

Theoretical Statistics and Mathematics Unit, Indian Statistical Institute, Delhi Centre, 7 S J S Sansanwal Marg, New Delhi 110 016, India

E-mail address, K R Parthasarathy: `krp@isid.ac.in`

Theoretical Statistics and Mathematics Unit, Indian Statistical Institute, Delhi Centre, 7 S J S Sansanwal Marg, New Delhi 110 016, India

E-mail address, Ritabrata Sengupta: `rb@isid.ac.in`