Hack's law in a drainage network model: a Brownian web approach

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Abstract

Hack (1957), while studying the drainage system in the Shenandoah valley and the adjacent mountains of Virginia, observed a power law relation $l \sim a^{0.6}$ between the length l of a stream from its source to a divide and the area a of the basin that collects the precipitation contributing to the stream as tributaries. We study the tributary structure of Howard's drainage network model of headward growth and branching studied by Gangopadhyay *et al.* (2004). We show that the exponent of Hack's law is 2/3 for Howard's model. Our study is based on a scaling of the process whereby the limit of the watershed area of a stream is area of a Brownian excursion process. To obtain this we define a dual of the model and show that under diffusive scaling, both the original network and its dual converge jointly to the standard Brownian web and its dual.

Key words: Brownian excursion, Brownian meander, Brownian web, Hack's law.

AMS 2000 Subject Classification: 60D05.

1 Introduction

River basin geomorphology is a very old subject of study initiated by Horton (1945). Hack (1957), studying the drainage system in the Shenandoah valley and the adjacent mountains of Virginia, observed a power law relation

$$l \sim a^{0.6} \tag{1}$$

between the length l of a stream from its source to a divide and the area of the basin a that collects the precipitation contributing to the stream as tributaries. Hack also corroborated this power law through the data gathered by Langbein (1947) of nearly 400 different streams in northeastern United States. This empirical relation (1) is widely accepted nowadays albeit with a different exponent (see Gray (1961), Muller (1973)) and is

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called Hack's law. Mandelbrot (1982) mentions Hack's law to strengthen his contention that "*if all rivers as well as their basins are mutually similar, the fractal length-area argument predicts (river's length)$^{1/D}$ is proportional to (basin's area)$^{1/2}$*" where $D > 1$ is the fractal dimension of the river. In this connection it is worth remarking that the Hurst exponent in fractional Brownian motion and in time series analysis arose from the study of the Nile basin by Hurst (1927) where he proposed the relation $l_\perp = l_\parallel^{0.9}$ as that governing the width, $l_\perp$, and the length, $l_\parallel$, of the smallest rectangular region containing the drainage system.

Various statistical models of drainage networks have been proposed (see Rodriguez-Iturbe *et al.* (1997) for a detailed survey). In this paper we study the tributary structure of a 2-dimensional drainage network called the Howard's model of headward growth and branching (see Rodriguez-Iturbe *et al.* (1997)). Our study is based on a scaling of the process and we obtain the watershed area of a stream as the area of a Brownian excursion process. This gives a statistical explanation of Hack's law and justifies the remark of Giacometti *et al.* (1996): "*From the results we suggest that a statistical framework referring to the scaling invariance of the entire basin structure should be used in the interpretation of Hack's law.*"

We first present an informal description of the model: suppose that the vertices of the d-dimensional lattice $\mathbb{Z}^d$ are open or closed with probability p $(0 < p < 1)$ and $1 - p$ respectively, independently of all other vertices. Each open vertex $\mathbf{u} \in \mathbb{Z}^d$ represents a water source and connects to a unique open vertex $\mathbf{v} \in \mathbb{Z}^d$. These edges represents the channels through which water can flow. The connecting vertex $\mathbf{v}$ is chosen so that the d-th co-ordinate of $\mathbf{v}$ is one more than that of $\mathbf{u}$ and $\mathbf{v}$ has the minimum L_1 distance from $\mathbf{u}$. In case of non-uniqueness of such a vertex, we choose one of the closest open vertices with equal probability, independently of everything else.

Let V denote the set of *open* vertices and $h(\mathbf{u})$ denote the uniquely chosen vertex to which $\mathbf{u}$ connects, as described above. Set $\langle \mathbf{u}, h(\mathbf{u}) \rangle$ as the edge (channel) connecting $\mathbf{u}$ and $h(\mathbf{u})$. From the construction it follows that the random graph, $\mathcal{G} = (V, E)$ with edge set $E := \{ \langle \mathbf{u}, h(\mathbf{u}) \rangle : \mathbf{u} \in V \}$, does not contain any circuit. This model has been studied by Gangopadhyay *et al.* (2004) and the following results were obtained:

Theorem 1.1. *Let* $0 < p < 1$.

   (i) *For* $d = 2$ *and* $d = 3$, $\mathcal{G}$ *consists of one single tree almost surely, and, for* $d \geq 4$, $\mathcal{G}$ *is a forest consisting of infinitely many disjoint trees almost surely.*

   (ii) *For any* $d \geq 2$, *the graph* $\mathcal{G}$ *contains no bi-infinite path almost surely.*

In this paper we consider only $d = 2$. Before proceeding further we present a formal description for $d = 2$ which will be used later. Fix $0 < p < 1$ and let $\{B_\mathbf{u} : \mathbf{u} = (\mathbf{u}(1), \mathbf{u}(2)) \in \mathbb{Z}^2\}$ be an i.i.d. collection of Bernoulli random variables with success probability p. Set $V = \{\mathbf{u} \in \mathbb{Z}^2 : B_\mathbf{u} = 1\}$. Let $\{U_\mathbf{u} : \mathbf{u} \in \mathbb{Z}^2\}$ be another i.i.d. collection of random variables, independent of the collection of random variables $\{B_\mathbf{u} : \mathbf{u} \in \mathbb{Z}^2\}$, taking values in the set $\{1, -1\}$, with $\mathbb{P}(U_\mathbf{u} = 1) = \mathbb{P}(U_\mathbf{u} = -1) = 1/2$. For a vertex

$(x, t) \in \mathbb{Z}^2$, we consider $k_0 = \min\{|k| : k \in \mathbb{Z}, B_{(x+k,t+1)} = 1\}$. Clearly, k_0 is almost surely finite. Now, we define,

$$h(x, t) := \begin{cases} (x + k_0, t + 1) \in V & \text{if } (x - k_0, t + 1) \notin V \\ (x - k_0, t + 1) \in V & \text{if } (x + k_0, t + 1) \notin V \\ (x + U_{(x,t)} k_0, t + 1) \in V & \text{if } (x \pm k_0, t + 1) \in V. \end{cases}$$

For any $k \geq 0$, let

$$h^{k+1}(x, t) := h(h^k(x, t)) \text{ with } h^0(x, t) := (x, t),$$

$$C_k(x, t) := \begin{cases} \{(y, t - k) \in V : h^k(y, t - k) = (x, t)\} & \text{if } (x, t) \in V, \\ \emptyset & \text{otherwise,} \end{cases}$$

$$C(x, t) := \cup_{k \geq 0} C_k(x, t).$$

Here $h^k(x, t)$ represents the 'k-th generation progeny' of (x, t), the sets $C_k(x, t)$ and $C(x, t)$ denote, respectively, the set of k-th generation ancestors and the set of all ancestors of (x, t); $C(x, t) = \emptyset$ if $(x, t) \notin V$. In the terminology of drainage network, $C(x, t)$ represents the region of precipitation, the water from which is channelled through the open point (x, t) (see Figure 1). From Theorem 1.1 (ii), we have that $C(x, t)$ is finite almost surely.
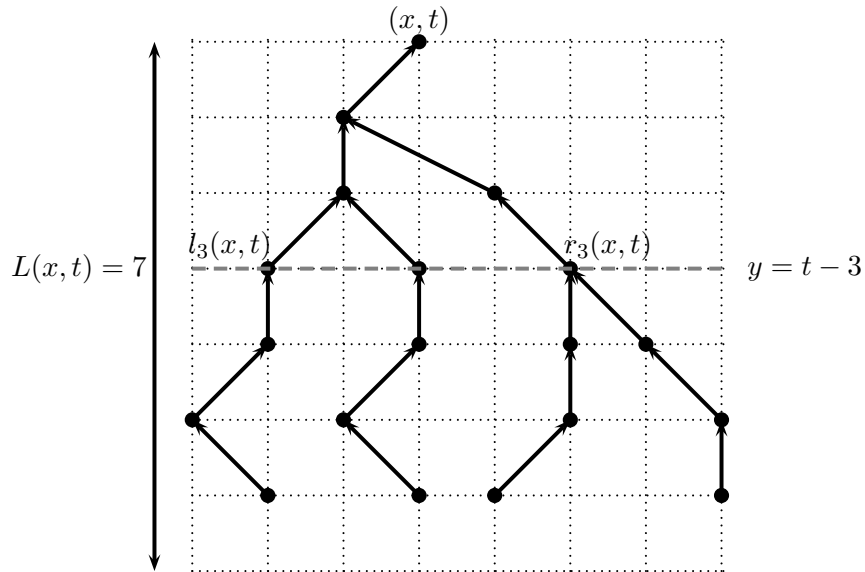

Figure 1: The bold vertices on the line $y = t - 3$ constitute the set $C_3(x, t)$ and all the bold vertices together constitute the cluster $C(x, t)$.

Now, we define

$$L(x, t) := \inf\{k \geq 0 : C_k(x, t) = \emptyset\},$$

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as the 'length of the channel', which as earlier is finite almost surely. We observe that for any $(x,t) \in \mathbb{Z} \times \mathbb{Z}$, $L(x,t) \geq 0$ and the distribution of $L(x,t)$ does not depend upon (x,t). Our first result is about the length of the channel. We remark here that Newman *et al.* (2005) has a similar result in a set-up which allows crossing of paths.

Theorem 1.2. *We have*

$$\lim_{n \to \infty} \sqrt{n}\mathbb{P}(L(0,0) > n) = \frac{1}{\gamma_0 \sqrt{\pi}}.$$

where $\gamma_0^2 := \gamma_0^2(p) = \frac{(1-p)(2-2p+p^2)}{p^2(2-p)^2}$.

Next we define

$$r_k(x,t) := \begin{cases} \max\{u : (u, t-k) \in C_k(x,t)\} & \text{if } (x,t) \in V \text{ and } 0 \leq k < L(x,t), \\ 0 & \text{otherwise}, \end{cases}$$

$$l_k(x,t) := \begin{cases} \min\{u : (u, t-k) \in C_k(x,t)\} & \text{if } (x,t) \in V \text{ and } 0 \leq k < L(x,t), \\ 0 & \text{otherwise}, \end{cases}$$

$$D_k(x,t) := r_k(x,t) - l_k(x,t).$$

The quantity $D_k(x,t)$ denotes the *width* of the set of all k-th generation ancestors of (x,t). We define the *width process* $D_n^{(x,t)}(s)$ and the *cluster process* $K_n^{(x,t)}(s)$ for $s \geq 0$ as follows : for $k = 0, 1, \ldots$ and $k/n \leq s \leq (k+1)/n$,

$$D_n^{(x,t)}(s) := \frac{D_k(x,t)}{\gamma_0 \sqrt{n}} + \frac{(ns - [ns])}{\gamma_0 \sqrt{n}}(D_{k+1}(x,t) - D_k(x,t))$$

$$K_n^{(x,t)}(s) := \frac{\#C_k(x,t)}{\gamma_0 \sqrt{n}} + \frac{(ns - [ns])}{\gamma_0 \sqrt{n}}(\#C_{k+1}(x,t) - \#C_k(x,t)) \tag{2}$$

where $\gamma_0 > 0$ is as in the statement of Theorem 1.2. In other words, $D_n^{(x,t)}(s)$ (respectively $K_n^{(x,t)}(s)$) is defined $D_k(x,t)/(\gamma_0 \sqrt{n})$ (respectively $\#C_k(x,t)/(\gamma_0 \sqrt{n})$) at time points $s = k/n$ and, at other time points defined by linear interpolation. The distributions of both $D_n^{(x,t)}$ and $K_n^{(x,t)}$ are independent of (x,t).

To describe our results we need to introduce two processes, Brownian meander and Brownian excursion, studied by Durrett *et al.* (1977). Let $\{W(s) : s \geq 0\}$ be a standard Brownian motion with $W(0) = 0$. Let $\tau_1 := \sup\{s \leq 1 : W(s) = 0\}$ and $\tau_2 := \inf\{s \geq 1 : W(s) = 0\}$. Note that $\tau_1 < 1$ and $\tau_2 > 1$ almost surely. The standard Brownian meander, $W^+(s)$, and the standard Brownian excursion, $W_0^+(s)$, are given by

$$W^+(s) := \frac{|W(\tau_1 + s(1 - \tau_1))|}{\sqrt{1 - \tau_1}}, \quad s \in [0,1] \tag{3}$$

$$W_0^+(s) := \frac{|W(\tau_1 + s(\tau_2 - \tau_1))|}{\sqrt{\tau_2 - \tau_1}}, \quad s \in [0,1]. \tag{4}$$

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Both of these processes are continuous non-homogeneous Markov process (see Durrett *et al.* (1977) and references therein). Further, $W^+(0) = 0$ and, for $x \geq 0$, $\mathbb{P}(W^+(1) \leq x) = 1 - \exp(-x^2/2)$, i.e. $W^+(1)$ follows a Rayleigh distribution.

We also need some random variables obtained as functionals of these two processes. In particular, let

$$I_0^+ := \int_0^1 W_0^+(t)dt \quad \text{and} \quad M_0^+ := \max\{W_0^+(t) : t \in [0,1]\}.$$

Janson *et al.* (2007) showed that, as $x \to \infty$,

$$\mathbb{P}(I_0^+ > x) \sim \frac{6\sqrt{6}}{\sqrt{\pi}} x \exp(-6x^2) \text{ and, the density, } f_{I_0^+}(x) \sim \frac{72\sqrt{6}}{\sqrt{\pi}} x^2 \exp(-6x^2).$$

The random variable M_0^+ is continuous, having a strictly positive density on $(0,\infty)$ (see Durrett *et al.* (1977)) and for $x > 0$,

$$\mathbb{P}(M_0^+ \leq x) = 1 + 2\sum_{k=1}^{\infty} \exp(-(2kx)^2/2)[1 - (2kx)^2] \text{ with } \mathbb{E}(M_0^+) = \sqrt{\pi/2}.$$

For $f \in C[0,\infty)$ let $f|_{[0,1]}$ denotes the restriction of f over $[0,1]$. Our next result is about the weak convergence of the width process $D_n^{(0,0)}|_{[0,1]}$ and the cluster process $K_n^{(0,0)}|_{[0,1]}$ under diffusive scaling. Here and subsequently, as is commonly used in statistics, we use the notation $X|Y$ to denote the conditional random variable X given Y.

Theorem 1.3. *As $n \to \infty$, we have*

(i) $D_n^{(0,0)}|_{[0,1]}\big|\mathbf{1}_{\{L(0,0)>n\}} \Rightarrow \sqrt{2}W^+$,

(ii) $\sup\{|pD_n^{(0,0)}(s) - K_n^{(0,0)}(s)| : s \in [0,1]\}\big|\mathbf{1}_{\{L(0,0)>n\}} \xrightarrow{P} 0$.

The following corollary is an immediate consequence of Theorem 1.3:

Corollary 1.3.1. *For $u > 0$, as $n \to \infty$ we have*

(i) $\sqrt{n}\mathbb{P}(\#C_n(0,0) > \sqrt{n}\gamma_0 u) \to \frac{1}{\gamma_0\sqrt{\pi}}\exp(-u^2/4p^2)$,

(ii) $\mathbb{P}(\sum_{k=0}^n \#C_k(0,0) > n^{\frac{3}{2}}\gamma_0 u\big|L(0,0) > n) \to \mathbb{P}(p\sqrt{2}I^+ > u)$.

Before we proceed to state Theorem 1.4 we recall some results regarding random vectors whose distribution functions have regularly varying tails (see Resnick (2007) page 172). A random vector Z on $(0,\infty)^d$ with a distribution function F has a regularly varying tail if, as $n \to \infty$, there exists a sequence $b_n \to \infty$ such that $n\mathbb{P}\{Z/b_n \in \cdot\} \xrightarrow{v} \nu(\cdot)$ for some $\nu \in M_+$ where $M_+ := \{\mu : \mu$ is a non-negative Radon measure on $(0,\infty)^d\}$. Here $\xrightarrow{v}$ denotes vague convergence. It is in this context that Theorem 1.4 obtains a regularly varying tail for the distribution of $(L(x,t), (\#C(x,t))^{2/3})$; which justifies that

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the exponent of Hack's law is 2/3 for Howard's model. In addition we obtain a scaling law, with a Hack exponent of 1/2, for the length of the stream *vis-à-vis* the maximum width of the region of precipitation, i.e.,

$$D_{\max}(0,0) := \max\{D_k(0,0) : 0 \le k < L(0,0)\}. \tag{5}$$

It should be noted that Leopold *et al.* (1962) obtained an exponent of 0.64 through computer simulations.

Theorem 1.4. *Let* $\mathbf{E} := (0,\infty) \times (0,\infty)$. *There exist measures* μ *and* ν *on the Borel* σ-*algebra on* $\mathbf{E}$ *such that for any Borel set* $B \subseteq \mathbf{E}$ *we have*

$$\sqrt{n}\mathbb{P}\Big[\frac{1}{n}\big(L(0,0),(\#C(0,0))^{2/3}\big) \in B\Big] \to \mu(B) \tag{6}$$

$$\sqrt{n}\mathbb{P}\Big[\frac{1}{n}\big(L(0,0),(D_{max}(0,0))^{1/2}\big) \in B\Big] \to \nu(B) \tag{7}$$

with μ *and* ν *being given by*

$$\mu(B) = \int\int_B \frac{3\sqrt{v}}{4\sqrt{2\pi}\gamma_0^2 pt^3} f_{I_0^+}\Big(\frac{v^{\frac{3}{2}}}{\gamma_0 p\sqrt{2t^3}}\Big)dvdt$$

$$\nu(B) = \int\int_B \frac{v}{\sqrt{2\pi}\gamma_0^2 pt^2} f_{M_0^+}\Big(\frac{v^2}{\gamma_0 p\sqrt{2t}}\Big)dvdt,$$

where $f_{I_0^+}$ *and* $f_{M_0^+}$ *denote the density functions of* I_0^+ *and* M_0^+ *respectively. Moreover, for* $\lambda, \tau > 0$, *we have*

$$\sqrt{n}\mathbb{P}\Big[\frac{1}{n}\big(L(0,0),(\#C(0,0))^{\alpha}\big) \in (\tau,\infty) \times (\lambda,\infty)\Big] = \begin{cases} 0 & \text{if } \alpha < 2/3 \\ (\pi\tau\gamma_0^2)^{-1/2} & \text{if } \alpha > 2/3 \end{cases} \tag{8}$$

and

$$\sqrt{n}\mathbb{P}\Big[\frac{1}{n}\big(L(0,0),(D_{max}(0,0))^{\alpha}\big) \in (\tau,\infty) \times (\lambda,\infty)\Big] = \begin{cases} 0 & \text{if } \alpha < 1/2 \\ (\pi\tau\gamma_0^2)^{-1/2} & \text{if } \alpha > 1/2. \end{cases} \tag{9}$$

The estimates of the densities $f_{I_0^+}$ and $f_{M_0^+}$ imply that μ and ν are finite measures on $\mathbf{E}$. An immediate consequence of the above theorem is the following:

Corollary 1.4.1. *As* $n \to \infty$ *for* $u > 0$, *we have*

(i) $\sqrt{n}\mathbb{P}(\#C(0,0) > \sqrt{2n^3}\gamma_0 pu) \to \frac{1}{2\sqrt{\pi}\gamma_0} \int_0^\infty t^{-\frac{3}{2}}\bar{F}_{I_0^+}(ut^{-\frac{3}{2}})dt,$

(ii) $\sqrt{n}\mathbb{P}(D_{max}(0,0) > \sqrt{2n}\gamma_0 pu) \to \frac{1}{2\sqrt{\pi}\gamma_0} \int_0^\infty t^{-\frac{3}{2}}\bar{F}_{M_0^+}(ut^{-\frac{1}{2}}))dt$

where $F_{I_0^+}$ *and* $F_{M_0^+}$ *are the distribution functions of* I_0^+ *and* M_0^+ *respectively and* $\bar{F}_{I_0^+} := 1 - F_{I_0^+}$, $\bar{F}_{M_0^+} := 1 - F_{M_0^+}$.

The proofs of the above theorems are based on a scaling of the process. In the next section we define a dual graph and show that as processes, under a suitable scaling, the original and the dual processes converge jointly to the Brownian web and its dual in distribution (the double Brownian web). This invariance principle is used in Sections 3 and 4 to prove the theorems. In this connection it is worth noting that in Proposition 2.7, we have provided an alternate characterization of the dual of Brownian web which is of independent interest. This characterization is suitable for proving the joint convergence of coalescing non-crossing path family and its dual to the double Brownian web and has been used in Theorem 2.9 to achieve the required convergence.

We should mention here that the Brownian web appears as a universal scaling limit for various network models (see Fontes *et al.* (2004), Ferrari *et al.* (2005), Coletti *et al.* (2009)). It is reasonable to expect that with suitable modifications our method will give similar results in other network models. Our results will hold for any network model which admits a dual and satisfies (i) conditions listed in Remark 2.1, (ii) the scaled model and its dual converges weakly to the double Brownian web (see Section 2) and (iii) a certain sequence of counting random variables are uniformly integrable (see Lemma 3.3). In this sense our result can be considered as a universality class result.

2   Dual process and the double Brownian web

2.1   Dual process

For the graph $\mathcal{G}$ we now describe a dual process such that the set of ancestors $C(x, t)$ (defined in the previous section) of a vertex $(x, t) \in V$ is bounded by two dual paths. The dependency inherent in the graph $\mathcal{G}$ implies that, although the cluster is bounded by two dual paths, these paths are *not given by independent random walks*. The dual vertices are precisely the mid-points between two consecutive open vertices on each horizontal line $\{y = n\}, n \in \mathbb{Z}$ with each dual vertex having a unique offspring dual vertex in the negative direction of the y-axis. Before giving a formal definition, we direct the attention of the reader to Figure 2.

For $\mathbf{u} \in \mathbb{Z}^2$, we define,

$$\begin{aligned} J_{\mathbf{u}}^+ &:= \inf\{k : k \geq 1, (\mathbf{u}(1) + k, \mathbf{u}(2)) \in V\} \\ J_{\mathbf{u}}^- &:= \inf\{k : k \geq 1, (\mathbf{u}(1) - k, \mathbf{u}(2)) \in V\}. \end{aligned} \tag{10}$$

Next, we define $r(\mathbf{u}) := (\mathbf{u}(1) + J_{\mathbf{u}}^+, \mathbf{u}(2))$ and $l(\mathbf{u}) := (\mathbf{u}(1) - J_{\mathbf{u}}^-, \mathbf{u}(2))$, as the first open point to the right (*open right neighbour*) and the first open point to the left (*open left neighbour*) of $\mathbf{u}$ respectively. For $(x, t) \in V$, let $\widehat{r}(x, t) := (x + J_{(x,t)}^+/2, t)$ and $\widehat{l}(x, t) := (x - J_{(x,t)}^-/2, t)$ denote respectively the right dual neighbour and the left dual neighbour of (x, t) in the dual vertex set. Finally, the dual vertex set is given by

$$\widehat{V} := \{\widehat{r}(x, t), \widehat{l}(x, t) : (x, t) \in V\}.$$

For a vertex $(u, s) \in \widehat{V}$, let $(v, s - 1) \in \widehat{V}$ be such that the straight line segment joining (u, s) and $(v, s - 1)$ does not cross any edge in $\mathcal{G}$. The dual edges are edges joining all
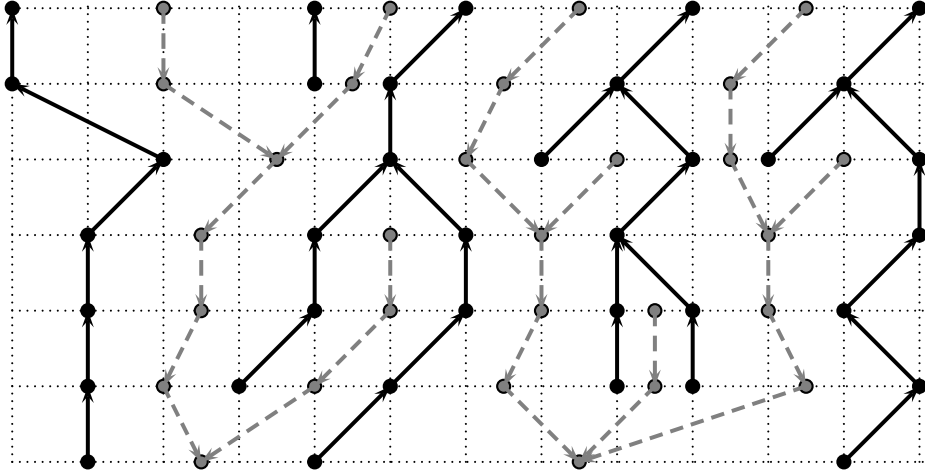
Figure 2: The black points are open vertices, the gray points are the vertices of the dual process and the gray (dashed) paths are the dual paths

such (u, s) and $(v, s - 1)$. Formally, for $(u, s) \in \widehat{V}$, we define

$$
\begin{aligned}
a^l(u, s) &:= \sup\{z : (z, s - 1) \in V, h(z, s - 1)(1) < u\} \\
a^r(u, s) &:= \inf\{z : (z, s - 1) \in V, h(z, s - 1)(1) > u\}
\end{aligned}
\tag{11}
$$

and set $\widehat{h}(u, s) := ((a^l(u, s) + a^r(u, s))/2, s - 1)$. Note that $(a^r(u, s), s - 1)$ and $(a^l(u, s), s - 1)$ are the nearest vertices in V to the right and left respectively of the dual vertex $\widehat{h}(u, s)$. Finally the edge set of the dual graph $\widehat{\mathcal{G}} := (\widehat{V}, \widehat{E})$ is given by

$$
\widehat{E} := \{\langle (u, s), \widehat{h}(u, s) \rangle : (u, s) \in \widehat{V}\}.
$$

Remark 2.1: Note that the vertex set of the dual graph is a subset of $\frac{1}{2}\mathbb{Z} \times \mathbb{Z}$. Before we proceed, we list some properties of the graph $\mathcal{G}$ and its dual $\widehat{\mathcal{G}}$.

(1) $\mathcal{G}$ uniquely specifies the dual graph $\widehat{\mathcal{G}}$ and the dual edges do not intersect the original edges. The construction ensures that $\widehat{\mathcal{G}}$ does not contain any circuit.

(2) For $(x, t) \in V$, the cluster $C(x, t)$ is enclosed within the dual paths starting from $\widehat{r}(x, t)$ and $\widehat{l}(x, t)$. The boundedness of $C(x, t)$ for every $(x, t) \in V$ implies that these two dual paths coalesce, thus $\widehat{\mathcal{G}}$ is a single tree;

(3) Since paths starting from any two open vertices in the original graph coalesce and the dual edges do not cross the original edges, there is no bi-infinite path in $\widehat{\mathcal{G}}$. $\quad\square$

We now obtain a Markov process from the dual paths. Fix $(u, s) \in \widehat{V}$ and for $k \geq 1$, set $\widehat{h}^k(u, s) := \widehat{h}(\widehat{h}^{k-1}(u, s))$ where $\widehat{h}^0(u, s) := (u, s)$. Let $\widehat{h}^k(u, s) := (\widehat{X}_k^{(u,s)}, s - k)$ for $k \geq 0$. Given $\widehat{X}_k^{(u,s)} = v$, we have $\widehat{X}_{k+1}^{(u,s)} = \widehat{X}_1^{(v,s-k)} = (a^l(v, s - k) + a^r(v, s - k))/2$. To find the distribution of $\widehat{X}_1^{(v,s-k)}$ we note that

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(a) if $v \notin \mathbb{Z}$, then $v - 1/2 \in \mathbb{Z}$ and

$$a^r(v, s - k) = v - 1/2 + J^+_{(v-1/2, s-k-1)} \text{ and } a^l(v, s - k) = v + 1/2 - J^-_{(v+1/2, s-k-1)};$$

(b) if $v \in \mathbb{Z}$ and $(v, s - k - 1) \notin V$ then

$$a^r(v, s - k) = v + J^+_{(v, s-k-1)} \text{ and } a^l(v, s - k) = v - J^-_{(v, s-k-1)};$$

(c) if $v \in \mathbb{Z}$ and $(v, s - k - 1) \in V$, then note that the open right neighbour $r(v, s - k)$ and open left neighbour $l(v, s - k)$, flanking the dual vertex $(v, s - k)$ from both sides, are equidistant from $(v, s - k - 1)$. Thus, either $U_{(v, s-k-1)} = 1$, in which case $a^r(v, s - k) = v$ and $a^l(v, s - k) = v - J^-_{(v, s-k-1)}$, or $U_{(v, s-k-1)} = -1$, in which case $a^l(v, s - k) = v$ and $a^r(v, s - k) = v + J^+_{(v, s-k-1)}$.

We note that in all the three cases above, $\widehat{X}^{(u,s)}_{k+1}$ is a function of $\widehat{X}^{(u,s)}_k$ and the collection of random variables $\{(B_{\mathbf{u}}, U_{\mathbf{u}}) : \mathbf{u}(2) = s - k - 1 \in \mathbb{Z}\}$. Thus by the random mapping representation (see, for example, Levin *et al.* (2008)) we have

Proposition 2.2. *For $(u, s) \in \widehat{V}$ the process $\{\widehat{X}^{(u,s)}_k : k \geq 0\}$ is a time homogeneous Markov process.*

Before we proceed, we make the following observations about the transition probabilities of the Markov process. Let G be a geometric random variable taking values in $\{1, 2, \dots\}$, i.e., $\mathbb{P}(G = l) = p(1 - p)^{l-1}$ for $l \geq 1$. For any $\mathbf{u} \in \mathbb{Z} \times \mathbb{Z}$, the random variables $J^+_{\mathbf{u}}$ and $J^-_{\mathbf{u}}$ are i.i.d. copies of the geometric random variable G independent of $B_{\mathbf{u}}$. Further, if $\mathbf{u}_1, \mathbf{u}_2 \in \mathbb{Z}^2$ are such that $\mathbf{u}_1(1) \geq \mathbf{u}_2(1) - 1$ and $\mathbf{u}_1(2) = \mathbf{u}_2(2)$, the random variables $J^+_{\mathbf{u}_1}$ and $J^-_{\mathbf{u}_2}$ are also independent. Now, for $u \notin \mathbb{Z}$ and, $v \in \mathbb{Z}/2$, we have

$$\mathbb{P}(\widehat{X}^{(u,s)}_1 - \widehat{X}^{(u,s)}_0 = v \big| \widehat{X}^{(u,s)}_0 = u) = \mathbb{P}(J^+_{(u-1/2, s-1)} - J^-_{(u+1/2, s-1)} = 2v)$$
$$= \mathbb{P}(G_1 - G_2 = 2v) \qquad (12)$$

where G_1 and G_2 are i.i.d. copies of G, defined above. If $u \in \mathbb{Z}$ and $v \in \mathbb{Z}/2$, we have, using notation from (c) above

$$\mathbb{P}(\widehat{X}^{(u,s)}_1 - \widehat{X}^{(u,s)}_0 = v \big| \widehat{X}^{(u,s)}_0 = u)$$
$$= (1 - p)\mathbb{P}(G_1 - G_2 = 2v) + p\mathbb{P}(G = 2v)/2 + p\mathbb{P}(G = -2v)/2 \qquad (13)$$

where G_1 and G_2 are as above. It is therefore obvious that the transition probabilities of $\widehat{X}^{(u,s)}_k$ depend on whether the present state is an integer or not.

From equations (12) and (13), it immediately follows that

Proposition 2.3. *For any $(u, s) \in \widehat{V}$, $\{\widehat{X}^{(u,s)}_k : k \geq 0\}$ is an L^2-martingale with respect to the filtration $\mathcal{F}_k := \sigma(\{B_{\mathbf{u}}, U_{\mathbf{u}} : \mathbf{u} \in \mathbb{Z}^2, \mathbf{u}(2) \geq s - k\})$.*

2.2 Dual Brownian web

In this section we briefly describe the dual Brownian web $\widehat{\mathcal{W}}$ associated with $\mathcal{W}$ and present an alternate characterization of the dual Brownian web $\widehat{\mathcal{W}}$.

The Brownian web (studied extensively by Arratia (1979), Arratia (1981), Toth *et al.* (1998), Fontes *et al.* (2004)) may be viewed as a collection of one-dimensional coalescing Brownian motions starting from every point in the space time plane $\mathbb{R}^2$. We recall relevant details from Fontes *et al.* (2004).

Let $\mathbb{R}_c^2$ denote the completion of the space time plane $\mathbb{R}^2$ with respect to the metric

$$\rho((x_1, t_1), (x_2, t_2)) := |\tanh(t_1) - \tanh(t_2)| \vee \left| \frac{\tanh(x_1)}{1 + |t_1|} - \frac{\tanh(x_2)}{1 + |t_2|} \right|.$$

As a topological space $\mathbb{R}_c^2$ can be identified with the continuous image of $[-\infty, \infty]^2$ under a map that identifies the line $[-\infty, \infty] \times \{\infty\}$ with the point $(*, \infty)$, and the line $[-\infty, \infty] \times \{-\infty\}$ with the point $(*, -\infty)$. A path π in $\mathbb{R}_c^2$ with starting time $\sigma_\pi \in [-\infty, \infty]$ is a mapping $\pi : [\sigma_\pi, \infty] \to [-\infty, \infty] \cup \{*\}$ such that $\pi(\infty) = *$ and, when $\sigma_\pi = -\infty$, $\pi(-\infty) = *$. Also $t \mapsto (\pi(t), t)$ is a continuous map from $[\sigma_\pi, \infty]$ to $(\mathbb{R}_c^2, \rho)$. We then define Π to be the space of all paths in $\mathbb{R}_c^2$ with all possible starting times in $[-\infty, \infty]$. The following metric, for $\pi_1, \pi_2 \in \Pi$

$$d_\Pi(\pi_1, \pi_2) := |\tanh(\sigma_{\pi_1}) - \tanh(\sigma_{\pi_2})| \vee \sup_{t \geq \sigma_{\pi_1} \wedge \sigma_{\pi_2}} \left| \frac{\tanh(\pi_1(t \vee \sigma_{\pi_1}))}{1 + |t|} - \frac{\tanh(\pi_2(t \vee \sigma_{\pi_2}))}{1 + |t|} \right|$$

makes Π a complete, separable metric space.

Remark 2.4: Convergence in this metric can be described as locally uniform convergence of paths as well as convergence of starting times. Therefore, for any $\epsilon > 0$ and $m > 0$, we can choose $\epsilon_1(= f(\epsilon, m)) > 0$ such that for $\pi_1, \pi_2 \in \Pi$ with $\{(\pi_i(t), t) : t \in [\sigma_{\pi_i}, m]\} \subseteq [-m, m] \times [-m, m]$ for $i = 1, 2$, $d_\Pi(\pi_1, \pi_2) < \epsilon_1$ implies that $\|(\pi_1(\sigma_{\pi_1}), \sigma_{\pi_1}) - (\pi_2(\sigma_{\pi_2}), \sigma_{\pi_2})\|_2 < \epsilon$ and $\sup\{|\pi_1(t) - \pi_2(t)| : t \in [\max\{\sigma_{\pi_1}, \sigma_{\pi_2}\}, m]\} < \epsilon$. We will use this later several times. □

Let $\mathcal{H}$ be the space of compact subsets of (Π, d_Π) equipped with the Hausdorff metric $d_\mathcal{H}$. The Brownian web $\mathcal{W}$ is a random variable taking values in the complete separable metric space $(\mathcal{H}, d_\mathcal{H})$.

Before introducing the dual Brownian web we require a similar metric space on the collection of backward paths. As in the definition of Π, let $\widehat{\Pi}$ be the collection of all paths $\widehat{\pi}$ with starting time $\sigma_{\widehat{\pi}} \in [-\infty, \infty]$ such that $\widehat{\pi} : [-\infty, \sigma_{\widehat{\pi}}] \to [-\infty, \infty] \cup \{*\}$ with $\widehat{\pi}(-\infty) = *$ and, when $\sigma_{\widehat{\pi}} = +\infty$, $\widehat{\pi}(\infty) = *$. As earlier $t \mapsto (\widehat{\pi}(t), t)$ is a continuous map from $[-\infty, \sigma_{\widehat{\pi}}]$ to $(\mathbb{R}_c^2, \rho)$. We equip $\widehat{\Pi}$ with the metric

$$d_{\widehat{\Pi}}(\widehat{\pi}_1, \widehat{\pi}_2) = |\tanh(\sigma_{\widehat{\pi}_1}) - \tanh(\sigma_{\widehat{\pi}_2})| \vee \sup_{t \leq \sigma_{\widehat{\pi}_1} \vee \sigma_{\widehat{\pi}_2}} \left| \frac{\tanh(\widehat{\pi}_1(t \wedge \sigma_{\widehat{\pi}_1}))}{1 + |t|} - \frac{\tanh(\widehat{\pi}_2(t \wedge \sigma_{\widehat{\pi}_2}))}{1 + |t|} \right|$$

making $(\widehat{\Pi}, d_{\widehat{\Pi}})$ a complete, separable metric space. The complete separable metric space of compact sets of paths of $\widehat{\Pi}$ is denoted by $(\widehat{\mathcal{H}}, d_{\widehat{\mathcal{H}}})$, where $d_{\widehat{\mathcal{H}}}$ is the Hausdorff metric on $\widehat{\mathcal{H}}$, and let $\mathcal{B}_{\widehat{\mathcal{H}}}$ be the corresponding Borel σ field.

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2.3 Properties of $(\mathcal{W}, \widehat{\mathcal{W}})$

The Brownian web and its dual $(\mathcal{W}, \widehat{\mathcal{W}})$ is a $(\mathcal{H} \times \widehat{\mathcal{H}}, \mathcal{B}_{\mathcal{H}} \times \mathcal{B}_{\widehat{\mathcal{H}}})$ valued random variable such that $\mathcal{W}$ and $\widehat{\mathcal{W}}$ uniquely determine each other almost surely with $\widehat{\mathcal{W}}$ being equally distributed as $-\mathcal{W}$, the Brownian web rotated 180^0 about the origin. The interaction between the paths in $\mathcal{W}$ and $\widehat{\mathcal{W}}$ is that of Skorohod reflection (see Soucaliuc *et al.* (2000)). We list some properties which hold almost surely.

(a) Let $D, \widehat{D} \subseteq \mathbb{R}^2$ be two deterministic dense sets. There exist unique paths $\pi^{(x,t)} \in \mathcal{W}$ and $\widehat{\pi}^{(y,s)} \in \widehat{\mathcal{W}}$ starting from any $(x,t) \in D$ and $(y,s) \in \widehat{D}$ respectively.

(b) As in Fontes *et al.* (2003), for $(\mathcal{W}, \widehat{\mathcal{W}})$ and $(x,t) \in \mathbb{R}^2$, we define

$$m_{\text{in}}(x,t) := \lim_{\epsilon \downarrow 0} \{\text{number of paths in } \mathcal{W} \text{ starting at some } t - \epsilon \text{ that pass}$$

$$\text{through } (x,t) \text{ and are disjoint in } (t - \epsilon, t)\};$$

$$m_{\text{out}}(x,t) := \lim_{\epsilon \downarrow 0} \{\text{number of paths in } \mathcal{W} \text{ starting at } (x,t) \text{ that are}$$

$$\text{disjoint in } (t, t + \epsilon)\}.$$

The type of a point (x,t) is given by $(m_{\text{in}}(x,t), m_{\text{out}}(x,t))$. Similarly we define $\widehat{m}_{\text{in}}(x,t)$ and $\widehat{m}_{\text{out}}(x,t)$ for the dual paths. It is known that (see Proposition 5.12, Theorem 5.16 of Fontes *et al.* (2003))

  (i) $m_{\text{in}}(x,t) + 1 = \widehat{m}_{\text{out}}(x,t)$ and $m_{\text{out}}(x,t) - 1 = \widehat{m}_{\text{in}}(x,t)$.

  (ii) Every deterministic point $(x,t) \in \mathbb{R}^2$ is of type $(0,1)$.

  (iii) For any deterministic time t, each point on $\mathbb{R} \times \{t\}$ is of either type $(0,1), (0,2)$ or $(1,1)$ in $\mathcal{W}$.

(c) For $\pi^1, \pi^2 \in \mathcal{W}$, let $t^{\pi^1, \pi^2} := \inf\{t : t > \max\{\sigma_{\pi^1}, \sigma_{\pi^2}\}, \pi^1(t) = \pi^2(t)\}$. Then, $t^{\pi^1, \pi^2} < \infty$ and for all $s > t^{\pi^1, \pi^2}$, $\pi^1(s) = \pi^2(s)$, i.e., the paths coalesce at the time of intersection. For $\{\pi_n : n \geq 1\} \subseteq \mathcal{W}$ with $d_\Pi(\pi_n, \pi) \to 0$, we have that $t^{\pi_n, \pi} \to \sigma_\pi$ as $n \to \infty$ (see Sun *et al.* (2008)).

(d) For $\pi^1 \in \mathcal{W}$ with $(\pi^1(\sigma_{\pi^1}), \sigma_{\pi^1})$ of type $(0,1), (0,2), (1,1), (2,1)$ or $(1,2)$ and for any $\epsilon > 0$, there exist paths $\pi^2, \pi^3 \in \mathcal{W}$ such that $\sigma_{\pi^2} < \sigma_{\pi^1} < \sigma_{\pi^3}$ and $\pi^2(t) = \pi^1(t) = \pi^3(t)$ for all $t \geq \sigma_{\pi^1} + \epsilon$ (follows from the proof of Lemma 3.4 of Sun *et al.* (2008)).

(e) For $\pi \in \mathcal{W}$, $\widehat{\pi} \in \widehat{\mathcal{W}}$ and

  (i) for no $s, t \in [\sigma_\pi, \sigma_{\widehat{\pi}}]$, we have $(\pi(s) - \widehat{\pi}(s))(\pi(t) - \widehat{\pi}(t)) < 0$, i.e., no forward path of $\mathcal{W}$ crosses a dual path of $\widehat{\mathcal{W}}$;

  (ii) $\int_{\sigma_\pi}^{\sigma_{\widehat{\pi}}} 1\{\pi(s) = \widehat{\pi}(s)\}ds = 0$, i.e., forward paths of $\mathcal{W}$ and dual paths of $\widehat{\mathcal{W}}$ "spend zero Lebesgue time together" (see Sun *et al.* (2008)).

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(f) For any $s > 0$, the sets $\{\pi(t + s) : \pi \in \mathcal{W}, \sigma_\pi \leq t\}$ and $\{\widehat{\pi}(t - s) : \widehat{\pi} \in \widehat{\mathcal{W}}, \sigma_{\widehat{\pi}} \geq t\}$ are locally finite.

We introduce some notation to study the sets $\{\pi(t + s) : \pi \in \mathcal{W}, \sigma_\pi \leq t\}$ and $\{\widehat{\pi}(t - s) : \widehat{\pi} \in \widehat{\mathcal{W}}, \sigma_{\widehat{\pi}} \geq t\}$. For a $(\mathcal{H}, B_\mathcal{H})$ valued random variable K and $t \in \mathbb{R}$ let $K^{t-} := \{\pi : \pi \in K \text{ and } \sigma_\pi \leq t\}$. Similarly for a $(\widehat{\mathcal{H}}, B_{\widehat{\mathcal{H}}})$ valued random variable $\widehat{K}$ and $t \in \mathbb{R}$ let $\widehat{K}^{t+} := \{\widehat{\pi} : \widehat{\pi} \in \widehat{K} \text{ and } \sigma_{\widehat{\pi}} \geq t\}$. For $t_1, t_2 \in \mathbb{R}$, $t_2 > t_1$ and a $(\mathcal{H}, B_\mathcal{H})$ valued random variable K, define

$$
\begin{aligned}
\mathcal{M}_K(t_1, t_2) &:= \{\pi(t_2) : \pi \in K^{t_1-}, \pi(t_2) \in [0, 1]\}; \\
\xi_K(t_1, t_2) &:= \#\mathcal{M}_K(t_1, t_2),
\end{aligned}
\tag{14}
$$

i.e., $\xi_K(t_1, t_2)$ denotes the number of distinct points in $[0, 1] \times t_2$ which are on some path in K^{t_1-}. We note that for $t > 0$, $\mathcal{M}_\mathcal{W}(t_0, t_0 + t) = \mathcal{N}_\mathcal{W}(t_0, t; 0, 1)$ as defined in Sun *et al.* (2008). It is known that for all $t > 0$ the random variable $\xi_\mathcal{W}(t_0, t_0 + t)$ is finite almost surely (see (E_1) in Theorem 1.3 in Sun *et al.* (2008)) with

$$
\mathbb{E}(\xi_\mathcal{W}(t_0, t_0 + t)) = \frac{1}{\sqrt{\pi t}}.
\tag{15}
$$

Moreover, from the earlier stated properties of $(\mathcal{W}, \widehat{\mathcal{W}})$ the proof of the following Proposition is straightforward.

Proposition 2.5. *For any $t_0 < t_1$ almost surely we have*

(i) $\mathcal{M}_\mathcal{W}(t_0, t_1) \cap \mathbb{Q} = \emptyset$;

(ii) *each point in $\mathcal{M}_\mathcal{W}(t_0, t_1)$ is of type $(1, 1)$;*

(iii) *for each $x \in \mathcal{M}_\mathcal{W}(t_0, t_1)$ there exists $\pi_1, \pi_2 \in \mathcal{W}$ with $\sigma_{\pi_1} < t_0$, $\sigma_{\pi_2} > t_0$ and $\pi_1(t_1) = \pi_2(t_1) = x$;*

(iv) *for each $x \in \mathcal{M}_\mathcal{W}(t_0, t_1)$ there exist exactly two paths $\widehat{\pi}_r^{(x,t_1)}$ and $\widehat{\pi}_l^{(x,t_1)}$ in $\widehat{\mathcal{W}}$ starting from (x, t_1) with $\widehat{\pi}_r^{(x,t_1)}(t) > \widehat{\pi}_l^{(x,t_1)}(t)$ for all $[t_0, t_1)$.*

There are several ways to construct $\widehat{\mathcal{W}}$ from $\mathcal{W}$. In this paper we follow the *wedge characterization* provided by Sun *et al.* (2008). For $\pi^r, \pi^l \in \mathcal{W}$ with coalescing time t^{π^r,π^l} and $\pi^r(\max\{\sigma_{\pi^r}, \sigma_{\pi^l}\}) > \pi^l(\max\{\sigma_{\pi^r}, \sigma_{\pi^l}\})$, the wedge with right boundary π^r and left boundary π^l, is an open set in $\mathbb{R}^2$ given by

$$
A = A(\pi^r, \pi^l) := \{(y, s) : \max\{\sigma_{\pi^l}, \sigma_{\pi^r}\} < s < t^{\pi^r,\pi^l}, \pi^l(s) < y < \pi^r(s)\}.
\tag{16}
$$

A path $\widehat{\pi} \in \widehat{\Pi}$, is said to *enter the wedge A from outside* if there exist t_1 and t_2 with $\sigma_{\widehat{\pi}} > t_1 > t_2$ such that $(\widehat{\pi}(t_1), t_1) \notin \bar{A}$ and $(\widehat{\pi}(t_2), t_2) \in A$.

From Theorem 1.9 in Sun *et al.* (2008) it follows that the dual Brownian web $\widehat{\mathcal{W}}$ associated with the Brownian web $\mathcal{W}$ satisfies the following wedge characterization.

Theorem 2.6. *Let $(\mathcal{W}, \widehat{\mathcal{W}})$ be a Brownian web and its dual. Then almost surely*

$$\widehat{\mathcal{W}} = \{\widehat{\pi} : \widehat{\pi} \in \widehat{\Pi} \text{ and does not enter any wedge in } \mathcal{W} \text{ from outside}\}.$$

Because of Theorem 2.6, for a $(\mathcal{H} \times \widehat{\mathcal{H}}, \mathcal{B}_{\mathcal{H}} \times \mathcal{B}_{\widehat{\mathcal{H}}})$ valued random variable $(\mathcal{W}, \mathcal{Z})$ to show that $\mathcal{Z} = \widehat{\mathcal{W}}$, it suffices to check that $\mathcal{Z}$ satisfies the wedge condition. Here we present an alternate condition which is easier to check.

Proposition 2.7. *Let $(\mathcal{W}, \mathcal{Z})$ be a $(\mathcal{H} \times \widehat{\mathcal{H}}, \mathcal{B}_{\mathcal{H}} \times \mathcal{B}_{\widehat{\mathcal{H}}})$ valued random variable such that*

(1) *for any deterministic $(x, t) \in \mathbb{R}^2$, there exists a path $\widehat{\pi}^{(x,t)} \in \mathcal{Z}$ starting at (x, t) and going backward in time almost surely;*

(2) *paths in $\mathcal{Z}$ do not cross paths in $\mathcal{W}$ almost surely, i.e., there does not exist any $\pi \in \mathcal{W}, \widehat{\pi} \in \mathcal{Z}$ and $t_1, t_2 \in (\sigma_\pi, \sigma_{\widehat{\pi}})$ such that $(\widehat{\pi}(t_1) - \pi(t_1))(\widehat{\pi}(t_2) - \pi(t_2)) < 0$ almost surely;*

(3) *paths in $\mathcal{Z}$ and paths in $\mathcal{W}$ do not coincide over any time interval almost surely, i.e., for any $\pi \in \mathcal{W}$ and $\widehat{\pi} \in \mathcal{Z}$ and for no pair of points $t_1 < t_2$ with $\sigma_\pi \leq t_1 < t_2 \leq \sigma_{\widehat{\pi}}$ we have $\widehat{\pi}(t) = \pi(t)$ for all $t \in [t_1, t_2]$ almost surely.*

Then $\mathcal{Z} = \widehat{\mathcal{W}}$ almost surely.

Proof: From conditions (2) and (3), we have that $\widehat{\pi} \in \mathcal{Z}$ does not enter any wedge in $\mathcal{W}$ from outside. Hence $\mathcal{Z} \subseteq \widehat{\mathcal{W}}$. To show $\widehat{\mathcal{W}} \subseteq \mathcal{Z}$, we first observe that since $\mathcal{Z}$ is compact, it is enough to show that for any $\widehat{\pi} \in \widehat{\mathcal{W}}$ and $\epsilon > 0$ and finitely many time points $t_k < t_{k-1} < \cdots < t_1 < \sigma_{\widehat{\pi}}$ with $t_i \in \mathbb{Q}$, there exists $\widehat{\pi}_{\mathcal{Z}} \in \mathcal{Z}$ such that $|\widehat{\pi}_{\mathcal{Z}}(t_i) - \widehat{\pi}(t_i)| < \epsilon$ for all $i = 1, \ldots, k$.

We recall here that for any $(x, t) \in \mathbb{Q} \times \mathbb{Q}$ there exists almost surely a unique path $\pi^{(x,t)} \in \mathcal{W}$ such that the finite dimensional distributions of $\{\pi^{(x,t)} : (x, t) \in \mathbb{Q} \times \mathbb{Q}\}$ are given by that of coalescing Brownian motions. Furthermore, by assumption (1), for every (x, t) in $\mathbb{Q} \times \mathbb{Q}$, there is a path $\widehat{\pi}_{\mathcal{Z}}^{(x,t)} \in \mathcal{Z}$ almost surely.

We use ideas introduced in Sun *et al.* (2008) to create a *fish-trap* using paths of $\mathcal{W}$, which will ensure that a path of $\mathcal{Z}$ lies close to the given path. In other words, we construct two collections of paths f_{left} and f_{right} in $\mathcal{W}$, each member of f_{left} lying to the left of the dual path $\widehat{\pi}$ in $\widehat{\mathcal{W}}$ and each member of f_{right} lying to the right of $\widehat{\pi}$. Since paths in $\mathcal{Z}$ cannot cross paths in $\mathcal{W}$, the construction also ensures that any path of $\mathcal{Z}$, starting at the right of the *top-most* member of f_{left} cannot weave through the paths of f_{left} and will always remain to the right of all the paths in f_{left}. Using the fact that there is a path in $\mathcal{Z}$, starting from every point having both co-ordinates rational, we will conclude the result.

We pick a rational $x_k^{(\text{left})} \in (\widehat{\pi}(t_k) - \epsilon/2, \widehat{\pi}(t_k))$ and start a path $\pi_k^{(\text{left})} \in \mathcal{W}$ from $(x_k^{(\text{left})}, t_k)$. We have $\pi_k^{(\text{left})}(t_{k-1}) < \widehat{\pi}(t_{k-1})$ almost surely. Now, we choose a rational $x_{k-1}^{(\text{left})}$ such that $\max\{\pi_k^{(\text{left})}(t_{k-1}), \widehat{\pi}(t_{k-1}) - \epsilon/2\} < x_{k-1}^{(\text{left})} < \widehat{\pi}(t_{k-1})$ and start a path $\pi_{k-1}^{(\text{left})} \in \mathcal{W}$ from $(x_{k-1}^{(\text{left})}, t_{k-1})$. We continue this way and construct the famliy of paths

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$\{\pi_j^{(\text{left})} : j = 2, 3, \ldots, k\}$ with starting point of the j th path being $(x_j^{(\text{left})}, t_j)$ for $j = 2, 3, \ldots, k$. Clearly each of these paths stays to the left of $\widehat{\pi}$. We construct similarly another collection of paths $\{\pi_j^{(\text{right})} : j = 2, 3, \ldots, k\}$ with starting point of the j th path being $(x_j^{(\text{right})}, t_j)$ for $j = 2, 3, \ldots, k$, whose paths stay to the right of $\widehat{\pi}$. This collection of paths constitutes the fish-trap. Now, consider $x_1 \in \mathbb{Q}$ such that $\max\{\pi_2^{(\text{left})}(t_1), \widehat{\pi}(t_1) - \epsilon/2\} < x_1 < \min\{\pi_2^{(\text{right})}(t_1), \widehat{\pi}(t_1) + \epsilon/2\}$. and start a path $\widehat{\pi}_{\mathcal{Z}}^{(x_1, t_1)} \in \mathcal{Z}$ from the point (x_1, t_1). Since no paths of $\mathcal{Z}$ and $\mathcal{W}$ cross each other, on $[t_k, t_1]$ the backward path $\widehat{\pi}_{\mathcal{Z}}^{(x_1, t_1)}$ must stay in between $\{\pi_j^{(\text{left})} : j = 2, 3, \ldots, k\}$ and $\{\pi_j^{(\text{right})} : j = 2, 3, \ldots, k\}$. Therefore, we have $|\widehat{\pi}_{\mathcal{Z}}^{(x_1, t_1)}(t_i) - \widehat{\pi}(t_i)| < \epsilon$ for $i = 1, \ldots, k$. This completes the proof. $\qquad\square$

2.4 Convergence to the double Brownian web

For any $(x, t) \in V$ the path $\pi^{(x,t)}$ in the random graph $\mathcal{G}$ is obtained as the piecewise linear function $\pi^{(x,t)} : [t, \infty) \to \mathbb{R}$ with $\pi^{(x,t)}(t+k) = h^k(x, t)(1)$ for every $k \geq 0$ and $\pi^{(x,t)}$ being linear in the interval $[t+k, t+k+1]$. Similarly for $(x, t) \in \widehat{V}$, the dual path $\widehat{\pi}^{(x,t)}$ is the piecewise linear function $\widehat{\pi}^{(x,t)} : (-\infty, t] \to \mathbb{R}$ with $\widehat{\pi}^{(x,t)}(t-k) = \widehat{h}^k(x, t)(1)$ for every $k \geq 0$ and $\widehat{\pi}^{(x,t)}$ being linear in the interval $[t-k-1, t-k]$. Let $\mathcal{X} := \{\pi^{(x,t)} : (x, t) \in V\}$ and $\widehat{\mathcal{X}} := \{\widehat{\pi}^{(x,t)} : (x, t) \in \widehat{V}\}$ be the collection of all possible paths and dual paths admitted by $\mathcal{G}$ and $\widehat{\mathcal{G}}$.

For a given $\gamma > 0$ and a path π with starting time σ_π, the scaled path $\pi_n(\gamma) : [\sigma_\pi/n, \infty] \to [-\infty, \infty]$ is given by $\pi_n(\gamma)(t) = \pi(nt)/(\sqrt{n}\gamma)$ for each $n \geq 1$. Thus, the starting time of the scaled path $\pi_n(\gamma)$ is $\sigma_{\pi_n(\gamma)} = \sigma_\pi/n$. Similarly for the backward path $\widehat{\pi}$, the scaled version is $\widehat{\pi}_n(\gamma) : [-\infty, \sigma_{\widehat{\pi}}/n] \to [-\infty, \infty]$ given by $\widehat{\pi}_n(\gamma)(t) = \widehat{\pi}(nt)/(\sqrt{n}\gamma)$ for each $n \geq 1$. For each $n \geq 1$, let $\mathcal{X}_n = \mathcal{X}_n(\gamma) := \{\pi_n^{(x,t)}(\gamma) : (x, t) \in V\}$ and $\widehat{\mathcal{X}}_n = \widehat{\mathcal{X}}_n(\gamma) := \{\widehat{\pi}_n^{(x,t)}(\gamma) : (x, t) \in \widehat{V}\}$ be the collections of all the n th order diffusively scaled paths and dual paths respectively.

The closure $\overline{\mathcal{X}}_n(\gamma)$ of $\mathcal{X}_n(\gamma)$ in (Π, d_Π) and the closure $\overline{\widehat{\mathcal{X}}}_n(\gamma)$ of $\widehat{\mathcal{X}}_n(\gamma)$ in $(\widehat{\Pi}, d_{\widehat{\Pi}})$ are $(\mathcal{H}, \mathcal{B}_{\mathcal{H}})$ and $(\widehat{\mathcal{H}}, \mathcal{B}_{\widehat{\mathcal{H}}})$ valued random variables respectively. Coletti *et al.* (2009) showed that

Theorem 2.8. *For $\gamma_0 := \gamma_0(p)$ as in Theorem 1.2, as $n \to \infty$, $\overline{\mathcal{X}}_n(\gamma_0)$ converges weakly to the standard Brownian Web $\mathcal{W}$.*

Our main result of this section is the joint invariance principle for $\{(\overline{\mathcal{X}}_n(\gamma_0), \overline{\widehat{\mathcal{X}}}_n(\gamma_0)) : n \geq 1\}$ considered as $(\mathcal{H} \times \widehat{\mathcal{H}}, \mathcal{B}_{\mathcal{H}} \times \mathcal{B}_{\widehat{\mathcal{H}}})$ valued random variables.

Theorem 2.9. *$\{(\bar{\mathcal{X}}_n(\gamma_0), \bar{\widehat{\mathcal{X}}}_n(\gamma_0)) : n \geq 0\}$ converges weakly to $(\mathcal{W}, \widehat{\mathcal{W}})$ as $n \to \infty$.*

We require the following propositions to prove Theorem 2.9. We say that $\{\widehat{W}^{(x,t)}(u) : u \leq t\}$ is a Brownian motion going *back in time* if $\widehat{W}^{(x,t)}(t-s) := W(t+s)$, $s \geq 0$ where $\{W(u) : u \geq t\}$ is a Brownian motion with $W(t) = x$.

Proposition 2.10. *For any deterministic point $(x, t) \in \mathbb{R}^2$, there exists a sequence of paths $\widehat{\theta}_n^{(x,t)} \in \widehat{\mathcal{X}}_n(\gamma_0)$ which converges in distribution to $\widehat{W}^{(x,t)}$.*
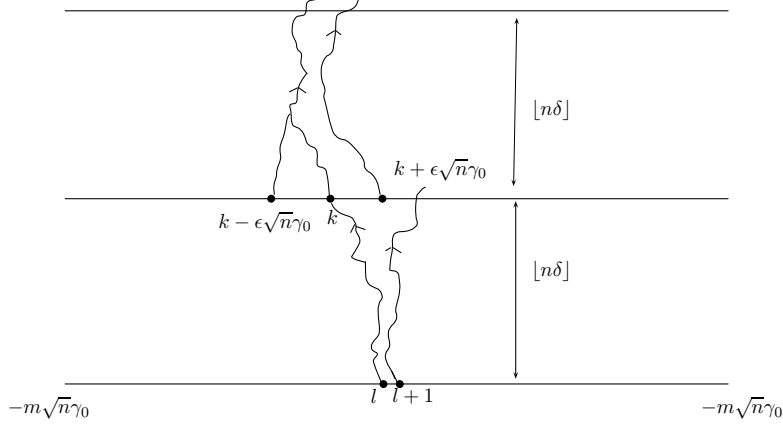
Figure 3: The vertices $(l, 0)$ and $(l + 1, 0)$ and the corresponding vertex $(k, \lfloor n\delta \rfloor)$ as required in the proof of Lemma 2.11.

Proof : For any $(x, t) \in \mathbb{R}^2$ fix $t_n = \lfloor nt \rfloor$ and $x_n = \max\{\lfloor \sqrt{n}\gamma_0 x \rfloor + j : j \leq 0, (\lfloor \sqrt{n}\gamma_0 x \rfloor + j, t_n) \in \widehat{V}\}$. Let $\widehat{\theta}_n^{(x,t)} \in \widehat{\mathcal{X}}_n(\gamma_0)$ be the scaling of the path $\widehat{\pi}^{(x_n,t_n)} \in \widehat{\mathcal{X}}$.

Since $\mathcal{G}$ is invariant under translation by lattice points and $\widehat{\mathcal{G}}$ is uniquely determined by $\mathcal{G}$, the conditional distribution of $\{(x_n, t_n) + \widehat{h}^j(0, 0) : j \geq 0\}$ given $(0, 0) \in \widehat{V}$ is the same as that of $\{\widehat{h}^j(x_n, t_n) : j \geq 0\}$. We observe that $(x_n/(\sqrt{n}\gamma_0), t_n/n) \to (x, t)$ as $n \to \infty$ almost surely. Hence, it suffices to prove that the scaled dual path starting from $(0, 0)$ given $(0, 0) \in \widehat{V}$ converges in distribution to $\widehat{W}^{(0,0)}$.

From Proposition 2.3 we see that $\widehat{X}_j^{(0,0)} = \widehat{h}^j(0, 0)(1)$ is an L^2 martingale with respect to the filtration $\sigma(\{B_{(z,s)}, U_{(z,s)} : z \in \mathbb{Z}, s \geq -k\})$. Let

$$\eta_n(u) := s_n^{-1} \big[\widehat{X}_j^{(0,0)} + (\widehat{X}_{j+1}^{(0,0)} - \widehat{X}_j^{(0,0)})(us_n^2 - s_j^2)/(s_{j+1}^2 - s_j^2) \big]$$

for $u \in [0, \infty)$ and $s_j^2 \leq us_n^2 < s_{j+1}^2$, where $s_n^2 = \sum_{j=1}^n \mathbb{E}((\widehat{X}_j^{(0,0)} - \widehat{X}_{j-1}^{(0,0)})^2)$. We know η_n converges in distribution to a standard Brownian motion (see Theorem 3, Brown (1971)). Since $s_n^2/(n\gamma_0^2) \to 1$, it can be seen that $\sup_{u \in [0,M]} |\eta_n(u) - \widehat{\theta}_n^{(0,0)}(-u)| \to 0$ in probability for any $M > 0$. So by Slutsky's theorem, we conclude that $\widehat{\theta}_n^{(0,0)}$ converges in distribution to a standard Brownian motion going backward in time. $\square$

The next result helps in estimating the probability that a direct path and a dual path stay close to each other for some time period. Given $m \in \mathbb{N}$ and $\epsilon, \delta > 0$ we define the event

$$\begin{aligned}
B_n^\epsilon = B_n^\epsilon(\delta, m) := \big\{&\text{there exist } \pi_1^n, \pi_2^n, \pi_3^n \in \mathcal{X}_n \text{ such that } \sigma_{\pi_1^n}, \sigma_{\pi_2^n} \leq 0, \sigma_{\pi_3^n} \leq \lfloor n\delta \rfloor/n, \\
&\pi_1^n(0) \in [-m, m], |\pi_1^n(0) - \pi_2^n(0)| < \epsilon, \text{ with } \pi_1^n(\lfloor n\delta \rfloor/n) \neq \pi_2^n(\lfloor n\delta \rfloor/n), \\
&\text{and } |\pi_1^n(\lfloor n\delta \rfloor/n) - \pi_3^n(\lfloor n\delta \rfloor/n)| < \epsilon, \text{ with } \pi_1^n(2\lfloor n\delta \rfloor/n) \neq \pi_3^n(2\lfloor n\delta \rfloor/n) \big\}.
\end{aligned}$$

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Lemma 2.11. *For any $m \in \mathbb{N}$ and $\epsilon, \delta > 0$, we have*

$$\mathbb{P}(B_n^\epsilon(\delta, m)) \leq C_1(\delta, m)\epsilon$$

where $C_1(\delta, m)$ is a positive constant, depending only on δ and m.

Proof : Let D_n^ϵ be the unscaled version of the event B_n^ϵ, i.e.,

$$D_n^\epsilon := \big\{ \text{there exist } (x, 0), (y, 0), (z, \lfloor n\delta \rfloor) \in V \text{ such that } x \in [-m\sqrt{n}\gamma_0, m\sqrt{n}\gamma_0],$$
$$|x - y| < \sqrt{n}\epsilon\gamma_0 \text{ and } h^{\lfloor n\delta \rfloor}(x, 0) \neq h^{\lfloor n\delta \rfloor}(y, 0),$$
$$|h^{\lfloor n\delta \rfloor}(x, 0)(1) - z| < \sqrt{n}\epsilon\gamma_0, h^{2\lfloor n\delta \rfloor}(x, 0) \neq h^{\lfloor n\delta \rfloor}(z, \lfloor n\delta \rfloor) \big\}.$$

On the event D_n^ϵ there exists $l \in [-m\sqrt{n}\gamma_0, m\sqrt{n}\gamma_0] \cap \mathbb{Z}$ such that the unscaled paths starting from $(l, 0)$ and $(l+1, 0)$ (as in Figure 2.4) do not meet in time $\lfloor n\delta \rfloor$ – an event which occurs with probability at most $C_2/\sqrt{n\delta}$ for some constant $C_2 > 0$ (see Theorem 4 of Coletti *et al.* (2009)). Supposing $h^{\lfloor n\delta \rfloor}(l, 0)(1) = k$, two unscaled paths, one starting from a vertex $\lfloor \sqrt{n}\epsilon\gamma_0 \rfloor$ distance to the left of k and the other starting from a vertex $\lfloor \sqrt{n}\epsilon\gamma_0 \rfloor$ distance to the right of k, do not meet in time $\lfloor n\delta \rfloor$ has a probability at most $C_2 2\sqrt{n}\epsilon\gamma_0/\sqrt{n\delta}$ for all $k \in \mathbb{Z}$. Thus summing over all possibilities of l and k and using the Markov property we have

$$\mathbb{P}(D_n^\epsilon) \leq \mathbb{P}(\cup_{l=-2m\sqrt{n}\gamma_0}^{2m\sqrt{n}\gamma_0} \cup_{k \in \mathbb{Z}} \{h^{\lfloor n\delta \rfloor}(l, 0)(1) = k \neq h^{\lfloor n\delta \rfloor}(l+1, 0)(1) \text{ and}$$
$$h^{\lfloor n\delta \rfloor}(k - \lfloor\sqrt{n}\epsilon\gamma_0\rfloor, \lfloor n\delta \rfloor) \neq h^{\lfloor n\delta \rfloor}(k + \lfloor\sqrt{n}\epsilon\gamma_0\rfloor, \lfloor n\delta \rfloor)\})$$
$$\leq \sum_{l=-2m\sqrt{n}\gamma_0}^{2m\sqrt{n}\gamma_0} \frac{2C_2\sqrt{n}\epsilon\gamma_0}{\sqrt{n\delta}} \sum_{k \in \mathbb{Z}} \mathbb{P}\{h^{\lfloor n\delta \rfloor}(l, 0)(1) = k \neq h^{\lfloor n\delta \rfloor}(l+1, 0)(1)\}$$
$$\leq \sum_{l=-2m\sqrt{n}\gamma_0}^{2m\sqrt{n}\gamma_0} \frac{2C_2\sqrt{n}\epsilon\gamma_0}{\sqrt{n\delta}} \mathbb{P}\{h^{\lfloor n\delta \rfloor}(l, 0)(1) \neq h^{\lfloor n\delta \rfloor}(l+1, 0)(1)\}$$
$$\leq \sum_{l=-2m\sqrt{n}\gamma_0}^{2m\sqrt{n}\gamma_0} \frac{2C_2\sqrt{n}\epsilon\gamma_0}{\sqrt{n\delta}} \frac{C_2}{\sqrt{n\delta}}$$
$$\leq C_1(\delta, m)\epsilon.$$

$\square$

Proof of Theorem 2.9: Since $\widehat{\mathcal{X}}$ consists of non-crossing paths only, Proposition 2.10 implies the tightness of the family $\{\widehat{\bar{\mathcal{X}}}_n : n \geq 1\}$ (see Proposition B.2 in the Appendix of Fontes *et al.* (2004)). The joint family $\{(\bar{\mathcal{X}}_n, \widehat{\bar{\mathcal{X}}}_n) : n \geq 1\}$ is tight since each of the two marginals is tight. To prove Theorem 2.9 it suffices to show that for any subsequential limit $(\mathcal{W}, \mathcal{Z})$ of $\{(\bar{\mathcal{X}}_n, \widehat{\bar{\mathcal{X}}}_n) : n \geq 1\}$, the random variable $\mathcal{Z}$ satisfies the conditions given in Proposition 2.7.

Consider a convergent subsequence of $\{(\bar{\mathcal{X}}_n, \widehat{\bar{\mathcal{X}}}_n) : n \geq 1\}$ such that $(\mathcal{W}, \mathcal{Z})$ is its weak limit and by Skorohod's representation theorem, we may assume that the convergence
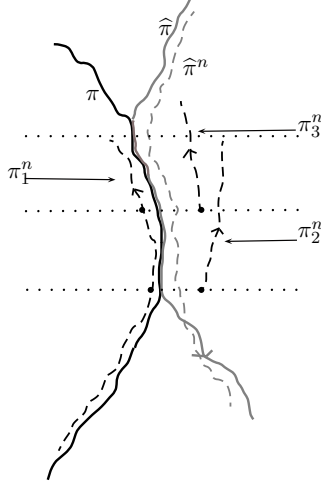
Figure 4: The event $A(\delta, m)$. The bold paths are from $(\mathcal{W}, \widehat{\mathcal{W}})$ and the approximating dashed paths are from $(\mathcal{X}_n, \widehat{\mathcal{X}}_n)$.

happens almost surely. For ease of notation, we denote the convergent subsequence by itself.

From Proposition 2.10 it follows that for any deterministic $(x, t) \in \mathbb{R}^2$ there exists a path $\widehat{\pi} \in \mathcal{Z}$ starting at (x, t) going backward in time almost surely.

Next we need to show that paths in $\mathcal{Z}$ do not cross paths in $\mathcal{W}$ almost surely. It is enough to consider paths in a compact set $[-m, m] \times [-m, m]$ for $m \in \mathbb{N}$. Now, suppose that a backward path $\widehat{\pi} \in \mathcal{Z}$ crosses a forward path $\pi \in \mathcal{W}$ in $[-m, m] \times [-m, m]$. More precisely there exist $\widehat{\pi} \in \mathcal{Z}$ and $\pi \in \mathcal{W}$ such that we have the following:

(a) $m > \sigma_{\widehat{\pi}} > \sigma_{\pi} > -m$, $-m \leq \pi(t), \widehat{\pi}(t) \leq m$ for all $t \in [\sigma_{\pi}, \sigma_{\widehat{\pi}}]$,

(b) there exist $\sigma_{\pi} < t_1 < t_2 < \sigma_{\widehat{\pi}}$ such that $(\pi(t_1) - \widehat{\pi}(t_1))(\pi(t_2) - \widehat{\pi}(t_2)) < 0$.

By continuity, we can choose $\epsilon' > 0$ so that $\left[(\pi(t_1) + u_1) - (\widehat{\pi}(t_1) + u_2)\right]\left[(\pi(t_2) + u_3) - (\widehat{\pi}(t_2) + u_4)\right] < 0$ for all $-\epsilon' < u_1, u_2, u_3, u_4 < \epsilon'$. Choose $\epsilon = \min\{(\sigma_{\widehat{\pi}} - t_2)/3, (t_1 - \sigma_{\pi})/3, \epsilon'\}$ and set $\epsilon_1 = f(\epsilon, m)$, as described in Remark 2.4.

From the almost sure convergence of $(\bar{\mathcal{X}}_n, \bar{\widehat{\mathcal{X}}}_n)$ to $(\mathcal{W}, \mathcal{Z})$, for any realization ω of the above event, we may choose $n_0 (= n_0(\omega))$ so that there exists $(\pi^{n_0}, \widehat{\pi}^{n_0}) \in \bar{\mathcal{X}}_{n_0} \times \bar{\widehat{\mathcal{X}}}_{n_0}$ with $d_{\Pi}(\pi, \pi^{n_0}) < \epsilon_1$ and $d_{\widehat{\Pi}}(\widehat{\pi}, \widehat{\pi}^{n_0}) < \epsilon_1$, which in turn implies that $\max\{\sup\{|\pi(t) - \pi^{n_0}(t)| : t \in [t_1, t_2]\}, \sup\{|\widehat{\pi}(t) - \widehat{\pi}^{n_0}(t)| : t \in [t_1, t_2]\}\} < \epsilon'$. Thus, by our choice of ϵ', we obtain that $(\pi^{n_0}(t_1) - \widehat{\pi}^{n_0}(t_1))(\pi^{n_0}(t_2) - \widehat{\pi}^{n_0}(t_2)) < 0$, i.e., the paths $(\pi^{n_0}, \widehat{\pi}^{n_0}) \in \bar{\mathcal{X}}_{n_0} \times \bar{\widehat{\mathcal{X}}}_{n_0}$ cross each other, yielding a contradiction.

Now, to prove that condition (3) in Proposition 2.7 is satisfied, we define the following

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event: for $\delta > 0$ and positive integer $m \geq 1$, let

$$A(\delta, m) := \big\{\text{there exist paths } \pi \in \mathcal{W} \text{ and } \widehat{\pi} \in \mathcal{Z} \text{ with } \sigma_\pi, \sigma_{\widehat{\pi}} \in (-m, m),$$
$$\text{and there exists } t_0 \text{ such that } \sigma_\pi < t_0 < t_0 + \delta < \sigma_{\widehat{\pi}},$$
$$\text{and } -m < \pi(t) = \widehat{\pi}(t) < m \text{ for all } t \in [t_0, t_0 + \delta]\big\}.$$

It is enough to show that for any fixed $\delta > 0$ and for $m \geq 1$, we have $\mathbb{P}\big(A(\delta, m)\big) = 0$. For any $0 < \epsilon < 1$, we also define

$$A^\epsilon(\delta, m) := \big\{\text{there exist paths } \pi \in \mathcal{W} \text{ and } \widehat{\pi} \in \mathcal{Z} \text{ with } \sigma_\pi, \sigma_{\widehat{\pi}} \in (-m, m), \text{ and}$$
$$\text{there exists } t_0 \text{ such that } \sigma_\pi < t_0 < t_0 + \delta < \sigma_{\widehat{\pi}} \text{ and } \pi(t), \widehat{\pi}(t) \in (-m, m)$$
$$\text{for } t \in [t_0, t_0 + \delta] \text{ and } \sup\{|\pi(t) - \widehat{\pi}(t)| : t \in [t_0, t_0 + \delta]\} < \epsilon\big\}.$$

Clearly, we have $A(\delta, m) \subseteq \cap_{\epsilon > 0} A^\epsilon(\delta, m)$. Further, we have $A^\epsilon(\delta, m)$ is decreasing in ϵ, so that $\mathbb{P}(A(\delta, m)) \leq \lim_{\epsilon \downarrow 0} \mathbb{P}(A^\epsilon(\delta, m))$.

Now, for every $n \geq 1$ and $j \geq 1$, set $h_n = \lfloor n\delta/3 \rfloor / n$ and $t_n^j = -m + jh_n$. Let

$$B_n^\epsilon(\delta, m; j) := \big\{\text{there exist } \pi_1^n, \pi_2^n, \pi_3^n \in \mathcal{X}_n \text{ such that } \sigma_{\pi_1^n}, \sigma_{\pi_2^n} \leq t_n^j, \sigma_{\pi_3^n} \leq t_n^{j+1},$$
$$\pi_1^n(t_n^j) \in [-2m, 2m], |\pi_1^n(t_n^j) - \pi_2^n(t_n^j)| < 4\epsilon, \text{ with } \pi_1^n(t_n^{j+1}) \neq \pi_2^n(t_n^{j+1})$$
$$\text{and } |\pi_1^n(t_n^{j+1}) - \pi_3^n(t_n^{j+1})| < 4\epsilon, \text{ with } \pi_1^n(t_n^{j+2}) \neq \pi_3^n(t_n^{j+2})\big\}.$$

We observe that the event $B_n^\epsilon(\delta, m; j)$ is a translation of the event $B_n^\epsilon(\delta, 2m)$, considered in Lemma 2.11, where the starting points of the paths are shifted up by t_n^j, with δ and ϵ replaced by $\delta/3$ and 4ϵ respectively. Hence, by translation invariance of our model and Lemma 2.11, we have $\mathbb{P}(B_n^\epsilon(\delta, m; j)) = \mathbb{P}(B_n^{4\epsilon}(\delta/3, 2m)) \leq 4C_1(\delta/3, 2m)\epsilon$ for all $n \geq 1$. We show that $A^\epsilon(\delta, m) \subseteq \liminf_{n \to \infty} \cup_{j=1}^{\lfloor \frac{6m}{\delta} \rfloor} B_n^\epsilon(\delta, m; j)$, which implies that

$$\mathbb{P}(A(\delta, m)) \leq \lim_{\epsilon \to 0} \mathbb{P}(A^\epsilon(\delta, m)) \leq \limsup_{\epsilon \to 0} \mathbb{P}\Big(\liminf_{n \to \infty} \cup_{j=1}^{\lfloor \frac{6m}{\delta} \rfloor} B_n^\epsilon(\delta, m; j)\Big)$$

$$\leq \limsup_{\epsilon \to 0} \liminf_{n \to \infty} \sum_{j=1}^{\lfloor \frac{6m}{\delta} \rfloor} \mathbb{P}(B_n^\epsilon(\delta, m; j)) \leq \limsup_{\epsilon \to 0} \frac{6m}{\delta} 4C_1(\delta/3, m)\epsilon = 0.$$

For any realization ω in $A^\epsilon(\delta, m)$, we have $\pi \in \mathcal{W}$ and $\widehat{\pi} \in \mathcal{Z}$, with their starting times $\sigma_\pi, \sigma_{\widehat{\pi}}$ in $(-m, m)$ such that $\sigma_\pi < t_0 < t_0 + \delta < \sigma_{\widehat{\pi}}$, $\sup\{|\pi(t) - \widehat{\pi}(t)| : t \in [t_0, t_0 + \delta]\} < \epsilon$, and $-m < \pi(t), \widehat{\pi}(t) < m$ for $t \in [t_0, t_0 + \delta]$. We choose $\epsilon' = \min\{\epsilon/2, (\sigma_{\widehat{\pi}} - t_0 - \delta)/3, (t_0 - \sigma_\pi)/3\}$ and set $\epsilon_1 = f(\epsilon', m)$, as in Remark 2.4. Using the almost sure convergence of $(\chi_n, \widehat{\chi}_n)$ to $(\mathcal{W}, \mathcal{Z})$, we choose $n_0(= n_0(\omega)) > 6/\delta$ such that for all $n \geq n_0$, there exist $\pi_1^n \in \chi_n$ and $\widehat{\pi}^n \in \widehat{\chi}_n$ with $\max\{d_\Pi(\pi, \pi_1^n), d_{\widehat{\Pi}}(\widehat{\pi}, \widehat{\pi}^n)\} < \epsilon_1$. From the choice of ϵ_1 (see Remark 2.4), it is ensured that $\sigma_{\widehat{\pi}^n} > t_0 + \delta$ and $\sigma_{\pi_1^n} < t_0$ and $\sup\{|\pi_1^n(t) - \pi(t)|, |\widehat{\pi}^n(t) - \widehat{\pi}(t)| : t \in [t_0, t_0 + \delta]\} < \epsilon/2$. Therefore, we have that $\sup\{|\pi_1^n(t) - \widehat{\pi}^n(t)| : t \in [t_0, t_0 + \delta]\} < 2\epsilon$ for all $n \geq n_0$.

Now we divide the time interval $[-m, m]$ with parallel horizontal lines at $y = t_n^j$ for $j = 1, 2, \ldots$ Now, from the choice of n_0, there must exist $1 \leq j \leq \lfloor \frac{6m}{\delta} \rfloor$ such that

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the path $\pi_1^n(t)$ starts below t_n^j and $\widehat{\pi}^n(t)$ starts above t_n^{j+2} and the paths stay within 2ϵ distance of each other in the time interval $[t_n^j, t_n^{j+2}]$. Now $(\pi_1^n(t_n^j), t_n^j)$ and $(\widehat{\pi}^n(t_n^j), t_n^j)$ are scaled open vertex and scaled dual vertex respectively. If $\pi_1^n(t_n^j) < \widehat{\pi}^n(t_n^j)$, by our definition of the dual vertex, we must have at least one more scaled open vertex, say (α_n^1, t_n^j), such that $\widehat{\pi}^n(t_n^j) < \alpha_n^1 < \widehat{\pi}^n(t_n^j) + 2\epsilon$, so that $\pi_1^n(t_n^j) < \alpha_n^1 < \pi_1^n(t_n^j) + 4\epsilon$. In such a case, the scaled path π_2^n, starting from the scaled open point (α_n^1, t_n^j), will not meet the path π_1^n, at least till time point t_n^{j+1}, i.e., $\pi_1^n(t_n^{j+1}) \neq \pi_2^n(t_n^{j+1})$, since the scaled dual path $\widehat{\pi}^n$ staying in between π_1^n and π_2^n starts before t_n^{j+2}. If $\pi_2^n(t_n^{j+1}) < \pi_1^n(t_n^{j+1}) + 4\epsilon$, we take π_3^n as the continuation of π_2^n from t_n^{j+1}. Again the paths π_1^n and π_3^n will not meet before time point t_n^{j+2}. If $\pi_2^n(t_n^{j+1}) \geq \pi_1^n(t_n^{j+1}) + 4\epsilon$, using the same logic as above, there exists another scaled open vertex (α_n^2, t_n^{j+1}) such that $\pi_1^n(t_n^{j+1}) < \alpha_n^2 < \pi_1^n(t_n^{j+1}) + 4\epsilon$ and the scaled path π_3^n starting from the scaled open point (α_n^2, t_n^{j+1}) does not meet π_1^n until t_n^{j+2}. In the case $\widehat{\pi}^n(t_n^j) < \pi_1^n(t_n^j)$, similar argument holds. Therefore, the event $\cup_{j=1}^{\lfloor \frac{6m}{\delta} \rfloor} B_n^\epsilon(\delta, m; j)$ occurs. $\qquad\square$

Remark 2.12: Modifying the proof of Lemma 2.11 suitably, it can be shown that the probability of the event $\mathbb{P}(A^\epsilon(\delta, m))$ decays faster than any power of ϵ. $\qquad\square$

3   Proof of Theorem 1.2

Let $\xi := \xi_{\mathcal{W}}(0, 1)$ and $\xi_n := \xi_{\bar{\mathcal{X}}_n}(0, 1)$ be as defined in (14). The proof of Theorem 1.2 follows from the following Proposition.

Proposition 3.1. $\mathbb{E}[\xi_n] \to \mathbb{E}[\xi]$ as $n \to \infty$.

We first complete the proof of Theorem 1.2 assuming Proposition 3.1.
Proof of Theorem 1.2: Using the translation invariance of our model, we have,

$$\sqrt{n}\gamma_0 \mathbb{P}(L(0,0) > n) = \sum_{k=0}^{\lfloor \sqrt{n}\gamma_0 \rfloor} \mathbb{E}(\mathbf{1}_{\{L(k,n)>n\}}) \times \frac{\sqrt{n}\gamma_0}{\lfloor \sqrt{n}\gamma_0 \rfloor + 1}$$

$$= \mathbb{E}(\xi_n) \times \frac{\sqrt{n}\gamma_0}{\lfloor \sqrt{n}\gamma_0 \rfloor + 1} \to \mathbb{E}(\xi) = \frac{1}{\sqrt{\pi}} \text{ as } n \to \infty.$$

This proves Theorem 1.2. $\qquad\square$

Proposition 3.1 will be proved through a sequence of lemmas.

To state the next lemma we need to recall from Theorem 2.9 that $(\bar{\mathcal{X}}_n, \widehat{\bar{\mathcal{X}}}_n) \Rightarrow (\mathcal{W}, \widehat{\mathcal{W}})$ as $n \to \infty$. Using Skorohod's representation theorem we assume that we are working on a probability space where $d_{\mathcal{H}\times\widehat{\mathcal{H}}}((\bar{\mathcal{X}}_n, \widehat{\bar{\mathcal{X}}}_n), (\mathcal{W}, \widehat{\mathcal{W}})) \to 0$ almost surely as $n \to \infty$.

Lemma 3.2. For $t_1 > t_0$ we have

$$\mathbb{P}(\xi_{\bar{\mathcal{X}}_n}(t_0, t_1) \neq \xi_{\mathcal{W}}(t_0, t_1) \text{ for infinitely many } n) = 0.$$

Proof: We prove the lemma for $t_0 = 0$ and $t_1 = 1$, i.e., for $\xi_n = \xi_{\bar{\mathcal{X}}_n}(0,1)$ and $\xi_{\mathcal{W}}(0,1)$, the proof for general t_0, t_1 being similar. First we show that, for all $k \geq 0$,

$$\liminf_{n \to \infty} \mathbf{1}_{\{\xi_n \geq k\}} \geq \mathbf{1}_{\{\xi \geq k\}} \text{ almost surely.} \tag{17}$$

Indeed, for $k = 0$, both $\mathbf{1}_{\{\xi_n \geq k\}}$ and $\mathbf{1}_{\{\xi \geq k\}}$ equal 1. For $k \geq 1$, (17) follows from almost sure convergence of $(\bar{\mathcal{X}}_n, \widehat{\mathcal{X}}_n)$ to $(\mathcal{W}, \widehat{\mathcal{W}})$ and from the properties of the set $\mathcal{M}_{\mathcal{W}}(0,1)$ as described in Proposition 2.5.

To complete the proof, we need to show that $\mathbb{P}(\limsup_{n \to \infty}\{\xi_n > \xi\}) = 0$. This is equivalent to showing that $\mathbb{P}(\Omega_0^k) = 0$ for all $k \geq 0$, where

$$\Omega_0^k := \{\omega : \xi_n(\omega) > \xi(\omega) = k \text{ for infinitely many } n\}.$$

Consider $k = 0$ first. From Proposition 2.5 it follows that on the event $\xi = 0$, almost surely we can obtain $\gamma := \gamma(\omega) > 0$ such that $\mathcal{M}_{\mathcal{W}}(0,1) \cap (-\gamma, 1 + \gamma) = \emptyset$. From the almost sure convergence of $(\bar{\mathcal{X}}_n, \widehat{\mathcal{X}}_n)$ to $(\mathcal{W}, \widehat{\mathcal{W}})$, we have $\mathbb{P}(\Omega_0^0) = 0$.

For $k > 0$, on the event Ω_0^k we show a forward path $\pi \in \mathcal{W}$ coincides with a dual path $\widehat{\pi} \in \widehat{\mathcal{W}}$ for a positive time which leads to a contradiction. From Proposition 2.5, it follows that given $\eta > 0$, there exist $m_0 \in \mathbb{N}$ and $s_0 \in (1/m_0, 1)$ such that $\mathbb{P}(\xi_{\mathcal{W}}(1/m_0, 1) = \xi_{\mathcal{W}}(1/m_0, s_0) = \xi_{\mathcal{W}}(0,1) = k) > 1 - \eta$ i.e., the paths leading to any single point considered in $\mathcal{M}_{\mathcal{W}}(0,1) = \mathcal{M}_{\mathcal{W}}(1/m_0, 1)$ have coalesced before time s_0. Fix $0 < \epsilon < 1/m_0$ such that $(x - \epsilon, x + \epsilon) \subset (0,1)$ for all $x \in \mathcal{M}_{\mathcal{W}}(1/m_0, 1)$ and the ϵ-tubes around the k paths contributing to $\mathcal{M}_{\mathcal{W}}(s_0, 1)$, viz., $\pi_1(t), \ldots, \pi_k(t), t \in [s_0, 1]$, given by

$$T_\epsilon^i := \{(x,t) : \pi_i(t) - \epsilon \leq x \leq \pi_i(t) + \epsilon, s_0 \leq t \leq 1\} \text{ for } i = 1, \ldots, k,$$

are disjoint. Since we have almost sure convergence on the event Ω_0^k, there exists n_0 such that one of the k tubes must contain at least two paths, $\pi_1^{n_0}, \pi_2^{n_0}$ (say) of $\mathcal{X}_{n_0}$ which do not coalesce by time 1. From the construction of dual paths it follows that there exists at least one dual path $\widehat{\pi}^{n_0} \in \bar{\widehat{\mathcal{X}}}_{n_0}^{1+}$ lying between $\pi_1^{n_0}$ and $\pi_2^{n_0}$ for $t \in [s_0, 1]$ and hence we must have an approximating $\widehat{\pi} \in \widehat{\mathcal{W}}^{1+}$ close to $\widehat{\pi}^{n_0}$ for $t \in [s_0, 1]$. Since we have only finitely many disjoint k tubes, taking $\epsilon \to 0$ and using compactness of $\widehat{\mathcal{W}}$ we obtain that there exists $\widehat{\pi} \in \widehat{\mathcal{W}}$ such that $\widehat{\pi}(t) = \pi_i(t)$ for $t \in [s_0, 1]$ and for some $1 \leq i \leq k$. This violates property (d) of Brownian web and its dual listed earlier. Hence $\mathbb{P}(\Omega_0^k) = 0$ for all $k \geq 0$ and this completes the proof of Lemma. $\square$

Lemma 3.2 immediately gives the following corollary.

Corollary 3.2.1. *As $n \to \infty$, ξ_n converges in distribution to ξ.*

Corollary 3.2.1 alongwith the following lemma completes the proof of Proposition 3.1.

Lemma 3.3. *The family $\{\xi_n : n \in \mathbb{N}\}$ is uniformly integrable.*

Proof: For $m \in \mathbb{N}$, let $K_m = [-m, m]^2 \cap \mathbb{Z}^2$ and $\Omega_m := \{(0, 1), (0, -1), (1, 1), (1, -1)\}^{K_m}$. We assign the product probability measure $\mathbb{P}'$ whose marginals for $\mathbf{u} \in K_m$ are given by

$$\mathbb{P}'\{\zeta : \zeta(\mathbf{u}) = (a, b)\} = \begin{cases} \frac{p}{2} & \text{for } a = 1 \text{ and } b \in \{1, -1\} \\ \frac{(1-p)}{2} & \text{for } a = 0 \text{ and } b \in \{1, -1\}. \end{cases}$$

$\mathbb{P}'$ is the measure induced by the random variables $\{(B_{\mathbf{u}}, U_{\mathbf{u}}) : \mathbf{u} \in K_m\}$.

For $\zeta \in \Omega_m$ and for $K \subseteq K_m$, the K cylinder of ζ is given by $C(\zeta, K) := \{\zeta' : \zeta'(\mathbf{u}) = \zeta(\mathbf{u})$ for all $\mathbf{u} \in K\}$. For any two events $A, B \subseteq \Omega_m$, let

$$A \square B := \{\zeta : \text{ there exists } K = K(\zeta) \subseteq K_m \text{ such that } C(\zeta, K) \subseteq A,$$
$$\text{and } C(\zeta, K') \subseteq B \text{ for } K' = K_m \setminus K\}$$

denote the disjoint occurrence of A and B. Note that this definition is associative, i.e., for any $A, B, C \subseteq \Omega_m$ we have $(A \square B) \square C = A \square (B \square C)$.

Let

$$F_n^m := \{\text{there exist } (u_1, n), (u_2, n) \in \widehat{V} \text{ with } 0 \leq u_1 < u_2 \leq \sqrt{n}\gamma_0 \text{ and } (v_1^l, l), (v_2^l, l) \in V$$
$$\text{for all } 0 \leq l \leq n \text{ such that } -m \leq v_1^l < \widehat{h}^l(u_1, n)(1) < \widehat{h}^l(u_2, n)(1) < v_2^l \leq m\},$$
$$E_n^m(k) := \{\text{for } 1 \leq i \neq j \leq k, \text{ there exists } (x_i, 0) \in V \text{ with } h^n(x_i, 0)(1) \in [0, \sqrt{n}\gamma_0] \text{ and }$$
$$h^n(x_i, 0) \neq h^n(x_j, 0), \text{ and } -m \leq h^l(x_i, 0)(1) \leq m \text{ for all } 0 \leq l \leq n\}.$$

We claim that for all $k \geq 2$,

$$E_n^m(3k) \subseteq \underbrace{F_n^m \square F_n^m \square \cdots \square F_n^m}_{k \text{ times}}. \tag{18}$$

We prove it for $k = 2$. For general k, the proof is similar. Let $(u_i, n) \in \widehat{V}, 1 \leq i \leq 5$ and $(x_i, 0) \in V, 1 \leq i \leq 6$ be as in Figure 5. The region explored to obtain $\widehat{h}^j(u_i, n), 1 \leq j \leq n$ is contained within $\cup_{l=0}^{n-1}[h^l(x_i, 0)(1), h^l(x_{i+1}, 0)(1)] \times \{l\}$. Thus the regions explored to obtain the dual paths starting from $(u_1, n), (u_2, n)$ and the dual paths starting from $(u_4, n), (u_5, n)$ are disjoint (see Figure 5). Hence it follows that $E_n^m(6) \subseteq F_n^m \square F_n^m$.

Since the event $E_n^m(k)$ is monotonic in m, from (18) we get

$$\mathbb{P}(\xi_n \geq 3k) = \mathbb{P}(\lim_{m \to \infty} E_n^m(3k)) = \lim_{m \to \infty} \mathbb{P}(E_n^m(3k))$$
$$\leq \lim_{m \to \infty} \mathbb{P}(F_n^m \square \cdots \square F_n^m) = \lim_{m \to \infty} \mathbb{P}'(F_n^m \square \cdots \square F_n^m).$$

Applying BKR inequality (see Reimer (2000)) we get

$$\mathbb{P}(\xi_n \geq 3k) \leq \lim_{m \to \infty} (\mathbb{P}'(F_n^m))^k = (\mathbb{P}(\lim_{m \to \infty} F_n^m))^k = (\mathbb{P}(F_n))^k \tag{19}$$

where $F_n := \{\text{there exist } (u_1, n), (u_2, n) \in \widehat{V} \text{ with } 0 \leq u_1 < u_2 \leq \sqrt{n}\gamma_0 \text{ such that } \widehat{h}^n(u_1, n) \neq \widehat{h}^n(u_2, n)\}$.
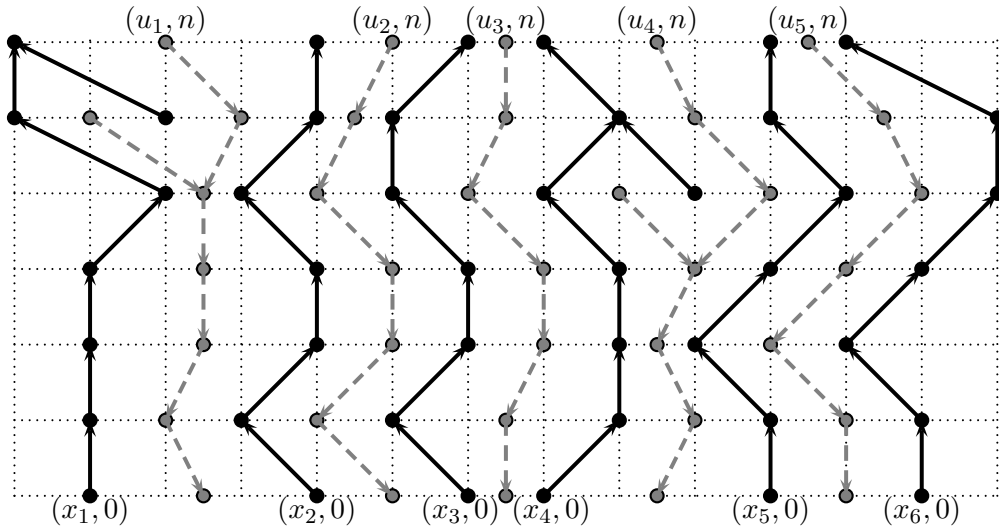
Figure 5: The event $E_n^m(6)$

For any $(x,t) \in \mathbb{R}^2$ fix $t_n = \lfloor nt \rfloor$ and $x_n = \max\{\lfloor \sqrt{n}\gamma_0 x \rfloor + j : j \leq 0, (\lfloor \sqrt{n}\gamma_0 x \rfloor + j, t_n) \in \widehat{V}\}$. Let $\widehat{\theta}_n^{(x,t)} \in \widehat{\mathcal{X}}_n(\gamma_0)$ be the scaling of the path $\widehat{\pi}^{(x_n,t_n)} \in \widehat{\mathcal{X}}$. Define

$$F_n' := \{\widehat{\theta}_n^{(0,1)} \text{ and } \widehat{\theta}_n^{(1,1)} \text{ do not coalesce in time } 1\}.$$

We observe that $F_n \subseteq F_n'$. Now $\mathbb{P}(F_n')$ converges to the probability that two independent Brownian motions starting at a distance 1 from each other do not meet by time 1. Since $\lim_{n\to\infty} \mathbb{P}(F_n') < 1$, the family $\{\xi_n : n \in \mathbb{N}\}$ is uniformly integrable. $\qquad\square$

Remark 3.4: It is to be noted that Newman *et al.* (2005) also used ideas of negative correlation to establish the weak convergence of $\mathcal{M}_\mathcal{W}$ as a point process when $\mathcal{M}_\mathcal{W}$ is not necessarily restricted to an interval. In our case since we are interested in only the cardinality of $\mathcal{M}_\mathcal{W}$ our necessity for the negative correlation ideas come in only through the BKR inequality in a much less essential manner. $\qquad\square$

4 Proofs of Theorems 1.3 and 1.4

In this section we prove Theorems 1.3 and 1.4. The main idea of the proof is that the horizontal distance between the dual paths $\widehat{\pi}^{\widehat{r}(x,t)}$ and $\widehat{\pi}^{\widehat{l}(x,t)}$ (see Figure 6) form a Brownian excursion process after scaling. The cluster $C(x,t)$ being enclosed between these two paths, its size is related to the area under the Brownian excursion.

We need to introduce some notation. For $\tau > 0$ let $S^\tau, S^{\tau^+} : C[0,\infty) \to \mathbb{R}$ be defined by $S^\tau(f) := \inf\{t \geq 0 : f(t+s) \geq f(t) \text{ for all } 0 \leq s \leq \tau\}$ and $S^{\tau^+}(f) := \inf\{t \geq 0 :$
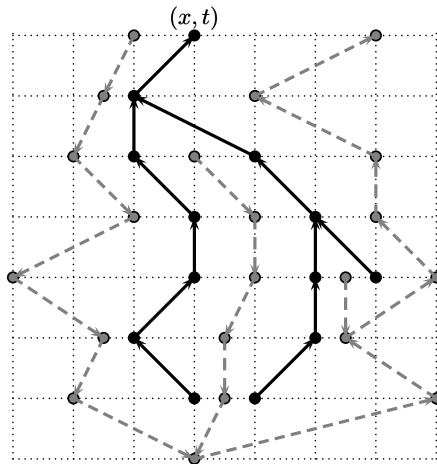
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Figure 6: The two dual paths $\widehat{\pi}^{\widehat{l}(x,t)}$ and $\widehat{\pi}^{\widehat{r}(x,t)}$ enclose the cluster $C(x,t)$. These dual paths after scaling are each Brownian paths.

$f(t+s) > f(t)$ for all $0 < s \leq \tau\}$. Let $T^{\tau^+} : C[0,\infty) \to C[0,\infty)$ be the map given by

$$T^{\tau^+}(f)(s) := \begin{cases} f(S^{\tau^+} + s) - f(S^{\tau^+}) & \text{if } S^{\tau^+} < \infty \\ f(s) & \text{otherwise.} \end{cases} \tag{20}$$

For a Brownian motion W with $W(0) = 0$ we define $W^\tau = T^{\tau^+}(W)$. From Bolthausen (1976), we have $S^{\tau^+} = S^\tau < \infty$ almost surely under the measure induced by W on $C[0,\infty)$ and $W^1|_{[0,1]} \overset{d}{=} W^+$ where W^+ is the standard Brownian meander process defined in (3). From the scaling property of Brownian motion it follows that $\{W^\tau(s) : s \in [0,\tau]\} \overset{d}{=} \{\sqrt{\tau}W^+(s/\tau) : s \in [0,\tau]\}$. Durrett *et al.* (1977) (Theorem 2.1) proved that $W|\mathbf{1}_{\{\min_{s\in[0,1]} W(s) \geq -\epsilon\}} \Rightarrow W^+$ as $\epsilon \downarrow 0$. Using this result and from the scaling property of W^τ, given above, straightforward calculations imply the following lemma and its corollary.

Lemma 4.1. *For $\tau > 0$ and W a standard Brownian motion on $[0,\infty)$ starting from 0, we have $W|\mathbf{1}_{\{\min_{t\in[0,\tau]} W(t) \geq -1/n\}} \Rightarrow W^\tau$ as $n \to \infty$.*

Proof: Let $C_b(C[0,\tau],\mathbb{R})$ be the space of all real valued bounded continuous functions on $C[0,\tau]$. Similarly $C_b(C[\tau,\infty),\mathbb{R})$ and $C_b(C[0,\infty),\mathbb{R})$ are defined. For $h_1 \in C_b(C[0,\tau],\mathbb{R})$ and $h_2 \in C_b(C[\tau,\infty),\mathbb{R})$ we define $h_1 \otimes h_2 \in C_b(C[0,\infty),\mathbb{R})$ given by $h_1 \otimes h_2(f) := h_1(f|_{[0,\tau]})h_2(f|_{[\tau,\infty)})$. Define $\mathcal{A} := \{h_1 \otimes h_2 : h_1 \in C_b(C[0,\tau],\mathbb{R}) \text{ and } h_2 \in C_b(C[\tau,\infty),\mathbb{R})\}$. By Theorem 4.5 of Ethier *et al.* (1986) $\mathcal{A}$ is a convergence determining class. Thus it suffices to show that $\mathbb{E}(h(W)|\{\min_{t\in[0,\tau]} W(t) \geq -1/n\}) \to \mathbb{E}(h(W^\tau))$ as $n \to \infty$ for $h \in \mathcal{A}$. For $x \in \mathbb{R}$ let $\mathbb{P}_x$ denote the probability measure of a Brownian motion on $[\tau,\infty)$ taking value x at time τ and $\mathbb{E}_x$ the expectation with respect to measure

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$\mathbb{P}_x$. For $h = h_1 \otimes h_2 \in \mathcal{A}$ we have

$$\mathbb{E}\big[h(W^\tau)\big]$$
$$= \mathbb{E}\big[h_1(W^\tau|_{[0,\tau]})h_2(W^\tau|_{[\tau,\infty)})\big]$$
$$= \mathbb{E}\big[h_1(W^\tau|_{[0,\tau]})h_2(W(S^\tau + t) - W(S^\tau) : t \geq \tau)\big]$$
$$= \mathbb{E}\big[h_1(W^\tau|_{[0,\tau]})\mathbb{E}(h_2(W(S^\tau + t) - W(S^\tau) : t \geq \tau)\big|\sigma(W(t) : 0 \leq t \leq S^\tau + \tau))\big]$$
$$= \mathbb{E}\big[h_1(W^\tau|_{[0,\tau]})\mathbb{E}(h_2(\{(W(S^\tau + t) - W(S^\tau + \tau)) + (W(S^\tau + \tau) - W(S^\tau)) :$$
$$t \geq \tau\})\big|\sigma(W(t) : 0 \leq t \leq S^\tau + \tau))\big]$$
$$= \mathbb{E}\big[h_1(W^\tau|_{[0,\tau]})\mathbb{E}_{W^\tau(\tau)}(h_2(\tilde{W}))\big],$$

where for $s \geq 0$, $\tilde{W}(\tau + s) := W^\tau(\tau) + W_1(s)$ and W_1 is a Brownian motion on $[0, \infty)$ independent of W^τ starting at 0. Since $S^\tau + \tau$ is a stopping time, the last equality follows from strong Markov property of Brownian motion. We now observe that

$$\mathbb{E}\big[h(W)\big| \min_{t \in [0,\tau]} W(t) \geq -1/n\big]$$
$$= \mathbb{E}\big[h_1(W|_{[0,\tau]})h_2(W|_{[\tau,\infty)})\big| \min_{t \in [0,\tau]} W(t) \geq -1/n\big]$$
$$= \mathbb{E}\big[h_1(W|_{[0,\tau]})\mathbf{1}_{\{\min_{t \in [0,\tau]} W(t) \geq -1/n\}}h_2(W|_{[\tau,\infty)})\big]\mathbb{P}(\min_{t \in [0,\tau]} W(t) \geq -1/n)^{-1}$$
$$= \mathbb{E}\big[h_1(W|_{[0,\tau]})\mathbf{1}_{\{\min_{t \in [0,\tau]} W(t) \geq -1/n\}}\mathbb{E}(h_2(W|_{[\tau,\infty)})\big|\sigma(W(t) : 0 \leq t \leq \tau))\big]$$
$$\mathbb{P}(\min_{t \in [0,\tau]} W(t) \geq -1/n)^{-1}$$
$$= \mathbb{E}\big[h_1(W|_{[0,\tau]})\mathbf{1}_{\{\min_{t \in [0,\tau]} W(t) \geq -1/n\}}\mathbb{E}_{W(\tau)}(h_2(\tilde{W}))\big]\mathbb{P}(\min_{t \in [0,\tau]} W(t) \geq -1/n)^{-1}$$
$$= \mathbb{E}\big[h_1(W|_{[0,\tau]})\mathbb{E}_{W(\tau)}(h_2(W'))\big| \min_{t \in [0,\tau]} W(t) \geq -1/n\big],$$

where $W'(\tau + s) := W(\tau) + W_2(s)$ for $s \geq 0$ and W_2 is a Brownian motion on $[0, \infty)$ independent of W starting at 0. The penultimate equality follows from Markov property. For $f \in C[0,\tau]$ the map $f \to h_1(f)\mathbb{E}_{f(\tau)}(h_2(W'))$ is continuous. From Theorem 2.1 of Durrett *et al.* (1977) and from the scaling property of W^τ it follows that $W|_{[0,\tau]}\big|\{\min_{t \in [0,\tau]} W(t) \geq -1/n\} \Rightarrow W^\tau|_{[0,\tau]}$. Hence we have

$$\mathbb{E}(h_1(W|_{[0,\tau]})\mathbb{E}_{W(\tau)}(h_2(W')))\big|\{\min_{t \in [0,\tau]} W(t) \geq -1/n\}) \to \mathbb{E}(h_1(W^\tau|_{[0,\tau]})\mathbb{E}_{W^\tau(\tau)}(h_2(\tilde{W})),$$

which completes the proof. $\qquad\square$

Define $\tilde{W}^\tau$ as the process on $C[0,\infty)$ given by

$$\tilde{W}^\tau(t) := \begin{cases} W^\tau(t) & \text{if } 0 \leq t \leq \tau \\ W^\tau(\tau) + \tilde{W}(t - \tau) & \text{otherwise} \end{cases}$$

where $\tilde{W}$ is a Brownian motion on $[0, \infty)$, independent of W^τ, with $\tilde{W}(0) = 0$. For $f \in C[0,\infty)$, let $t_f := \inf\{s > 0 : f(s) = 0\}$ with $t_f = \infty$ if $f(s) \neq 0$ for all $s > 0$.

Consider the mapping $H : C[0, \infty) \rightarrow C[0, \infty)$ given by $H(f)(t) := \mathbf{1}_{\{t \leq t_f\}} f(t)$. We define $W^{+,\tau} = H(W^\tau)$. Similar argument as that of Lemma 4.1 gives us the following corollary.

Corollary 4.1.1. *For $\tau > 0$ we have, $W^\tau \stackrel{d}{=} \tilde{W}^\tau$ and $W^{+,\tau} \stackrel{d}{=} H(\tilde{W}^\tau)$.*

Let $A \subset C[0, \infty)$ be such that

$$A := \{f \in C[0, \infty) : t_f < \infty \text{ and for every } \epsilon > 0 \text{ there exists}$$
$$s \in (t_f, t_f + \epsilon) \text{ with } f(s) < 0\}. \tag{21}$$

From Corollary 4.1.1, it follows that $\mathbb{P}(W^\tau \in A) = 1$. Hence H is continuous almost surely under the measure induced by W^τ on $C[0, \infty)$.

Next we obtain the distribution of $\int_0^\infty W^{+,\tau}(t)dt$. We first need the following lemma which is a minor modification of Lemma 2.4 in Bolthausen (1976).

Lemma 4.2. *$H \circ T^{\tau^+}$ is almost surely continuous under the measure induced by W on $C[0, \infty)$.*

Proof : We prove it for $\tau = 1$. For general $\tau > 0$ the proof is similar. Let $S^+ = S^{1^+}, T^+ = T^{1^+}$ and $S = S^1$. Since $\mathbb{P}(S^+(W) < \infty, T^+(W) \in A) = 1$, it suffices to show that S^+ is almost surely continuous. Let $f \in C[0, \infty)$ be such that $S^+(f) = S(f) < \infty$, $f(S^+(f) + 1) > f(S^+(f))$ and $T^+(f) \in A$. We first show that S^+ is a real valued measurable function on $C[0, \infty)$. This follows from the observation that for $u \geq 0$,

$$\{f : S^+(f) \leq u\} = \cup_{m \geq 1} \cap_{n \geq 1} \{f \in C[0, \infty) : \text{ there exists a rational } r \leq u \text{ such that}$$
$$i) f(r) < \min\{f(r + i/n) : 1 \leq i \leq n\},$$
$$ii) \min\{f(t) : 0 \leq t \leq r + 1/n\} < \min\{f(t) : r + 1/n \leq t \leq r + 1\} \text{ and}$$
$$iii) f(r) < f(r + 1) - 1/m\}.$$

We first show that for all $0 < \epsilon < 1$ there exists $\delta > 0$ with

$$S^+(f') \leq S^+(f) + \delta \text{ whenever } f' \text{ is such that } \sup\{|f(t) - f'(t)| : 0 \leq t \leq S^+(f) + 2\} < \delta.$$

We observe that there exists $\theta < \epsilon$ such that $f(t) > f(S^+)$ for all $t \in [S^+(f) + 1, S^+(f) + 1 + \theta]$. Choose $\delta = \inf\{f(t) - f(S^+(f)) : t \in [S^+(f) + \theta, S^+(f) + 1 + \theta]\}/3 > 0$. For $\sup\{|f(t) - f'(t)| : 0 \leq t \leq S^+(f) + 2\} < \delta$ we have that $S^+(f') \leq t' \leq S^+(f) + \theta$ where $t' = \sup\{t \in [S^+(f), S^+(f) + \theta] : f'(t) \leq f(S^+(f)) + \delta\} = \sup\{t \in [S^+(f), S^+(f) + 1 + \theta] : f'(t) \leq f(S^+(f)) + \delta\}$ (see Figure 7).

From the earlier arguments it follows that any sequence $\{S^+(f_n) : n \in \mathbb{N}\}$ such that $\lim_{n \rightarrow \infty} \sup\{|f(t) - f_n(t)| : 0 \leq t \leq S^+(f) + 2\} = 0$, has a convergent subsequence $\{S^+(f_{n_k}) : n_k \in \mathbb{N}\}$. To prove the other inequality we show that for any such convergent subsequence $\{S^+(f_{n_k}) : n_k \in \mathbb{N}\}$, we have $\lim_{n_k \rightarrow \infty} S^+(f_{n_k}) \geq S^+(f)$. Note that

$$\lim_{n \rightarrow \infty} (\inf\{S^+(f') : \sup\{|f(t) - f'(t)| : 0 \leq t \leq S^+(f) + 2\} \leq 1/n\}) = \lambda \leq S^+(f) = S(f).$$
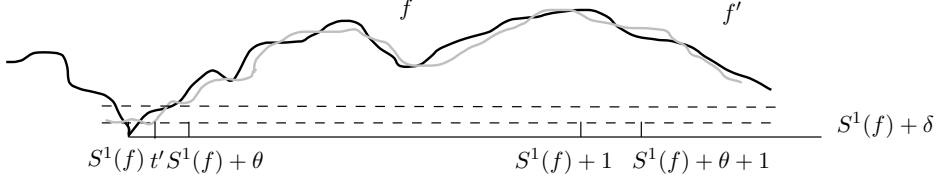
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Figure 7: The dark and light curves represent f and f' respectively.

Let $\{f_n : n \in \mathbb{N}\}$ be such that $\sup\{|f(t) - f_n(t)| : 0 \le t \le S^+(f) + 2\} \le 1/n$ and $\lim_{n \to \infty} S^+(f_n) = \lambda$. Fix $\delta > 0$. By the continuity of f and the uniform convergence of f_n to f on the interval $[0, S^+(f) + 2]$, there exists n_0 such that for $n \ge n_0$ we have

$$
\begin{aligned}
\inf\{f(t) : t \in [\lambda, \lambda + 1]\} &\ge \inf\{f(t) : t \in [S^+(f_n), S^+(f_n) + 1]\} - \delta \\
&\ge \inf\{f_n(t) : t \in [S^+(f_n), S^+(f_n) + 1]\} - 2\delta \\
&= f_n(S^+(f_n)) - 2\delta \\
&\ge f(S^+(f_n)) - 3\delta \ge f(\lambda) - 4\delta.
\end{aligned}
$$

This shows that $S(f) \le \lambda$. Since $S^+(f) = S(f)$, this completes the proof. $\square$

Let $X_1, X_2, \ldots$ be i.i.d. random variables with $\mathbb{E}(X_1) = 0$ and $V(X_1) = 1$ and $S_k := \sum_{i=1}^{k} X_i$ be the associated random walk with $S_0 = 0$. Let $T_n^{\tau^+} := \inf\{k : S_{k+i} > S_k$ for $i = 1, 2, \ldots, \lfloor n\tau \rfloor\}$. Clearly $T_n^{\tau^+} < \infty$ almost surely and we set $Z_n^k := S_{T_n^{\tau^+}+k} - S_{T_n^{\tau^+}}$ for $k \ge 0$. The following lemma and its proof is a minor modification of Lemma 3.1 in Bolthausen (1976).

Lemma 4.3. *For any $a_1, \ldots, a_m \in \mathbb{R}$ we have*

$$\mathbb{P}(S_k \le a_k \text{ for } k = 1, \ldots, m | S_k > 0 \text{ for } k = 1, \ldots, \lfloor n\tau \rfloor) = \mathbb{P}(Z_n^k \le a_k \text{ for } k = 1, \ldots, m).$$

Proof : We prove it for $\tau = 1$. For general $\tau > 0$ the proof is similar. Let $B_j := \cup_{s=0}^{j-1}\{S_s < S_r$ for $s + 1 \le r \le \min\{j, s + n\}\}$.

$$
\begin{aligned}
&\mathbb{P}(S_{T_n^{1^+}+k} - S_{T_n^{1^+}} \le a_k \text{ for } k = 1, \ldots, m) \\
&= \sum_{j=0}^{\infty} \mathbb{P}(S_{j+k} - S_j \le a_k \text{ for } k = 1, \ldots, m | T_n^{1^+} = j)\mathbb{P}(T_n^{1^+} = j) \\
&= \sum_{j=0}^{\infty} \mathbb{P}(S_{j+k} - S_j \le a_k \text{ for } k = 1, \ldots, m | S_{j+k} > S_j \text{ for } k = 1, \ldots, n, B_j^c)\mathbb{P}(T_n^{1^+} = j) \\
&= \mathbb{P}(S_k \le a_k \text{ for } k = 1, \ldots, m | S_k > 0 \text{ for } k = 1, \ldots, n) \sum_{j=0}^{\infty} \mathbb{P}(T_n^{1^+} = j) \\
&= \mathbb{P}(S_k \le a_k \text{ for } k = 1, \ldots, m | S_k > 0 \text{ for } k = 1, \ldots, n),
\end{aligned}
$$

where the last step follows from the fact that $\mathbb{P}(T_n^{1^+} < \infty) = 1$. This completes the proof. $\square$

Lemma 4.4. *For $\tau, \lambda > 0$, we have*

$$\mathbb{P}(\int_0^\infty W^{+,\tau}(t)dt > \lambda) = \frac{\sqrt{\tau}}{2} \int_\tau^\infty t^{-\frac{3}{2}} \bar{F}_{I_0^+}(\lambda t^{-\frac{3}{2}})dt.$$

Proof : We give here a straightforward proof using random walk. Let $\{S_n : n \geq 0\}$ be a symmetric random walk with variance 1 starting at $S_0 = 0$. From Lemma 4.2 we have that $H \circ T^{\tau^+}$ is almost surely continuous under the measure induced by W^τ on $C[0,\infty)$. From Donsker's invariance principle and from the continuous mapping theorem it follows that for $\lambda > 0$, a continuity point of $\int_0^\infty W^{+,\tau}(t)dt$, we have

$$\mathbb{P}(\int_0^\infty W^{+,\tau}(t)dt > \lambda) = \lim_{n\to\infty} \mathbb{P}(\int_0^\infty H(T^{\tau^+}(Y_n))(t)dt > \lambda),$$

where

$$Y_n(t) := \frac{S_k}{\sqrt{n}} + \frac{(nt - [nt])}{\sqrt{n}}(S_{k+1} - S_k) \text{ for } \frac{k}{n} \leq t < \frac{k+1}{n}. \tag{22}$$

Similar argument as in Lemma 3.1 in Bolthausen (1976) gives us

$$\mathbb{P}(\int_0^\infty H(T^{\tau^+}(Y_n))(t)dt > \lambda) = \mathbb{P}(\int_0^\infty H(Y_n)(t)dt > \lambda \big| \min_{t\in[0,\tau]} Y_n(t) \geq 0, t_0 > n\tau)$$

where $t_0 := \inf\{n > 0 : S_n = 0\}$ is the first return time to 0 of the random walk. Hence for $\lambda > 0$, a continuity point of $W^{+,\tau}$, we obtain

$$\mathbb{P}(\int_0^\infty W^{+,\tau}(t)dt > \lambda)$$

$$= \lim_{n\to\infty} \mathbb{P}(\int_0^\infty H(Y_n)(t)dt > \lambda \big| \min_{t\in[0,\tau]} Y_n(t) \geq 0, t_0 > n\tau)$$

$$= \lim_{n\to\infty} \sum_{j=1}^\infty \frac{n^{\frac{3}{2}}\mathbb{P}(t_0 = \lfloor n\tau \rfloor + j)}{n(\sqrt{n}P(t_0 > n\tau))} \mathbb{P}(\int_0^\infty H(Y_n)(t)dt > \lambda \big| \min_{t\in[0,\tau]} Y_n(t) \geq 0, t_0 = \lfloor n\tau \rfloor + j)$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{n}\mathbb{P}(t_0 > n\tau)} \int_{\lfloor n\tau \rfloor/n}^\infty g_n(t) f_n(t)dt$$

where for $t \geq \lfloor n\tau \rfloor/n$, $f_n(t) = \mathbb{P}(\int_0^\infty H(Y_n)(u)du > \lambda \big| \min_{t\in[0,\tau]} Y_n(t) \geq 0, t_0 = \lfloor nt \rfloor + 1)$ and $g_n(t) = n^{\frac{3}{2}}\mathbb{P}(t_0 = \lfloor nt \rfloor + 1)$. It is known that (see Kaigh (1976))

$$\lim_{n\to\infty} \sqrt{n}\mathbb{P}(t_0 > n) = \sqrt{\frac{2}{\pi}} \text{ and } \lim_{n\to\infty} n^{\frac{3}{2}}\mathbb{P}(t_0 = n) = \frac{1}{\sqrt{2\pi}}.$$

Hence from Theorem 2.6 Kaigh (1976) together with the continuous mapping theorem and the scaling property of the Brownian motion we have $\mathbb{P}(\int_0^\infty W^{+,\tau}(t)dt > \lambda) = \frac{\sqrt{\tau}}{2} \int_\tau^\infty t^{-\frac{3}{2}} \bar{F}_{I_0^+}(\lambda t^{-\frac{3}{2}})dt$. Finally I_0^+ being a continuous random variable (see Janson *et al.* (2007)), it follows that the random variable $\int_0^\infty W^{+,\tau}(t)dt$ is continuous. This completes the proof. $\quad\square$

4.1 Proof of Theorem 1.3

Recall that $\widehat{r}(x,t)$ and $\widehat{l}(x,t)$ denote the right and left dual neighbours, respectively, of $(x,t) \in V$. Let $\widehat{D}_k(x,t) := \widehat{h}^k(\widehat{r}(x,t))(1) - \widehat{h}^k(\widehat{l}(x,t))(1)$ where $\hat{h}$ is as defined after (11). Consider the continuous function $\widehat{D}_n^{(x,t)} \in C[0,\infty)$ given by

$$\widehat{D}_n^{(x,t)}(s) := \frac{\widehat{D}_k(x,t)}{\gamma_0 \sqrt{n}} + \frac{(ns - [ns])}{\gamma_0 \sqrt{n}}(\widehat{D}_{k+1}(x,t) - \widehat{D}_k(x,t)) \text{ for } \frac{k}{n} \leq s \leq \frac{k+1}{n}. \quad (23)$$

Fix $\tau > 0$. For an $\mathcal{H} \times \widehat{\mathcal{H}}$ valued random variable $(K, \widehat{K})$ and for $x \in \mathcal{M}_K(0,\tau)$ let $\widehat{\pi}_r^{(x,\tau)}$ be defined as

$$\widehat{\pi}_r^{(x,\tau)} := \begin{cases} \widehat{\pi} & \text{if } \sigma_{\widehat{\pi}} = \tau \text{ and there does not exist } \widehat{\pi}_1 \in \widehat{K}^{\tau+} \text{ with } x < \widehat{\pi}_1(\tau) < \widehat{\pi}(\tau) \\ \widehat{\pi}_0 & \text{otherwise} \end{cases}$$

where $\widehat{\pi}_0$ denotes the constant zero function with $\sigma_{\widehat{\pi}_0} = \tau$. In other words $\widehat{\pi}_r^{(x,\tau)} \in \widehat{K}^{\tau+}$ is such that among all $\widehat{\pi} \in \widehat{K}^{\tau+}$, $\widehat{\pi}_r^{(x,\tau)}(\tau)$ is closest to (x,τ) on the right. Similarly $\widehat{\pi}_l^{(x,\tau)}$ is defined as the path closest to (x,τ) on the left.

For $\widehat{\pi} \in \widehat{\Pi}$ with $\sigma_{\widehat{\pi}} \geq \tau$, let $g(\widehat{\pi}) \in C[0,\infty)$ be given by $g(\widehat{\pi})(t) := \widehat{\pi}(\tau - t)$ for $t \geq 0$. Fix $f \in C_b[0,\infty)$ and define

$$\kappa_{(K,\widehat{K})}(\tau, f) := \sum_{x \in \mathcal{M}_K(0,\tau)} f(g(\widehat{\pi}_r^{(x,\tau)}) - g(\widehat{\pi}_l^{(x,\tau)})).$$

For the ease of notation let $\kappa(\tau, f) := \kappa_{(\mathcal{W},\widehat{\mathcal{W}})}(\tau, f)$, and $\kappa_n(\tau, f) := \kappa_{(\bar{\mathcal{X}}_n, \bar{\widehat{\mathcal{X}}}_n)}(\tau, f)$. Comparing with the definitions introduced in (14), for $m_f = \sup\{|f(s)| : s \in [0,\infty)\}$ we have

$$\kappa(\tau, f) \leq m_f \xi_{\mathcal{W}}(0,\tau), \ \kappa_n(\tau, f) \leq m_f \xi_{\bar{\mathcal{X}}_n}(0,\tau) \text{ for all } n \geq 1. \quad (24)$$

From Proposition 2.5, we know that for each $x \in \mathcal{M}_{\mathcal{W}}(0,\tau)$, there exist $\widehat{\pi}_r^{(x,\tau)}, \widehat{\pi}_l^{(x,\tau)} \in \widehat{\mathcal{W}}$ both starting from (x,τ) with $\widehat{\pi}_r^{(x,\tau)}(0) > \widehat{\pi}_l^{(x,\tau)}(0)$.

The following lemma is the main tool for establishing Theorem 1.3 and Theorem 1.4.

Lemma 4.5. *For $\tau > 0$ and $f \in C_b[0,\infty)$ we have*

$$\lim_{n \to \infty} \mathbb{E}[\kappa_n(\tau, f)] = \mathbb{E}[\kappa(\tau, f)]. \quad (25)$$

Proof : From (24) and Lemma 3.3 it follows that the family $\{\kappa_n(\tau, f) : n \in \mathbb{N}\}$ is uniformly integrable. Hence it suffices to show that $\kappa_n(\tau, f)$ converges in distribution to $\kappa(\tau, f)$ as $n \to \infty$. We assume that we are working on a probability space such that $(\bar{\mathcal{X}}_n, \bar{\widehat{\mathcal{X}}}_n)$ converges to $(\mathcal{W}, \widehat{\mathcal{W}})$ almost surely in $(\mathcal{H} \times \widehat{\mathcal{H}}, d_{\mathcal{H} \times \widehat{\mathcal{H}}})$. From Lemma 3.2 we have $\lim_{n \to \infty} \xi_{\bar{\mathcal{X}}_n}(0,\tau) = \xi_{\mathcal{W}}(0,\tau)$ almost surely, and hence from (24) for $\xi_{\mathcal{W}}(0,\tau) = 0$, we have $\kappa_n(\tau, f) = \kappa(\tau, f) = 0$ for all n large. Next we consider the case $\xi_{\mathcal{W}}(0,\tau) = k \geq 1$. Suppose $\mathcal{M}_{\mathcal{W}}(0,\tau) = \{x_1, \ldots, x_k\}$. From Lemma 3.2 we have that $\mathcal{M}_{\bar{\mathcal{X}}_n}(0,\tau) =$

$\{x_1^n, \ldots, x_k^n\}$ for all large n and $\lim_{n\to\infty} x_i^n = x_i$ for all $1 \le i \le k$. Fix $T \ge 0$. To complete the proof it is enough to show that $\sup\{|\widehat{\pi}_r^{(x_i,\tau)}(\tau - s) - \widehat{\pi}_r^{(x_i^n,\tau)}(\tau - s)| \vee |\widehat{\pi}_l^{(x_i,\tau)}(\tau - s) - \widehat{\pi}_l^{(x_i^n,\tau)}(\tau - s)| : s \in [0, \tau + T]\} \to 0$ as $n \to \infty$ for all $1 \le i \le k$.

We observe that for $y_i \in (\widehat{\pi}_r^{(x_i,\tau)}(0), \widehat{\pi}_l^{(x_i,\tau)}(0)) \cap \mathbb{Q}$ there exists $\pi^{(y_i,0)} \in \mathcal{W}$ such that $\pi^{(y_i,0)}(\tau) = x_i$. We choose $\epsilon = \epsilon(\omega) > 0$ so that for all $1 \le i \le k$

(a) $(x_i - \epsilon, x_i + \epsilon) \subset (0,1)$, $(x_i - 2\epsilon, x_i + 2\epsilon) \cap \mathcal{M}_\mathcal{W}(0,\tau) = \{x_i\}$ and

(b) $(\widehat{\pi}_r^{(x_i,\tau)}(0) - \pi^{(y_i,0)}(0)) \wedge (\pi^{(y_i,0)}(0) - \widehat{\pi}_l^{(x_i,\tau)}(0)) > 2\epsilon$.

Let $n_0 = n_0(\omega)$ be such that, for all $n \ge n_0$,

(i) $\xi_{\bar{\mathcal{X}}_n}(0,\tau) = \xi_\mathcal{W}(0,\tau)$ and

(ii) for all $1 \le i \le k$ there exist $\widehat{\pi}_i^{1,n}, \widehat{\pi}_i^{2,n} \in \bar{\widehat{\mathcal{X}}}_n^{\tau+}$ and $\pi_i^n \in \bar{\mathcal{X}}_n^{0-}$ such that $\sup\{|\widehat{\pi}_i^{1,n}(\tau - s) - \widehat{\pi}_r^{(x_i,\tau)}(\tau - s)| \vee |\widehat{\pi}_i^{2,n}(\tau - s) - \widehat{\pi}_l^{(x_i,\tau)}(\tau - s)| \vee |\pi_i^n(\tau - s) - \pi^{(y_i,0)}(\tau - s)| : s \in [0, \tau + T]\} < \epsilon$.

The choice of n_0 ensures that $\mathcal{M}_{\bar{\mathcal{X}}_n}(0,\tau) \cap (x_i - \epsilon, x_i + \epsilon) = \{x_i^n\}$. From the choice of ϵ it follows that if either $\widehat{\pi}_r^{(x_i^n,\tau)}(0) < \widehat{\pi}_l^{(x_i,\tau)}(0) + \epsilon$ or $\widehat{\pi}_r^{(x_i^n,\tau)}(0) > \widehat{\pi}_r^{(x_i,\tau)}(0) - \epsilon$ then due to uniqueness of x_i^n we can not have $\pi_i^n \in \bar{\mathcal{X}}_n^{0-}$ approximating $\pi^{(y_i,\tau)}$. This contradicts the choice of n_0. From the earlier observation and the fact that there exist exactly two dual paths in $\widehat{\mathcal{W}}^{\tau+}$ starting from (x_i, τ), it also follows that if $|\widehat{\pi}_r^{(x_i^n,\tau)}(s) - \widehat{\pi}_r^{(x_i,\tau)}(\tau - s)| \vee |\widehat{\pi}_l^{(x_i^n,\tau)}(s) - \widehat{\pi}_l^{(x_i,\tau)}(\tau - s)| > \epsilon$ for some $s \in [0, \tau + T]$ then we can not have $\widehat{\pi}_l$ and $\widehat{\pi}_r$ both in $\widehat{\mathcal{W}}^{\tau+}$ approximating $\widehat{\pi}_l^{(x_i^n,\tau)}$ and $\widehat{\pi}_r^{(x_i^n,\tau)}$ respectively. This again contradicts the choice of n_0. Hence we must have $\widehat{\pi}_r^{(x_i^n,\tau)}(\tau - s) = \widehat{\pi}_i^{1,n}(\tau - s)$ and $\widehat{\pi}_l^{(x_i^n,\tau)}(\tau - s) = \widehat{\pi}_i^{2,n}(\tau - s)$ for all $s \in [0, \tau + T]$ for all $n \ge n_0$. This completes the proof. $\quad\square$

The next lemma calculates $\mathbb{E}[\kappa(\tau, f)]$.

Lemma 4.6. *For $\tau > 0$ and $f \in C_b[0,\infty)$ we have $\mathbb{E}[\kappa(\tau, f)] = \mathbb{E}(f(\sqrt{2}W^{+,\tau}))/\sqrt{\pi\tau}$.*

Proof : Let $I_n \subset \{0, 1, \ldots, n-1\}$ given by $I_n := \{i : 0 \le i \le n-1, \widehat{\pi}^{(i/n,\tau)}, \widehat{\pi}^{((i+1)/n,\tau)} \in \widehat{\mathcal{W}}$ such that $\widehat{\pi}^{(i/n,\tau)}(0) < \widehat{\pi}^{((i+1)/n,\tau)}(0)\}$. We define

$$\zeta_n(\tau, f) = \sum_{i \in I_n} f(g(\widehat{\pi}^{((i+1)/n,\tau)} - \widehat{\pi}^{(i/n,\tau)})).$$

From Proposition 2.5 we know $\mathcal{M}_\mathcal{W}(0,\tau) \cap \mathbb{Q} = \emptyset$. For each $x \in \mathcal{M}_\mathcal{W}(0,\tau)$, set $l_n^x = \lfloor nx \rfloor/n$ and $r_n^x = l_n^x + (1/n)$. Since there are exactly two dual paths $\widehat{\pi}_r^{(x,\tau)}$ and $\widehat{\pi}_l^{(x,\tau)}$ starting from (x, τ) with $\widehat{\pi}_r^{(x,\tau)}(0) > \widehat{\pi}_l^{(x,\tau)}(0)$, from Proposition 3.2 (e) of Sun *et al.* (2008) it follows that $\{\widehat{\pi}^{(l_n^x,\tau)} : n \in \mathbb{N}\}$ and $\{\widehat{\pi}^{(r_n^x,\tau)} : n \in \mathbb{N}\}$ converge to $\widehat{\pi}_l^{(x,\tau)}$ and $\widehat{\pi}_r^{(x,\tau)}$ respectively in $(\widehat{\Pi}, d_{\widehat{\Pi}})$ as $n \to \infty$. Hence $\zeta_n(\tau, f) \to \kappa(\tau, f)$ almost surely as $n \to \infty$. For each $i \in I_n$ there exist $y_i \in (\widehat{\pi}^{(i/n,\tau)}(0), \widehat{\pi}^{((i+1)/n,\tau)}(0)) \cap \mathbb{Q}$ and $\pi^{(y_i,0)} \in \mathcal{W}$ such that $\pi^{(y_i,0)}(\tau) \in \mathcal{M}_\mathcal{W}(0,\tau)$. Hence for $m_f = \sup\{|f(t)| : t \ge 0\}$ we have

$\zeta_n(\tau, f) \le m_f \xi_{\mathcal{W}}(0, \tau)$ for all n. As $\mathbb{E}[\xi_{\mathcal{W}}(0, \tau)] < \infty$, the family $\{\zeta_n(\tau, f) : n \in \mathbb{N}\}$ is uniformly integrable and hence we have $\lim_{n\to\infty} \mathbb{E}[\zeta_n(\tau, f)] = \mathbb{E}[\kappa(\tau, f)]$. From the fact that $g(\widehat{\pi}^{((i+1)/n, \tau)}) - g(\widehat{\pi}^{(i/n, \tau)}) \overset{d}{=} H(1/n + \sqrt{2}W)$ where W denotes the standard Brownian motion on $[0, \infty)$, we have

$$\lim_{n\to\infty} \mathbb{E}[\zeta_n(\tau, f)]$$
$$= \lim_{n\to\infty} n\mathbb{E}[f(H(1/n + \sqrt{2}W))|1/n + \min_{t\in[0,\tau]} \sqrt{2}W(t) > 0]\mathbb{P}(1/n + \min_{t\in[0,\tau]} \sqrt{2}W(t) > 0)$$
$$= \lim_{n\to\infty} \mathbb{E}[f(H(1/n + \sqrt{2}W))|\min_{t\in[0,\tau]} \sqrt{2}W(t) > -1/n]n(2\Phi(1/\sqrt{2\tau}n) - 1)$$
$$= \mathbb{E}(f(\sqrt{2}W^{+,\tau}))/\sqrt{\pi\tau}.$$

where the last equality follows from Lemma 4.1, Slutsky's theorem and continuous mapping theorem. This completes the proof. $\square$

Now, to complete the proof of Theorem 1.3 we need the following lemmas.

Lemma 4.7. *For $\tau > 0$ we have $\widehat{D}_n^{(0,0)}|\mathbf{1}_{\{L(0,0)>n\tau\}} \Rightarrow \sqrt{2}W^{+,\tau}$ as $n \to \infty$.*

Proof : Using translation invariance of our model, we have

$$\mathbb{E}(f(\widehat{D}_n^{(0,0)})|\mathbf{1}_{\{L(0,0)>n\tau\}}) = \frac{\mathbb{E}[\kappa_n(\tau, f)]}{\mathbb{E}[\xi_{\bar{\mathcal{X}}_n}(0, \tau)]} \to \frac{\mathbb{E}[\kappa(\tau, f)]}{\mathbb{E}[\xi_{\mathcal{W}}(0, \tau)]} = \mathbb{E}(f(\sqrt{2}W^{+,\tau})).$$

This holds for all $f \in C_b[0, \infty)$ which completes the proof. $\square$

Lemma 4.8. *For $\tau > 0$ we have*

(a) $\sup\{|\widehat{D}_n^{(0,0)}(s) - D_n^{(0,0)}(s)| : s \ge 0\}|\mathbf{1}_{\{L(0,0)>n\tau\}} \overset{P}{\to} 0$ *as $n \to \infty$.*

(b) $\sup\{|K_n^{(0,0)}(s) - pD_n^{(0,0)}(s)| : s \ge 0\}|\mathbf{1}_{\{L(0,0)>n\tau\}} \overset{P}{\to} 0$ *as $n \to \infty$.*

Proof : For part (a) fix $0 < \alpha < 1/2$, $T \ge 0$ and we observe that

$$\mathbb{P}(\sup\{|\widehat{D}_k(0,0) - D_k(0,0)| : k \ge 0\} \ge n^\alpha, L(0,0) > n\tau)$$
$$\le \mathbb{P}(\max\{|\widehat{D}_k(0,0) - D_k(0,0)| : 0 \le k \le n(\tau + T) + 1\} \ge n^\alpha, L(0,0) > n\tau)$$
$$+ \mathbb{P}(L(0,0) > n(\tau + T)).$$

Because of Theorem 1.2 it is enough to show that $\sqrt{n}\,\mathbb{P}(\max\{|\widehat{D}_k(0,0) - D_k(0,0)| : 0 \le k \le n(\tau+T)+1\} \ge n^\alpha, L(0,0) > n\tau) \to 0$ as $n \to \infty$. Define $H_k^{(r)} = a^r(\widehat{h}^{k-1}(\widehat{r}(0,0))) - a^l(\widehat{h}^{k-1}(\widehat{r}(0,0)))$ and $H_k^{(l)} = a^r(\widehat{h}^{k-1}(\widehat{l}(0,0))) - a^l(\widehat{h}^{k-1}(\widehat{l}(0,0)))$ for $1 \le k \le n(\tau+T)+1$ where for $(x,t) \in \widehat{V}$, $a^l(x,t)$ and $a^r(x,t)$ are defined as in (11). From the construction of the dual process, we observe that $2[\widehat{D}_k(0,0) - D_k(0,0)] = H_k^{(r)} + H_k^{(l)}$ for $1 \le k \le n(\tau + T) + 1$. Thus, we have $\{\max\{|\widehat{D}_k(0,0) - D_k(0,0)| : 0 \le k \le n(\tau+T)+1\} \ge$

$n^\alpha, L(0,0) > n\tau\} \subseteq E_n \cup \{\widehat{r}(0,0) - \widehat{l}(0,0) \geq n^\alpha, (0,0) \in V\}$ where the event E_n is defined by

$$E_n := \bigcup_{k=1}^{\lfloor n(\tau+T)\rfloor+1} \{H_k^{(r)} \geq n^\alpha, (0,0) \in V\} \cup \bigcup_{k=1}^{\lfloor n(\tau+T)\rfloor+1} \{H_k^{(l)} \geq n^\alpha, (0,0) \in V\}. \quad (26)$$

For $\mathbb{P}\{H_k^{(r)} \geq n^\alpha, (0,0) \in V\}$, we have

$$\mathbb{P}\{H_k^{(r)} \geq n^\alpha, (0,0) \in V\} = \sum_{u_r \in \mathbb{Z}/2} \mathbb{P}(\widehat{h}^{k-1}(\widehat{r}(0,0)) = (u_r, -k+1), (0,0) \in V)$$

$$\times \mathbb{P}\{a^r(u_r, -k+1) - a^l(u_r, -k+1) \geq n^\alpha \mid \widehat{h}^{k-1}(\widehat{r}(0,0)) = (u_r, -k+1), (0,0) \in V\}.$$

The event $\{\widehat{h}^{k-1}(\widehat{r}(0,0)) = (u_r, -k+1), (0,0) \in V\}$ depends on $\{(B_\mathbf{u}, U_\mathbf{u}) : -k+1 \leq \mathbf{u}(2) \leq 0\}$ while, from the definition of $a^r(u_r, -k+1)$ and $a^l(u_r, -k+1)$ for $u_r \in \mathbb{Z}/2$ (see (11)), the event $\{a^r(u_r, -k+1) - a^l(u_r, -k+1) \geq n^\alpha\}$ depends only on $\{(B_\mathbf{u}, U_\mathbf{u}) : \mathbf{u}(2) = -k\}$ and hence is independent of the conditioning event. Further, we have $\mathbb{P}\{a^r(u_r, -k+1) - a^l(u_r, -k+1) \geq n^\alpha\} \leq (1-p)^{\lfloor n^\alpha \rfloor - 1}$. Similar argument holds for $\mathbb{P}\{H_k^{(l)} \geq n^\alpha, (0,0) \in V\}$. Therefore we have

$$\sqrt{n}\,\mathbb{P}(\max\{|\widehat{D}_k(0,0) - D_k(0,0)| : 0 \leq k \leq n(\tau+T)+1\} \geq n^\alpha, L(0,0) > n)$$
$$\leq \sqrt{n}\left[\mathbb{P}(E_n) + \mathbb{P}\{\widehat{r}(0,0) - \widehat{l}(0,0) \geq n^\alpha, (0,0) \in V\}\right]$$
$$\leq \sqrt{n}\left[2(n(\tau+T)+1)(1-p)^{\lfloor n^\alpha \rfloor - 1} + 2(1-p)^{\lfloor n^\alpha \rfloor - 1}\right] \to 0 \text{ as } n \to \infty.$$

This completes the proof of part (a).

For part (b), similar argument as explained in the beginning of part (a) shows that it is enough to show that for any $T \geq 0$, $\sqrt{n}\,\mathbb{P}(\max\{|\#C_k(0,0) - pD_k(0,0)| : 0 \leq k \leq n(\tau+T)+1\} \geq n^\alpha, L(0,0) > n\tau) \to 0$ as $n \to \infty$. We fix $\epsilon, \delta > 0$ and choose $m_0 > \epsilon/p$ so that $\mathbb{P}(\sup\{\sqrt{2}W^{+,\tau}(t) : t \in [0, \tau+T]\} \geq m_0) < \delta/8$. From Lemma 4.7 we have $\limsup_{n\to\infty} \mathbb{P}(\sup\{\widehat{D}_n^{(0,0)}(t) : t \in [0, \tau+T]\} \geq m_0 | L(0,0) > n\tau) \leq \delta/2$. Hence from the choice of m_0 we have that there exists n_1 such that for all $n \geq n_1$ we have $\mathbb{P}(\sup\{\widehat{D}_n^{(0,0)}(t) : t \in [0, \tau+T]\} \geq m_0 | L(0,0) > n\tau) < \delta$. Since $D_k(0,0) \leq \widehat{D}_k(0,0)$ for all $k \geq 0$, we also have $\mathbb{P}(\sup\{D_n^{(0,0)}(t) : t \in [0, \tau+T]\} \geq m_0) < \delta$ for all $n \geq n_1$. Let $A_n := \{\sup\{D_n^{(0,0)}(t) : t \in [0, \tau+T]\} \geq m_0\}$.

We have $\{\sup\{|pD_n^{(0,0)}(t) - K_n^{(0,0)}(t)| : t \in [0, \tau+T]\} \geq \epsilon\} \subseteq \Big\{\{\max\{|pD_k(0,0) - \#C_k(0,0)| : 0 \leq k \leq \lfloor n(\tau+T)\rfloor+1\} \geq \epsilon\gamma_0\sqrt{n}\} \cap A_n^c\Big\} \cup A_n$. Further, we can write $\Big\{\{\max\{|pD_k(0,0) - \#C_k(0,0)| : 0 \leq k \leq \lfloor n(\tau+T)\rfloor+1\} \geq \epsilon\gamma_0\sqrt{n}\} \cap A_n^c \cap \{L(0,0) > n\tau\}\Big\} \subseteq \bigcup_{k=1}^{\lfloor n(\tau+T)\rfloor+1} F_k$ where

$$F_k := \{|pD_k(0,0) - \#C_k(0,0)| \geq \epsilon\gamma_0\sqrt{n}, 0 \leq D_k(0,0) \leq m_0\gamma_0\sqrt{n}, (0,0) \in V\}.$$

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To compute $\mathbb{P}(F_k)$ we obtain

$$\mathbb{P}(F_k) \leq \mathbb{P}\big(|pD_k(0,0) - \#C_k(0,0)| \geq \epsilon\gamma_0\sqrt{n}, \frac{\epsilon\gamma_0\sqrt{n}}{4} \leq D_k(0,0) \leq m_0\gamma_0\sqrt{n}, (0,0) \in V\big).$$

The inequality follows from the fact that $\max\{pD_k(0,0), \#C_k(0,0)\} \leq D_k(0,0) + 1$ and hence $|pD_k(0,0) - \#C_k(0,0)| \leq 2(D_k(0,0) + 1)$. Now, we condition on the possible positions of the left dual and the right dual paths at the $(k-1)$ th step. Set $A^{(k-1)}_{u_l,u_r} = \{\widehat{h}^{k-1}(\widehat{r}(0,0)) = (u_r, -k+1), \widehat{h}^{k-1}(\widehat{l}(0,0)) = (u_l, -k+1), (0,0) \in V\}$ for $u_l, u_r \in \mathbb{Z}/2$ with $u_l \leq u_r$. Then we have

$$\mathbb{P}(F_k) = \sum_{u_l \leq u_r \in \mathbb{Z}/2} \mathbb{P}(A^{(k-1)}_{u_l,u_r})\mathbb{P}\big\{|pD_k(0,0) - \#C_k(0,0)| \geq \epsilon\gamma_0\sqrt{n},$$
$$\frac{\epsilon\gamma_0\sqrt{n}}{4} \leq D_k(0,0) \leq m_0\gamma_0\sqrt{n}, \big|A^{(k-1)}_{u_l,u_r}\big\}.$$

To compute the conditional probability we split the event by specifying the values $a^l(u_r, -k+1)$ and $a^r(u_l, -k+1)$. Let us denote $G_{i_1,i_2} := \{a^l(u_r, -k+1) = i_2, a^r(u_l, -k+1) = i_1\}$ for $i_1, i_2 \in \mathbb{Z}$. On this event G_{i_1,i_2}, we have $D_k(0,0) = \mathbf{1}_{\{i_2 > i_1\}}(i_2 - i_1)$. Further, we observe that $\#C_k(0,0) = 2 + Z_k$ where $Z_k := \#\{j : i_1 < j < i_2, (j, -k) \in V\}$. Let us denote $\Sigma = \{(i_1, i_2) \in \mathbb{Z}^2 : i_2 > i_1, \frac{\epsilon\gamma_0\sqrt{n}}{4} \leq (i_2 - i_1) \leq m_0\gamma_0\sqrt{n}\}$. Hence, we can write the conditional probability above as

$$\mathbb{P}\big\{|pD_k(0,0) - \#C_k(0,0)| \geq \epsilon\gamma_0\sqrt{n}, \frac{\epsilon\gamma_0\sqrt{n}}{4} \leq D_k(0,0) \leq m_0\gamma_0\sqrt{n}\big|A^{(k-1)}_{u_l,u_r}\big\}$$
$$\leq \sum_{(i_1,i_2)\in\Sigma} \mathbb{P}\big(G_{i_1,i_2} \cap \{|Z_k - p(i_2 - i_1 - 1)| \geq \frac{\epsilon\gamma_0\sqrt{n}}{2}\}\big|A^{(k-1)}_{u_l,u_r}\big)$$

Also we observe that $A^{(k-1)}_{u_l,u_r}$ depends on $\{(B_\mathbf{u}, U_\mathbf{u}) : -k + 1 \leq \mathbf{u}(2) \leq 0\}$ while the event G_{i_1,i_2} depends on $\{(B_\mathbf{u}, U_\mathbf{u}) : \mathbf{u}(1) \geq i_2 \text{ or } \mathbf{u}(1) \leq i_1, \mathbf{u}(2) = -k\}$, the event $\{|Z_k - p(i_2 - i_1 - 1)| \geq \frac{\epsilon\gamma_0\sqrt{n}}{2}\}$ depends on $\{(B_\mathbf{u}, U_\mathbf{u}) : i_1 < \mathbf{u}(1) < i_2, \mathbf{u}(2) = -k\}$ and Z_k follows binomial distribution with parameter $(i_2 - i_1 - 1, p)$. Hence, the events are independent. Thus, using Chernoff bound (see Theorem 4.3 Motwani *et al.* (1995)) we conclude that

$$\mathbb{P}(F_k) \leq \sum_{u_l \leq u_r \in \mathbb{Z}/2} \mathbb{P}(A^{(k-1)}_{u_l,u_r}) \sum_{(i_1,i_2)\in\Sigma} \mathbb{P}(G_{i_1,i_2}) 2\exp\big(\frac{-(\epsilon/(m_0p))^2 m_0\gamma_0\sqrt{n}p}{16}\big)$$
$$\leq 2\exp\big(-\frac{\epsilon^2\gamma_0\sqrt{n}}{16m_0p}\big),$$

and we have

$$\mathbb{P}\big(\{\max_{0 \leq k \leq \lfloor n(\tau+T)\rfloor + 1}\{|pD_k(0,0) - \#C_k(0,0)|\} \geq \epsilon\gamma_0\sqrt{n}\} \cap A^c_n \cap \{L(0,0) > n\tau\}\big)$$
$$\leq \sum_{k=0}^{\lfloor n(\tau+T)\rfloor + 1} \mathbb{P}(F_k) \leq 2(n(\tau + T) + 1)\exp\big(-\frac{\epsilon^2\gamma_0\sqrt{n}}{16m_0p}\big).$$

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Because of Theorem 1.2 we have

$$\mathbb{P}\big(\big\{\max_{0\leq j\leq\lfloor n(\tau+T)\rfloor+1}|pD_j(0,0)-\#C_j(0,0)|\geq\sqrt{n}\gamma_0\epsilon\big\}\cap A_n^c\cap\big|L(0,0)>n\tau\big)\to 0$$

as $n\to\infty$. This completes the proof. $\square$

Proof of Theorem 1.3 : We remarked that $W^1|_{[0,1]}=W^{+,1}|_{[0,1]}\stackrel{d}{=}W^+$. The proof of Theorem 1.3 follows from Lemmas 4.7 and 4.8 and Slutsky's Theorem with the choice of $\tau=1$. $\square$

4.2 Proof of Theorem 1.4

For $\lambda>0$, let $\bar{\lambda}:=\lambda^{3/2}(\gamma_0 p)^{-1}$. We show that

Lemma 4.9. *For $\tau,\lambda>0$,*

$$\lim_{n\to\infty}\sqrt{n}\mathbb{P}\big(L(0,0)>n\tau,\sum_{k=0}^{\infty}\#C_k(0,0)>(\lambda n)^{3/2}\big)$$
$$=\frac{1}{\gamma_0\sqrt{\pi\tau}}\mathbb{P}\big(\sqrt{2}\int_0^{\infty}W^{+,\tau}(t)dt>\bar{\lambda}\big)=\frac{1}{2\gamma_0\sqrt{\pi}}\int_{\tau}^{\infty}\bar{F}_{I_0^+}(\bar{\lambda}t^{-\frac{3}{2}})t^{-\frac{3}{2}}dt.$$

Proof : For $f\in C[0,\infty)$ let $I(f):=\int_0^{\infty}H(f)(t)dt$. Since $\mathbb{P}(W^{\tau}\in A)=1$ where A is defined as in (21), I is almost surely continuous under the measure induced by W^{τ} on $C[0,\infty)$. The proof follows from Theorem 1.3 (ii) and the continuous mapping theorem. $\square$

From the previous lemma we derive the following.

Corollary 4.9.1. *For $\lambda>0$, we have*

$$\lim_{n\to\infty}\sqrt{n}\mathbb{P}\big(\#C(0,0)>(\lambda n)^{3/2}\big)=\frac{1}{2\gamma_0\sqrt{\pi}}\int_0^{\infty}\bar{F}_{I_0^+}(\bar{\lambda}t^{-\frac{3}{2}})t^{-\frac{3}{2}}dt.$$

Proof : For any $\tau>0$ we have, $\mathbb{P}(\#C(0,0)>(n\lambda)^{3/2})\geq\mathbb{P}(L(0,0)>n\tau,\#C(0,0)>(n\lambda)^{3/2})$ and hence $\liminf_{n\to\infty}\sqrt{n}\mathbb{P}(\#C(0,0)>(n\lambda)^{3/2})\geq\frac{1}{2\gamma_0\sqrt{\pi}}\int_0^{\infty}\bar{F}_{I_0^+}(\bar{\lambda}t^{-\frac{3}{2}})t^{-\frac{3}{2}}dt$.

We observe that

$$\sqrt{n}\mathbb{P}(L(0,0)\leq n\tau,\#C(0,0)>(n\lambda)^{3/2})$$
$$\leq\sqrt{n}\mathbb{P}(\sum_{k=0}^{\lfloor n\tau\rfloor}\widehat{D}_k(0,0)>(n\lambda)^{3/2})$$
$$\leq\sqrt{n}\mathbb{E}[\sum_{k=0}^{\lfloor n\tau\rfloor}\widehat{D}_k(0,0)](n\lambda)^{-3/2}$$
$$=\sqrt{n}(\lfloor n\tau\rfloor+1)\mathbb{E}(\widehat{D}_0(0,0))(n\lambda)^{-3/2},$$

where we have used the fact that $\{\widehat{D}_k(0,0)=\widehat{h}^k(\widehat{r}(0,0))(1)-\widehat{h}^k(\widehat{l}(0,0))(1):k\geq 0\}$ is a martingale (see Proposition 2.3). From the earlier discussions it also follows that

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$\mathbb{E}(\widehat{D}_0(0,0)) \leq 2\mathbb{E}(G) = 2(1-p)p^{-1}$ where G is a geometric random variable. Thus $\limsup_{n\to\infty} \sqrt{n}\mathbb{P}(L(0,0) \leq n\tau, \#C(0,0) > (n\lambda)^{3/2}) = 0$ as $\tau \to 0$, which completes the proof. $\qquad\square$

Proof of Theorem 1.4 : From Lemma 6.1 of Resnick (2007) page 174, it follows that Lemma 4.9 together with Corollary 4.9.1 and Theorem 1.2 prove (6).

Fix $\tau > 0$, $\lambda > 0$. For $\alpha < 2/3$, $\delta > 0$ and for all large n, we have $\mathbb{P}(L(0,0) > n\tau, \#C(0,0) > (n\lambda)^{1/\alpha}) \leq \mathbb{P}(L(0,0) > n\tau, \#C(0,0) > (n\delta)^{3/2})$. Fix any $\epsilon > 0$ and choose $\delta = \delta(\epsilon) > 0$ so that $\frac{1}{\gamma_0\sqrt{\pi\tau}}\mathbb{P}(\sqrt{2}\int_0^\infty W^{+,\tau}(t)dt > \bar{\delta}) < \epsilon$, where $\bar{\delta} = \delta^{3/2}(\gamma_0 p)^{-1}$. From Lemma 4.9 we have $\limsup_{n\to\infty} \sqrt{n}\mathbb{P}(L(0,0) > n\tau, \#C(0,0) > (n\lambda)^{1/\alpha}) < \epsilon$.

On the other hand from property (a) of W^+ and property (g) of W^τ it follows that $\mathbb{P}(\int_0^\infty W^{+,\tau}(t)dt > 0) = 1$ for $\tau > 0$. Now for $\alpha > 2/3$ and $\delta > 0$ we have $\mathbb{P}(L(0,0) > n\tau, \#C(0,0) > (n\lambda)^{1/\alpha}) \geq \mathbb{P}(L(0,0) > n\tau, \#C(0,0) > (n\delta)^{3/2})$ for all large n. Again from Lemma 4.9 we have $\liminf_{n\to\infty} \sqrt{n}\mathbb{P}(L(0,0) > n\tau, \#C(0,0) > (n\lambda)^{1/\alpha}) \geq \frac{1}{\gamma_0\sqrt{\pi\tau}}\mathbb{P}(\sqrt{2}\int_0^\infty W^{+,\tau}(t)dt > \bar{\delta})$. Since $\limsup_{n\to\infty} \sqrt{n}\mathbb{P}(L(0,0) > n\tau, \#C(0,0) > (n\lambda)^{1/\alpha}) \leq \lim_{n\to\infty} \sqrt{n}\mathbb{P}(L(0,0) > n\tau) = \frac{1}{\gamma_0\sqrt{\pi\tau}}$, letting $\delta \to 0$, it follows that $\lim_{n\to\infty} \sqrt{n}\mathbb{P}(L(0,0) > n\tau, \#C(0,0) > (n\lambda)^{1/\alpha}) = \frac{1}{\gamma_0\sqrt{\pi\tau}}$ for $\alpha > 2/3$. This completes the proof of (8).

The argument for $(L(0,0), (D_{\max}(0,0))^{1/2})$ being similar is omitted. $\qquad\square$

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