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On the Fundamental Theorem of Asset Pricing

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Dedicated to the memory of G Kallianpur

Abstract

It is well known that existence of equivalent martingale measure (EMM) is *essentially* equivalent to absence of arbitrage. In this paper, we give an overview of this connection and also include work that we had done with Professor Kallianpur, which was not published as an article, but is included in the book by Kallianpur on Option pricing. This is the concept of No Approximate Arbitrage with Controlled Risk - NAACR which turns out to be equivalent to the existence of equivalent martingale measure. This seems to be the only result characterizing EMM in terms of simple strategies. Moreover, the proof of this assertion is purely functional analytic, without invoking semimartingales and stochastic integration.

1 Introduction

An important result in Mathematical finance - often called the *fundamental theorem of asset pricing* - states that existence of an equivalent (local) martingale measure is *essentially* equivalent to abscence of arbitrage opportunities. This result is the basis of the theory of *pricing by arbitrage*. While it is easy to prove that the existence of an equivalent (local) martingale measure - written as EMM - rules out existence of arbitrage opportunities in the class of admissible integrands, it is well known the converse is not true. One must rule out approximate arbitrage opportunities (suitably defined) to characterize EMM property.

This question has been discussed in discrete time as well as in continuous time, over finite as well as infinite horizon and for finitely many stock prices or commodities as well as infinite collection of commodities. See [9], [5], [3], and references therein. Various notions of ruling out approximate arbitrage have been proposed. In most papers, the approach is via a separation theorem (due to Kreps-Yan).

The Kreps - Yan theorem ([9], [14], [13]) says that existence of EMM is equivalent to a property called No Free Lunch (NFL) - which defines approximate arbitrage in terms of limits of nets in weak* topology. This was not considered suitable as it does not lend itself to an economic interpretation. Moreover, it allows the strategy to use positions that are very risky (see [3]).

In [2] Approximate arbitrage was defined as a position that can be approximated by any investor keeping his/her risks as low as possibe. A characterisation of existence of EMM was given in terms of abscence of arbitrage in the sense described in previous sentence. This manuscript was unpublished, but the material was essentially included in the book [7]. We will describe this work with details. This work relies on Orlicz spaces and we describe the basic properties of these spaces. Also, an interesting result on weak* closure of a convex set in $\boldsymbol{L}^\infty$ is included, which is of independent interest. We have also included a proof of the Kreps-Yan theorem.

We first describe the class of claims attainable by simple strategies over finite time horizon. In a sense, this class is the natural starting point as these are the strategies that can be implemented in practice. Also, to define this class, we do not need to assume that the stock price process is a semimartingale. Of course, once we show that there exists EMM, it follows that the stock price process is a semimartingale.

Throughout the article, we fix a probability space $(\Omega, \mathcal{F}, P)$ and we assume that $\mathcal{F}$ contains all P - null sets. All random variables considered are defined on this space and $\boldsymbol{L}^p$ referes to $\boldsymbol{L}^p(\Omega, \mathcal{F}, \mathbb{P})$.

2 Trading strategy and arbitrage opportunity

Consider a market consisting of d-stocks with stock prices at time t being given by $S_t^1, \ldots, S_t^d$. We assume that all processes are defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We assume that there is a riskless asset, bond,whose price is S_t^0. (Typically $S_t^0 = \exp(rt)$ or $S_t^0 = \exp(\int_0^t r_u du)$). Let

$$\tilde{S}_t^i = S_t^i / S_t^0$$

be the discounted price process.

Let $(\mathcal{F}_t)$ be the filtration generated by $(S_s^0, S_s^1, \ldots, S_s^d) : 0 \le s \le t$, namely $\mathcal{F}_t$ is the smallest σ-field with respect to which the random variables $(S_s^0, S_s^1, \ldots, S_s^d) : 0 \le s \le t$ are measurable.

We assume that for each i, $\tilde{S}_t^i$ is locally bounded, *i.e.* there exists a sequence $\{\tau_k\}$ of $(\mathcal{F}_t)$-stopping times, τ_k increasing to ∞, such that

$$|\tilde{S}_{t \wedge \tau_k}^i| \le c_{k,i} \quad t \ge 0 \tag{2.1}$$

for some constants $c_{k,i} < \infty$. By replacing τ_k by $\tau_k \wedge k$ if necessary, we assume that τ_k are bounded stopping times.

We consider invetment stratagies that involve infusion of capital at time zero but at subsequent times, no fresh investment is made nor is any money taken out for consumption. Thus, at subsequent times, money is moved from one asset to another at prevailing market prices.

We are considering a frictionless market- where transaction costs are zero and short selling is allowed. Short selling means a promise to sell something that you do not have. Short selling a bond tantamounts to taking a loan. Thus, it is implicit that deposits (buying bonds) or loans (short selling bonds) have the same rate of interest. This is an ideal market and practioners make adjustments for deviations from the same.

A simple investement strategy is a process $\pi_t = (\pi_t^1, \pi_t^2, \dots \pi_t^d)$ where

$$\pi_t^i = \sum_{j=0}^{m-1} a_j^i I_{(\sigma_j, \sigma_{j+1}]}(t),\ 1 \le i \le d \tag{2.2}$$

where a_j^i are $\mathcal{F}_{\sigma_j}$ measurable bounded random variables, $\sigma_0 \le \sigma_1 \le \dots \le \sigma_m$ are $(\mathcal{F}_t)$- stopping times with $\sigma_m \le \tau_k$ for some k, τ_k as in (2.1). Thus, for $1 \le i \le d$, π^i is constant over each of the intervals $(\sigma_0, \sigma_1]$, $(\sigma_1, \sigma_2]$, $\dots$, $(\sigma_{m-1}, \sigma_m]$. π_t^i is the number of shares of the i^{th} stock the investor will hold at time t. The adjective simple refers to the restriction that the investor changes his/her holdings only finitely many times. Since he/she cannot be allowed to forsee the future, his decision must be based only on information available to him/her at that instant. Hence $\{\sigma_j : 0 \le j \le m\}$ above should be stopping times and a_j^i is required to be $\mathcal{F}_{\sigma_j}$ measurable.

Since we are considering discounted prices, buying or selling bonds do not change value of an investors' holdings. The change in value is entirely due to the fluctuations in the price of stocks. Thus it can be seen that the discounted value process (with zero initial investment) for the simple strategy (π_t) given by (2.2) is

$$\tilde{V}_t(\pi) = \sum_{i=1}^{d} \sum_{j=0}^{m-1} a_j^i (\tilde{S}^i_{\sigma_{j+1} \wedge t} - \tilde{S}^i_{\sigma_j \wedge t}). \tag{2.3}$$

Note that in view of our assumptions, $\tilde{V}_t(\pi)$ is bounded for all simple investment strategies. Let

$$\boldsymbol{K}_s = \{\tilde{V}_t(\pi) : \pi \text{ is a simple strategy}\}.$$

$\boldsymbol{K}_s$ is the class of all (discounted) *positions attainable via simple strategies over finite horizons.* The subscript s reflects that we are considering simple strategies.

Remark : While here we have assumed that $t \in [0, \infty)$, we can consider $t \in [0, T]$ by requiring that $S_t^i = S_{t \wedge T}^i$ for all t, $i = 0, 1, 2, \ldots d$. Likewise, we can consider discrete time model with prices changing only at integer times by requiring that

$$S_t^i = S_{[t]}^i,\ 0 \leq t < \infty,\ \ i = 0, 1, \ldots, d.$$

Similarly, if we are considering a market consisting of infinitely many stocks S_t^α, $\alpha \in \Delta$, where Δ is an arbitrary index set, we can take $\mathcal{F}_t$ to be the filtration generated by $\{S_u^\alpha,\ \alpha \in \Delta,\ 0 \leq u \leq t\}$ and then define simple investment strategy π^α for the αth stock. The class $\boldsymbol{K}_s^\alpha$ of attainable claims via simple investment strategies over finite horizons on the stock S^α can be defined as above. i.e.

$$\boldsymbol{K}_s^\alpha = \{\tilde{V}_t(\pi^\alpha) : \pi^\alpha \text{ is a simple strategy}\}.$$

The class

$$\boldsymbol{K}_s = \text{ linear span } \{\cup_{\alpha \in \Delta} \boldsymbol{K}_s^\alpha\}$$

then represents positions attainable via simple strategies over infinitely many stocks over finite horizon. The discussion that follows depends on $\boldsymbol{K}_s$ alone and the underlying number of stocks plays no role.

A position $Z \in \boldsymbol{K}_s$ is said to be an arbitrage opportunity if $P(Z \geq 0) = 1$ and $P(Z > 0) > 0$. If such a position is attainable via a strategy π, then all investors would love to follow the strategy π and without any chance of losing money (risk), aim to make money. Such a behaviour would disturb the equilibrium, pushing up price of whatever this strategy requires to be bought. Thus one rules out existence of such positions. Formally, one imposes the following condition on a market in equilibrium:

Definition **2.1** $\boldsymbol{K}_s$ (or $\tilde{S}$) is said to satisfy the condition of *No Arbitrage* (written as NA) if

$$\boldsymbol{K}_s \cap L_+^\infty = \{0\}. \tag{2.4}$$

Here, for $1 \leq p \leq \infty$ $L_+^p = \{Z \in L^p, \mathbb{P}(Z \geq 0) = 1\}$. It has been found that it is useful to introduce positions that can be improved upon by a strategy. With this idea let us introduce

$$\boldsymbol{C}_s = \{W :\ \exists Z \in \boldsymbol{K}_s \text{ such that } W \leq Z\}.$$

It is easy to see that (2.4) implies

$$\boldsymbol{C}_s \cap L^\infty_+ = \{0\}. \tag{2.5}$$

3 Equivalent Martingale Measures

We will first explore a sufficient condition for NA.

Definition **3.1** A probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ is said to be an equivalent martingale measure (EMM)- (for $\tilde{S}$) if $\mathbb{Q} \equiv \mathbb{P}$ ($i.e.$ $\mathbb{Q} << \mathbb{P}$ and $\mathbb{P} << \mathbb{Q}$) and $\tilde{S}^i_t$ is a local martingale on $(\Omega, \mathcal{F}, \mathbb{Q})$ for $1 \le i \le d$.

Here is a simple observation.

Lemma **3.2** *Let $\mathbb{Q}$ be given by $\frac{d\mathbb{Q}}{d\mathbb{P}} = f$ with $\mathbb{P}(f > 0) = 1, f \in L^1(P)$. Then $\mathbb{Q}$ is an EMM for $\tilde{S}$ if and only if*

$$\mathbb{E}_{\mathbb{Q}}[W] \le 0 \ \ \forall W \in \boldsymbol{C}_s. \tag{3.1}$$

As a consequence, if an EMM $\mathbb{Q}$ exists, then NA holds.

Proof : If $\mathbb{Q}$ is an EMM, then for a simple startegy π, $\tilde{V}_t(\pi)$ is a local martingale on $(\Omega, \mathcal{F}, \mathbb{Q})$ and hence a martingale as it is bounded. In particular,

$$\mathbb{E}_{\mathbb{Q}}(\tilde{V}_t(\pi)) = \mathbb{E}_{\mathbb{Q}}(\tilde{V}_0(\pi)) = 0.$$

Thus

$$\mathbb{E}_{\mathbb{Q}}(Z) = 0 \quad \forall Z \in \boldsymbol{K}_s$$

and as a result, $\mathbb{E}_{\mathbb{Q}}(W) \le 0 \quad \forall W \in \boldsymbol{C}_s$.

Conversely, suppose (3.1) is satisfied. Since $Z \in \boldsymbol{K}_s$ implies $Z \in \boldsymbol{C}_s$ and also $-Z \in \boldsymbol{K}_s \subseteq \boldsymbol{C}_s$, (3.1) implies

$$\mathbb{E}_{\mathbb{Q}}[Z] = 0 \ \ \forall Z \in \boldsymbol{K}_s. \tag{3.2}$$

Fix $1 \le i \le d$ and a stopping time σ_1. We will show that

$$\mathbb{E}_{\mathbb{Q}}[\tilde{S}^i_{\sigma_1 \wedge \tau_k}] = \mathbb{E}_{\mathbb{Q}}[\tilde{S}^i_0] \tag{3.3}$$

where τ_k are as in (2.1). This will imply that

$$\tilde{S}^i_{t\wedge\tau_k} \text{ is a martingale for all } k$$

and hence that $\tilde{S}^i$ is a local martingale. This will complete the proof of the first part.

It remains to prove (3.3). Fix integers $k \geq 1$ and $1 \leq i \leq d$. Let $\sigma = \sigma_1 \wedge \tau_k$, $a^i = 1$ and $a^j = 0$ for $j \neq i, 1 \leq j \leq d$ and define

$$\pi^l(t) = a^l 1_{(0,\sigma]}(t), \;\; 1 \leq l \leq d.$$

Let $\pi_t = (\pi^1_t, \pi^2_t, \ldots \pi^d_t)$ be the corresponding investment strategy. Then for t such that $\tau_k \leq t$ (such a t exists as τ_k is bounded),

$$\tilde{V}_t(\pi) = \tilde{S}^i_\sigma - \tilde{S}^i_0.$$

Since $\tilde{V}_t(\pi) \in \boldsymbol{K}_s$, (3.2) implies (3.3).

Now suppose EMM Q exists. Let $Z \in \boldsymbol{K}_s$ be such that $P(Z \geq 0) = 1$. Then we have $Q(Z \geq 0) = 1$ and then $\mathbb{E}_{\mathbb{Q}}[Z] = 0$ implies $Q(Z = 0) = 1$ and as a conseqeunce, $P(Z = 0) = 1$. Thus NA holds. ∎

However, it is well known that the converse to the last part of the above lemma is not true. i.e. NA does not imply the existence of an EMM. (See [5], [3] and references therein). Here is one example of the well known phenomenon.

Example 1. Let $\Omega = \{-1, 1\}^N$ and let ξ_i be the coordinate mappings on Ω. Let P be the probability measure on Ω such that ξ_i's are independent and for $n \geq 1$, $P(\xi_n = 1) = \frac{1}{2} + \frac{1}{2\sqrt{n+1}}, P(\xi_n = -1) = \frac{1}{2} - \frac{1}{2\sqrt{n+1}}$.

For $n \geq 1$, let $S^0_n = (1+r)^n$ where r is the rate of interest. For $t \in \mathbb{R}$ let $S^0_t = S^0_{[t]}$. Let the stock price process S^1_t be given by $S^1_0 = 1$ and

$$S^1_t = \prod_{i=1}^{[t]} \left(1 + \frac{1}{2}\xi_i\right) S^0_t$$

The discounted stock price $\tilde{S}^1_t$ is then given by

$$\tilde{S}^1_t = \prod_{i=1}^{[t]} \left(1 + \frac{1}{2}\xi_i\right).$$

Let $\mathcal{F} = \sigma(\xi_i : i \geq 1)$ and $\mathcal{F}_n = \sigma(\xi_i : 1 \leq i \leq n)$. It is easy to see that on $(\Omega, \mathcal{F})$ there is a unique probability measure Q under which $(\tilde{S}^1_n, \mathcal{F}_n)$ is a martingale: it is the one under which ξ_i's are independent with $Q(\xi_i = 1) = Q(\xi_i = -1) = \frac{1}{2}$.

Let P^n, Q^n be restrictions of P, Q on $\mathcal{F}_n$. Then P^n and Q^n are equivalent. Further, if $g_n = \frac{dP^n}{dQ^n}$, then

$$g_n = \prod_{i=1}^{n} \left(1 + \frac{\xi_i}{\sqrt{i+1}}\right).$$

Moreover,

$$\begin{aligned}\int \sqrt{g_n} dQ^n =& \frac{1}{2} \prod_{i=1}^{n} \left[\sqrt{\left(1 + \frac{1}{\sqrt{i+1}}\right)} + \sqrt{\left(1 - \frac{1}{\sqrt{i+1}}\right)}\right] \\ \approx& \frac{1}{2} \prod_{i=1}^{n} \left(1 + \frac{c}{i+1}\right) \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Kakutani's theorem now implies that Q is orthogonal to P.

It follows that there is no probability measure equivalent to P under which $\tilde{S}^1_n$ is a martingale. This also implies that there is no probability measure equivalent to P under which $\tilde{S}^1_n$ is a local martingale. For if $\tilde{S}^1_n$ is a local martingale, boundedness of $\tilde{S}^1_n$ will imply that $\tilde{S}^1_n$ is actually a martingale. Hence EMM property does not hold for $\tilde{S}_t$ on $(\Omega, \mathcal{F}, P)$.

However, we will see that NA does hold. For this note that every $Z \in \boldsymbol{K}_s$ is $\mathcal{F}_m$ measurable for some m. Hence $\boldsymbol{K}_s \cap L^\infty_+ = \{0\}$. Indeed if $W \in \boldsymbol{K}_s \cap L^\infty_+(P)$, then W is $\mathcal{F}_m$ measurable for some m and hence $W \in \boldsymbol{K}_s \cap L^\infty_+(Q)$. But under Q, $\tilde{S}^1_t$ is a martingale and thus $\mathbb{E}_Q(W) = 0$, so that $Q(W = 0) = 1$. Finally this implies that $P(W = 0) = 1$.

So NA holds but EMM does not hold.

Let $g_0 = 1$. For $n \geq 1$ it can be verified that $\tilde{S}^1_{n+1} - \tilde{S}^1_n = \tilde{S}^1_n \frac{\xi_{n+1}}{2}$ and that

$$g_n - 1 = \sum_{m=0}^{n-1} \frac{2g_m}{\sqrt{m+1}\tilde{S}^1_m} (\tilde{S}^1_{m+1} - \tilde{S}^1_m).$$

Hence $Z_n = g_n - 1 \in \boldsymbol{K}_s$. Since $Q \perp P$, $g_n \to \infty$ a.s. P, and we get

$$P(Z_n \geq 1) \to 1.$$

Thus though there is no arbitrage opportunity in the class $\boldsymbol{K}_s$ of attainable claims, there is a sequence $\{Z_n\} \subset \boldsymbol{K}_s$ such that $P(Z_n \geq 1) \to 1$.

The example discussed above suggests that in order to have an equivalent (local) martingale measure one should rule out existence of sequences $\{Z_n\} \subset \boldsymbol{K}_s$ such that $P(Z_n \geq Z) \to 1$, $Z \in L^\infty_+$ and $P(Z = 0) < 1$. Let us tentatively call such sequences $\{Z_n\}$ as approximate arbitrage opportunities. However, existence of an equivalent martingale measure does not rule out approximate arbitrage opportunities as the following example shows.

Example 2. In the setup of Example 1, consider the stock prices $\tilde{S}^1_t$ on the probability space $(\Omega, \mathcal{F}, Q)$. Since $\tilde{S}^1_t$ is a Q martingale, the EMM property trivially holds. Let $f_0 = 1$ and

$$f_n = \prod_{i=1}^{n} (1 + \xi_i).$$

Then $f_m - f_{m-1} = 2 f_{m-1} \frac{(\tilde{S}^1_m - \tilde{S}^1_{m-1})}{\tilde{S}^1_{m-1}}$ and hence $W_n = 1 - f_n$ can be written as

$$\begin{aligned} W_n &= -\sum_{i=1}^{n} (f_i - f_{i-1}) \\ &= -\sum_{i=1}^{n} -\frac{2 f_{i-1}}{\tilde{S}^1_{i-1}} (\tilde{S}^1_i - \tilde{S}^1_{i-1}) \end{aligned}$$

and hence $W_n \in \boldsymbol{K}_s$. Note that $P(f_n = 2^n) = 2^{-n}$ and $P(f_n = 0) = 1 - 2^{-n}$. Thus $W_n \to 1$ a.s. $[P]$. This implies that $\{W_n\}$ is an approximate arbitrage opportunity since $P(W_n \geq 1 - \varepsilon) \to 1$ for every $\varepsilon > 0$.

In the first example, let us note that $Z_n \geq -1$, *i.e.* the risk associated with the approximate arbitrage opportunity $\{Z_n\}$ (namely Z_n^-) is bounded by 1. In the second example, Z_n^- is not bounded. Indeed $P(Z_n^- = 2^n - 1) = 2^{-n}$ for all n.

These comments suggest that to characterize the EMM property, one should rule out those approximate arbitrage opportunities for which the associated risks are controlled (in some appropriate sense). The next section shows us the way.

4 The Kreps - Yan Separation Theorem

Let $\boldsymbol{E} \subset \boldsymbol{L}^\infty$ be a linear subspace and let $\boldsymbol{D} = \{W : W = Z - Y, Z \in \boldsymbol{E}, Y \in \boldsymbol{L}_+^\infty\}$. Alternatively,

$$\boldsymbol{D} = \boldsymbol{E} - \boldsymbol{L}_+^\infty.$$

Let $\boldsymbol{D}^*$ be the closure of $\boldsymbol{D}$ in the weak* topology on $\boldsymbol{L}^\infty$ (*i.e.* $\sigma(\boldsymbol{L}^\infty, \boldsymbol{L}^1)$ *topology*). Note that $\boldsymbol{D}^*$ is a convex cone closed in the weak* topology. The following result is due to Kreps. This version is more general than the original version, and is due to Stricker [13] using results of Yan [14].

THEOREM **4.1** *The following are equivalent*

(i) $\exists f \in \boldsymbol{L}_+^1$, $P(f > 0) = 1$, *such that*

$$\int Zf \, d\mathbb{P} \leq 0 \ \ \forall Z \in \boldsymbol{D}. \tag{4.1}$$

(ii)

$$\boldsymbol{D}^* \cap \boldsymbol{L}_+^\infty = \{0\}. \tag{4.2}$$

PROOF : Suppose (i) is true. Then using the fact that $f \in \boldsymbol{L}^1 = (\boldsymbol{L}^\infty)^*$ we get

$$\int Zf \, d\mathbb{P} \leq 0 \ \ \forall Z \in \boldsymbol{D}^*. \tag{4.3}$$

Thus, if $W \in \boldsymbol{D}^* \cap \boldsymbol{L}_+^\infty$ then (4.3) implies $W = 0$.

For the other part, assume (4.2) is true. Then given $A \in \mathcal{F}$ with $\mathbb{P}(A) > 0$, consider $X = \mathbf{1}_A$. Then the closed convex set $\boldsymbol{D}^*$ and the compact set $\{X\}$ are disjoint and hence by Hahn-Banach Theorem, there exists $g^A \in \boldsymbol{L}^1$ and α such that

$$\int Zg^A d\mathbb{P} \leq \alpha \ \ \forall Z \in \boldsymbol{D}^* \quad \text{and} \quad \int \mathbf{1}_A g^A d\mathbb{P} > \alpha. \tag{4.4}$$

Since $0 \in \boldsymbol{E} \subseteq \boldsymbol{D}$, we have $0 \leq \alpha$. In view of (4.4), $\alpha < \infty$. Since $\boldsymbol{D}$ is a cone α can be chosen to be 0. i.e.

$$\int Wg^A \, d\mathbb{P} \leq 0 \ \ \forall W \in \boldsymbol{D}^* \tag{4.5}$$

and

$$\int \mathbf{1}_A g^A d\mathbb{P} > 0. \tag{4.6}$$

Moreover, since $-\mathbf{1}_B \in \boldsymbol{D}$ for all $B \in \mathcal{F}$, it follows that

$$\int \mathbf{1}_B g^A d\mathbb{P} \geq 0 \tag{4.7}$$

and hence that $\mathbb{P}(g^A \geq 0) = 1$ or $g^A \in \boldsymbol{L}^1_+$.

Let $\boldsymbol{U}$ be the class of all $f \in \boldsymbol{L}^1_+$ such that

$$\int Wf\, d\mathbb{P} \leq 0 \ \ \forall W \in \boldsymbol{D}^*$$

and let $\beta = \sup\{\mathbb{P}(f > 0) : \ f \in \boldsymbol{U}\}$. From the discussion above, it follows that $\beta > 0$. We first note that this supremum is attained. Let f_n be a sequence of functions in $\boldsymbol{U} \subseteq \boldsymbol{L}^1_+$ such that $P(f_n > 0) \to \beta$. Then, let

$$f = \sum_{n=1}^{\infty} \frac{1}{2^n(1 + a_n)} f_n$$

where $a_n = \int f_n d\mathbb{P}$. It follows that $f \in \boldsymbol{U}$ and $\mathbb{P}(f > 0) = \beta$. If $\beta < 1$, then take $A = \{f > 0\}^c$ and then obtain $g^A \in \boldsymbol{L}^1_+$ such that (4.5) and (4.6) are true. Then $f + g^A \in \boldsymbol{U}$ and $\mathbb{P}(f + g^A > 0) > \mathbb{P}(f > 0) = \beta$. This contradicts definition of β. Hence $\beta = 1$. Thus we have got $f \in U$ with $\mathbb{P}(f > 0) = 1$. ∎

The role played by requiring (4.2) as opposed to requiring

$$\boldsymbol{E}^* \cap \boldsymbol{L}^\infty_+ = \{0\} \tag{4.8}$$

where $\boldsymbol{E}^*$ is the closure of $\boldsymbol{E}$ in the weak* topology should be noted here. If we have (4.8), once again, given $A \in \mathcal{F}$ such that $\mathbb{P}(A) > 0$, we can get g^A such that (4.5) and (4.6) holds but we can no longer assert that(4.7) is true and as a consequence, g^A may not belong to $\boldsymbol{L}^1_+$.

As to the reason for taking closure in the weak* topology- we could have taken closure with respect to the norm topology but then the linear functional that the Hahn Banach theorem would yield may not be in L^1- as the dual of $\boldsymbol{L}^\infty$ (with supremum norm) contains all finitely additive measures as well.

5 No Free Lunch

As a consequence of the separation Theorem, we have

THEOREM **5.1** *The following are equivalent*

(i) *There exists an EMM $\mathbb{Q}$ for $\tilde{S}$*

(ii)

$$\boldsymbol{C}_s^* \cap \boldsymbol{L}_+^\infty = \{0\}. \tag{5.1}$$

This result follows immediately from Theorem 4.1 and Lemma 3.2.

Kreps called the condition (5.1) as *No Free Lunch* abbriviated as NFL. It can also be called NAA- *No Approximate Arbitrage.* However, here the approximation being in the weak* topology, the approximate arbitrage is in terms of a net $\{f_\alpha\}$ of positions. If EMM does not exist then the Theorem 5.1 yieds a random variable $f_0 \in \boldsymbol{L}_+^\infty$ with $\mathbb{P}(f_0 > 0) > 0$ such that there exist nets $\{g_\alpha \in \boldsymbol{K}_s\}_{\alpha \in \varDelta}$ and $\{f_\alpha \in \boldsymbol{C}_s\}_{\alpha \in \varDelta}$ with $f_\alpha \leq g_\alpha$ $\forall \alpha \in \varDelta$ and

$$\int f_\alpha h d\mathbb{P} \rightarrow \int f_0 h d\mathbb{P} \;\; \forall h \in \boldsymbol{L}^1. \tag{5.2}$$

This f_0 is the approximate arbitrage opportunity. This definition of approximate arbitrage (or free lunch, as defined by Kreps) was considered unsuitable as convergence via nets is difficult to comprehend and moreover, the positions f_α in (5.2) could be highly risky positions, as no control is imposed on the same.

Thus efforts continued to get versions of the result which involved only sequences and where the definition of approximate arbitrage imposes a control on the associated risk.

Let $1 \leq p \leq \infty$. Let us say that f_0 is an $\boldsymbol{L}^p$-approximate arbitrage if $\mathbb{P}(f_0 > 0) > 0$ and there exist sequences $\{g_n \in \boldsymbol{K}_s\}$, $\{f_n \in \boldsymbol{C}_s\}$, $f_n \leq g_n$ and $\|f_n - f_0\|_p \rightarrow 0$. Here $\|\cdot\|_p$ is the $\boldsymbol{L}^p$ norm. $1 < p < \infty$ and q be such that $\frac{1}{p} + \frac{1}{q} = 1$ Then it was shown by Ansel and Stricker [1] that $\tilde{S}$ does not admit an $\boldsymbol{L}^p$-approximate arbitrage if and only if there exists an EMM $\mathbb{Q}$ for $\tilde{S}$ with $\frac{d\mathbb{Q}}{d\mathbb{P}} \in \boldsymbol{L}^q$.

This follows from a $\boldsymbol{L}^p$ version of the Kreps-Yan Theorem in [1]. Kusuoka [8] also obtained a version of this result in Orlicz space. These versions do not require consideration of nets because the closure is being considered in the norm of the function space. Nonetheless these results do not quite characterise existence of EMM, but only EMM with a suitable density.

One remarkable result in this direction is due to Delbaen and Schachermayer [3]: Let $\bar{\boldsymbol{C}}_s$ denote the closure of $\boldsymbol{C}_s$ in $\boldsymbol{L}^\infty$ norm. Consider the condition

$$\bar{\boldsymbol{C}}_s \cap \boldsymbol{L}_+^\infty = \{0\}. \tag{5.3}$$

Another way of stating the condition (5.3) is as follows. If for $f_0 \in \boldsymbol{L}_+^\infty$ there exist sequences $\{g_n \in \boldsymbol{K}_s\}$, $\{f_n \in \boldsymbol{C}_s\}$, $f_n \leq g_n$ and $\|f_n - f_0\|_\infty \to 0$, then $f_0 = 0$. This condition has been called NFLVR (*No Free Lunch With Vanishing Risk*) with simple strategies. Yet another (equivalent) formulation of NFLVR is : If for $f_0 \in \boldsymbol{L}_+^\infty$ there exist sequences $\{g_n \in \boldsymbol{K}_s\}$, $\|g_n^-\|_\infty \to 0$, $\mathbb{P}(g_n \geq f_0 - \frac{1}{n}) \to 0$, then $f_0 = 0$. The latter is the rationale for the name Vanishing Risk.

It was shown by Delbaen and Schachermayer [3] that (5.3) implies that $\tilde{S}$ is a semimartingale. In that case, one can consider general (predictable) trading strategies $\pi_t = (\pi_t^1, \pi_t^2, \ldots \pi_t^d)$. The discounted value process $\tilde{V}_t(\pi)$ (with zero initial investment) for the strategy π is then given by

$$\tilde{V}_t(\pi) = \sum_{i=1}^{d} \int_0^t \pi_u^i d\tilde{S}_u^i. \tag{5.4}$$

The trading strategy π is called an admisible strategy if for some constant $\gamma > 0$

$$\mathbb{P}(\tilde{V}_t(\pi) \geq -\gamma \ \ \forall t) = 1.$$

The constant γ is interpreted as credit limit of the investor. It is well known that once we go to general strategies, EMM does not rule out arbitrage opportunity. However, EMM does imply that arbitrage opportunity cannot exist in the class of admissible strategies. We now define the analogues of $\boldsymbol{C}_s, K_s$ in terms of admissible strategies as follows:

$$\boldsymbol{K} = \{\tilde{V}_t(\pi) : \pi \text{ is an admissible strategy}\}$$

and

$$\boldsymbol{C} = \{W : \exists Z \in \boldsymbol{K}, W \leq Z\}.$$

Let $\bar{\boldsymbol{C}}$ denote the closure of $\boldsymbol{C}$ in the $\boldsymbol{L}^\infty$ norm.

DEFINITION **5.2** The process $\tilde{S}$ is said to satisfy NFLVR (No Free Lunch With Vanishing Risk) if

$$\bar{\boldsymbol{C}} \cap \boldsymbol{L}_+^\infty = \{0\}. \tag{5.5}$$

Using deep results in Stochastic Calculus, Delbaen and Schachermayer [3] showed that if (5.5) holds, then $\bar{\boldsymbol{C}}$ is closed in the weak* topology and hence one can conclude that EMM exists invoking the Kreps-Yan theorem.

Delbaen and Schachermayer [4] have shown by an example that NFLVR in the class of simple strategies does not imply existence of EMM even if one assumes that the underlying process is continuous.

In [2], Bhatt, Kallianpur and Karandikar had given a notion of absence of arbitrage in terms of sequences such that the associated risks remain bounded. It was also shown that absence of approximate arbitrage in this sense is equivalent to existence of EMM. This seems to be the only characterization of EMM in terms of simple strategies. The article was unpublished but parts were incorporated in [7].

6 Orlicz Spaces

We will need some results on Orlicz spaces which we state below. Φ is said to be a Young function if Φ is a continuous convex increasing function on $[0, \infty)$ with $\Phi(0) = 0$ and $\frac{\Phi(x)}{x} \uparrow \infty$.

For a Young function Φ, the function Ψ defined by

$$\Psi(y) = \sup\{xy - \Phi(x) : x \in [0, \infty)\} \text{ for } y \in [0, \infty) \tag{6.1}$$

is also a Young function. Ψ is called the conjugate function of Φ. From the definition of Φ, it follows that

$$xy \leq \Phi(x) + \Psi(y). \tag{6.2}$$

For a Young function Φ, we define three sets of random variables:

$$\begin{aligned}
\boldsymbol{J}_{\Phi} &= \{W : \mathbb{E}[\Phi(W)] < \infty\}, \\
\boldsymbol{E}_{\Phi} &= \{W : \mathbb{E}[\Phi(W/c)] < \infty \ \forall c \in R\}, \\
\boldsymbol{L}_{\Phi} &= \{W : \mathbb{E}[\Phi(W/c)] < \infty \ \text{ for some } c \in R\}.
\end{aligned}$$

Then $\boldsymbol{E}_{\Phi}$ and $\boldsymbol{L}_{\Phi}$ are linear speaces while J_{Φ} is a convex set but may fail to be a subspace.

For any random variable $Z \in \boldsymbol{L}_{\Phi}$, the Luxembourg norm $\|Z\|_{\Phi}$ is defined as follows:

$$\|Z\|_{\Phi} = \inf\{c > 0 : \mathbb{E}[\Phi(\tfrac{1}{c}|Z|)] \leq 1\}.$$

We list below some standard facts about Orlicz spaces. For proofs, we refer the reader to [10].

Theorem **6.1** *Let Φ be a Young function and Ψ be its conjugate. Then*

(i) $\boldsymbol{E}_{\Phi}$ *and* $\boldsymbol{L}_{\Phi}$ *are Banach spaces under the Luxembourg norm* $\|\cdot\|_{\Phi}$.

(ii) $\boldsymbol{L}^{\infty} \subseteq \boldsymbol{E}_{\Phi} \subseteq \boldsymbol{L}_{\Phi} \subseteq \boldsymbol{L}^{1}$.

(iii) For $X \in \boldsymbol{J}_{\Phi}$, *if* $\|X\|_{\Phi} \leq 1$ *then*

$$\mathbb{E}[\Phi(|X|)] \leq \|X\|_{\Phi}. \tag{6.3}$$

(iv) $Z_n, Z \in \boldsymbol{L}_{\Phi}$, $\|Z_n - Z\|_{\Phi} \to 0$ *implies* $\mathbb{E}[|Z_n - Z|] \to 0$.

(v) For $X \in \boldsymbol{L}_{\Phi}$ *and* $Y \in \boldsymbol{L}_{\Psi}$,

$$\mathbb{E}[|XY|] \leq 2\|X\|_{\Phi}\|Y\|_{\Psi} \tag{6.4}$$

(vi) $\boldsymbol{E}_{\Phi}^{*} = \boldsymbol{L}_{\Psi}$, *where* $Y \in \boldsymbol{L}_{\Psi}$ *acts on* $\boldsymbol{E}_{\Phi}$ *via*

$$X \to \mathbb{E}[XY].$$

Here is a simple observation about convergence in $\boldsymbol{L}_\Phi$. If $Z_n \to Z$ in $L(\Phi)$, then in view of (6.3), $\mathbb{E}\Phi(|Z_n - Z|) \to 0$. By Jensen's inequality, it then follows that

$$\Phi(\mathbb{E}(|Z_n - Z|)) \leq \mathbb{E}[\Phi(|Z_n - Z|)] \to 0.$$

Here is a result on characterisation of weak* closure of a convex set $\boldsymbol{D}$ in $\boldsymbol{L}^\infty$ in terms of norm closures of $\boldsymbol{D}$ in Orlicz norms. As far as we can make out, this is a new result. One half of the same is implicitly contained in [2], [7]

Theorem **6.2** *Let $\boldsymbol{D}$ be a convex subset of $\boldsymbol{L}^\infty$. Let $\boldsymbol{D}^*$ denote the closure of $\boldsymbol{D}$ in the weak* topology on $\boldsymbol{L}^\infty$. Let $\boldsymbol{D}^{[\Phi]}$ denote the closure of $\boldsymbol{D}$ in the $\|\cdot\|_\Phi$ norm. Then*

$$\boldsymbol{D}^* = \bigcap_{\Phi} \boldsymbol{D}^{[\Phi]} \tag{6.5}$$

where the intersection is taken over all Young functions Φ.

Proof : Let $Z \in \boldsymbol{D}^*$ and let $\{Z_\alpha : \alpha \in \Delta\}$ be a net such that $Z_\alpha \to Z$ in $\sigma(\boldsymbol{L}^\infty, \boldsymbol{L}^1)$ topology. Such a net exists as $\boldsymbol{D}^*$ is the closure of $\boldsymbol{D}$ in $\sigma(\boldsymbol{L}^\infty, \boldsymbol{L}^1)$ topology (which is also called the weak* topology). Let Φ be a Young function and let Ψ be its convex conjugate. Since

$$\boldsymbol{L}^\infty \subseteq \boldsymbol{E}_\Phi, \text{ and } \boldsymbol{L}_\Psi \subseteq \boldsymbol{L}^1$$

it follows that $Z_\alpha \to Z$ in $\sigma(\boldsymbol{E}_\Phi, \boldsymbol{L}_\Psi)$ topology. Since $\boldsymbol{D}$ is a convex set, its closure in the $\|\cdot\|_\Phi$ norm is the same as the closure in the $\sigma(\boldsymbol{E}_\Phi, \boldsymbol{L}_\Psi)$, which is the weak topology on $\boldsymbol{E}_\Phi$. (See Rudin Theorem 3.3.12) Hence $Z \in \boldsymbol{D}^{[\Phi]}$. Since this holds for all Φ, it follows that

$$\boldsymbol{D}^* \subseteq \bigcap_{\Phi} \boldsymbol{D}^{[\Phi]}. \tag{6.6}$$

We will prove the other part by contradiction. So suppose in (6.6) the inclusion is strict, namely there exists $Z \in \cap_\Phi \boldsymbol{D}^{[\Phi]}$ but $Z \notin \boldsymbol{D}^*$. Then Applying the Hahn-Banach Theorem to the closed convex set $\boldsymbol{D}^*$ (closed in weak*

topology by construction) and the compact set $\{Z\}$, we get that there exists a separating linear function in the dual, namely $\exists V \in \boldsymbol{L}^1$ such that

$$\sup_{X \in \boldsymbol{D}^*} \mathbb{E}[XV] = a < \mathbb{E}[ZV].$$

In particular

$$\sup_{X \in \boldsymbol{D}} \mathbb{E}[XV] \leq a < \mathbb{E}[ZV]. \tag{6.7}$$

Now $V \in \boldsymbol{L}^1$ implies that there exists a Young function Ψ such that $\mathbb{E}[\Psi(|V|)] < \infty$. Let Φ be the convex conjugate of Ψ. Now $X \to \mathbb{E}[XV]$ is a linear functional on $\boldsymbol{E}_\Phi$ (as $V \in J_\Psi \subseteq L_\Psi$). Hence (6.7) implies

$$\sup_{X \in \boldsymbol{D}^{[\Phi]}} \mathbb{E}[XV] \leq a < \mathbb{E}[ZV]. \tag{6.8}$$

But this is a contradiction since $Z \in \cap_\Phi \boldsymbol{D}^{[\Phi]}$. Thus we must have equality in (6.6). ∎

7 Approximate Arbitrage and EMM

When one rules out free lunch with vanishing risk, it amounts to ruling out approximate arbitrage with *risk* being taken as the absolute lower bound of the payoff. In the Economics and finance litreture, there are other approaches to quantifying risk: if the loss for, say a game, is modelled as L (a posiitve random variable, quantifying loss) then the risk is taken as

$$\mathbb{E}[\Phi(L)]$$

where Φ is an increasing function, often assumed to be convex. Here the point is that the function Φ could vary from investor to investor, depending upon her/his preferences. See [6], [12].

With this view, let us define approximate aribtrage as follows. Let $\mathcal{R}$ be the class of increasing convex functions from $[0, \infty)$ onto $[0, \infty)$ such that $\Phi(0) = 0$ and $\frac{\Phi(x)}{x} \uparrow \infty$. Thus $\mathcal{R}$ is the class of Young functions. $\Phi \in \mathcal{R}$ is to be thought of as a risk function, where risk associated with loss W is $\mathbb{E}\Phi(W)$ or risk associated with reward R is $\mathbb{E}[\Phi(R^-)]$.

We call a position $Z \in L^\infty_+$, $\mathbb{P}(Z > 0) > 0$ an approximable arbitrage opportunity if every investor can come as close to the position Z as desired using simple strategies over finite horizons and keeping the associated risk as small as desired irrespective of risk preferences. We make this notion more precise.

DEFINITION **7.1** A position Z is an approximate arbitrage with controlled risk if $Z \in \boldsymbol{L}^\infty_+$, $\mathbb{P}(Z > 0) > 0$ and for every $\Phi \in \mathcal{R}$, there exist $\{Z_n : n \geq 1\} \subset \boldsymbol{K}_s$ with $\mathbb{P}(Z_n \geq Z - \frac{1}{n}) \to 1$ and $\mathbb{E}[\Phi(Z_n^-)] \to 0$.

If no such Z exists, we say that $\tilde{S}$ (or $\boldsymbol{K}_s$) satisfies *no approximate arbitrage with controlled risk* - NAACR property.

Note that NAACR property has been defined only in terms of a sequence of simple strategies.

With this the main result of this article can be stated as

THEOREM **7.2** *Suppose $\tilde{S}$ is a locally bounded process. Then the process $\tilde{S}$ admits an equivalent (local) martingale measure if and only if $\tilde{S}$ satisfies NAACR property.*

PROOF : Let $\boldsymbol{A}$ denote the class of $Z \in \boldsymbol{L}^\infty_+$ such that for every $\Phi \in \mathcal{R}$ there exist $Z_n \in \boldsymbol{K}_s$, with

$$\mathbb{P}(Z_n \geq Z - \tfrac{1}{n}) \to 1 \tag{7.1}$$

and

$$\mathbb{E}[\Phi(Z_n^-)] \to 0. \tag{7.2}$$

Thus NAACR is equivalent to $\boldsymbol{A} = \{0\}$. Let $\boldsymbol{C}_s^{[\Phi]}$ denote the closure of $\boldsymbol{C}_s$ in the $\|\cdot\|_\Phi$ norm. As seen in Theorem 6.2, we have

$$\boldsymbol{C}_s^* = \bigcap_\Phi \boldsymbol{C}_s^{[\Phi]}. \tag{7.3}$$

Thus, in view of the Theorem 5.1, to complete the proof suffices to show that

$$\left(\bigcap_\Phi \boldsymbol{C}_s^{[\Phi]}\right) \bigcap \boldsymbol{L}^\infty_+ = \boldsymbol{A}. \tag{7.4}$$

If $Z \in \left(\boldsymbol{C}_s^{[\Phi]}\right) \bigcap \boldsymbol{L}_+^\infty$, then recalling that $\boldsymbol{C}_s^{[\Phi]}$ is the closure in norm, we get that there exist $W_j \in \boldsymbol{C}_s$, with $\|W_j - Z\|_\Phi \to 0$. Let $X_j \in \boldsymbol{K}_s$ be as in the definition of $\boldsymbol{C}_s$ i.e. we have, $W_j \leq X_j$. Using

$$X_j^- \leq W_j^- \leq (W_j - Z)^-$$

it follows that

$$\mathbb{E}[\Phi(X_j^-)] \leq \mathbb{E}[\Phi(|(W_j - Z)|)] \leq \|W_j - Z\|_\Phi \to 0.$$

Also, $(X_j - Z)^- \leq \|W_j - Z\| - \Phi$ and hence it follows that $(X_j - Z)^-$ converges to zero in probability. Thus by taking a suitable subsequence j_n, we can ensure that $Z_n = X_{j_n}$ satisfies (7.1). We have already seen that (7.2) holds for this choice. Hence

$$Z \in \left(\bigcap_\Phi \boldsymbol{C}_s^{[\Phi]}\right) \bigcap \boldsymbol{L}_+^\infty \text{ implies that } Z \in \boldsymbol{A}.$$

For the reverse inclusion, let $Z \in \boldsymbol{A}$ and $\Phi \in \mathcal{R}$ be fixed. For $k \geq 1$, let $\Phi_k(x) = \Phi(kx)$. Then $\Phi_k \in \mathcal{R}$ for all $k \geq 1$. Fix k. Let $\{X_j\} \subseteq \boldsymbol{K}_s$ be such that (7.1) holds and

$$\lim_{n\to\infty} \mathbb{E}[\Phi_k(X_j^-)] = 0. \tag{7.5}$$

Let $Y_j = X_j \wedge Z$. Then $Y_j \in \boldsymbol{C}_s$. Then (7.1) and $Y_j \leq Z$ implies

$$\mathbb{P}(Y_j \geq Z - \tfrac{1}{j}) \to 1.$$

and hence once again using $Y_j \leq Z$, we conclude

$$|Y_j - Z| \to 0 \text{ in probability.} \tag{7.6}$$

It can be seen that if $X_j \geq 0$ then $|Y_j - Z| \leq Z$ while if $X_j \leq 0$, then $|Y_j - Z| \leq Z + X_j^-$. Hence

$$|Y_j - Z| \leq Z + X_j^-$$

and as a consequence , using convexity of Φ_k we have

$$\Phi_k(\tfrac{1}{2}|Y_j - Z|) \leq \tfrac{1}{2}(\Phi_k(Z) + \Phi_k(X_j^-)). \tag{7.7}$$

Since Z is bounded and $\lim_{n\to\infty} \mathbb{E}[\Phi_k(X_j^-)] = 0$, the expression on RHS of (7.7) is uniformly integrable (recall k is fixed and X_j, Y_j may depend upon k) and thus

$$\{\Phi_k(\tfrac{1}{2}|Y_j - Z|) : \ j \geq 1\} \text{ is uniformly integrable.}$$

Continuity of Φ_k implies $\Phi_k(\frac{1}{2}|Y_j - Z|) \to 0$ in probability as $j \to \infty$ for each k. Thus

$$\mathbb{E}[\Phi(\tfrac{k}{2}|Y_j - Z|)] = \mathbb{E}[\Phi_k(\tfrac{1}{2}|Y_j - Z|)] \to 0 \text{ as } j \to \infty.$$

For each k, we choose j_k such that

$$\mathbb{E}[\Phi(\tfrac{k}{2}|Y_{j_k} - Z|)] \leq 1.$$

Then defining $W_k = Y_{j_k}$, it follows that (recall the definition of the norm on $\boldsymbol{L}_\Phi$)

$$\|W_k - Z\|_\Phi \leq \frac{2}{k}.$$

Thus $Z \in \boldsymbol{C}_s^{[\Phi]}$. Since this holds for all $\Phi \in \mathcal{R}$, we conclude

$$Z \in \bigcap_{\Phi} \boldsymbol{C}_s^{[\Phi]}.$$

This completes the proof as mentioned above. ∎

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