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Circular Distributions Arising from the Möbius Transformation of Wrapped Distributions

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Abstract

Recently, Kato and Jones (2010) used the Möbius transformation of the random variable following the von Mises distribution that provides a generalization of the wrapped Cauchy distribution and contains both symmetric and asymmetric families. In this paper we study the Möbius transformation of random variables following wrapped exponential and wrapped Laplace distributions, that also results in families containing both symmetric as well as asymmetric distributions. The resulting models are used on real data and contrasted with Kato and Jones model.

1 Introduction

Circular distributions play an important role in modeling directional data which arise in various fields. Thus it is important to seek methods for generating families of circular distributions that are capable of modeling variety of angular data. A common method for generating angular distributions is known as wrapping that involves a given distribution on the real line. Wrapping the density on the real line around the circumference of a circle with unit radius results in a probability distribution on the interval $[0, 2\pi]$, known as a *wrapped distribution*. That is, given a density g on $(-\infty, \infty)$ a wrapped circular density may be defined as

$$f(\theta) = \sum_{k=-\infty}^{\infty} g(\theta + 2k\pi),\ 0 \leq \theta < 2\pi. \tag{1.1}$$

This method produces well known symmetric angular distributions such as the *wrapped-normal* given by

$$f_{WN}(\theta) = \frac{1}{\sigma\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \exp\left\{-\frac{1}{2\sigma^2}(\theta + 2k\pi - \mu)^2\right\},\ 0 \leq \theta < 2\pi, \tag{1.2}$$

and *wrapped-Cauchy* given by

$$f_{WC}(\theta) = \frac{1}{2\pi} \frac{1-\rho^2}{1+\rho^2 - 2\rho\cos(\theta - \mu)}\ ,0 \leq \theta < 2\pi, \tag{1.3}$$

where μ, σ and ρ are the parameters of the model. There are other common two-parameter symmetric distributions used in the literature, of which we will mention von Mises-Fisher distribution and Cardioid distribution whose probability density functions are given by

$$f_{VM}(\theta) = \frac{1}{2\pi \mathcal{I}_0(\kappa)} \exp\{\kappa \cos(\theta - \mu)\} \tag{1.4}$$

and

$$f_{Cd}(\theta) = \frac{1}{2\pi}\{1 + 2\rho \cos(\theta - \mu)\}, \tag{1.5}$$

respectively, κ, μ and ρ being the unknown parameters. The reader may refer to Mardia (1972), Fisher (1993) and/or Jammalmadaka and Sengupta (2001) for such distributions and other succinct topics in circular data. In order to generate asymmetric circular distributions that may be more appropriate in some applications, wrapped versions of skewed distributions have been studied in the literature. For example, Pewsey (2000), Jammalamadaka and Kozubowski (2004), Sarma et al. (2011), Roy and Adnan (2012), and Rao and Devaaraj (2013) studied wrapped versions of skew-normal, exponential & asymmetric Laplace, lognormal & Weibull, Gompertz and gamma distributions, respectively.

Another general method of generating new families of circular distributions, using Möbius transformation, has been recently put forward by Kato and Jones (2010). Inspired by the interesting discovery by McCullagh (1996), that a Möbius transformation applied to the uniform random variable on a circle produces a wrapped-Cauchy distribution, Kato and Jones (2010), used the Möbius transformation of the random variable following the von Mises distribution. In addition to giving a richer family, this family has interesting connections to the wrapped Cauchy distribution. Kato and Jones (2013) extended distributions related to the wrapped Cauchy distribution via Brownian motion. The Bessel function involved in the von Mises distribution wards off some practitioners against its use and prompts in studying alternative families. Recently Kato and Jones (2014) introduced a new family of circular distributions using the characterization of trigonometric moments.

The general form of a Möbius transformation from circle $|z|= 1$ to circle $|z'|= 1$ is given by

$$z' = a\frac{z + b}{1 + z\bar{b}} \tag{1.6}$$

where $|a|= 1, |b|< 1$. That is, the Möbius transformation from, say, $\tilde{\Theta} \to \Theta$ is given by

$$e^{\mathrm{i}\Theta} = e^{\mathrm{i}\mu} \frac{e^{\mathrm{i}\tilde{\Theta}} + re^{\mathrm{i}\nu}}{re^{\mathrm{i}(\tilde{\Theta}-\nu)} + 1} \tag{1.7}$$

for $0 \leq \mu, \nu < 2\pi, 0 \leq r < 1$. Thus for a given $\tilde{\theta}$, the corresponding θ is given by

$$\theta = g(\tilde{\theta}) = \nu + \mu + 2 \arctan \left[\frac{1-r}{1+r} \tan \left(\frac{\tilde{\theta} - \nu}{2} \right) \right]. \tag{1.8}$$

Let $\tilde{\Theta}$ follow a circular distribution denoted by the density $f_{\tilde{\Theta}}$ then the corresponding density of Θ is given by

$$f_{\Theta}(\theta) = f_{\tilde{\Theta}}(\tilde{\theta})|_{\tilde{\theta}=g^{-1}(\theta)} \; \frac{1-r^2}{1+r^2-2r\cos(\theta-\gamma)} \tag{1.9}$$

where $\gamma = \mu + \nu$ and

$$g^{-1}(\theta) = \nu + 2 \arctan \left[\frac{1+r}{1-r} \tan \left(\frac{\theta - \gamma}{2} \right) \right] \quad \mathrm{mod} \;\; 2\pi.$$

Kato and Jones (2010) start with the von Mises distribution for $\tilde{\Theta}$ with $\mu = 0$ and the corresponding f_{Θ} is obtained as a four parameter family of distributions given by the density

$$f_{KJ}(\theta) = \frac{1}{2\pi \mathcal{I}_0(\kappa)} \exp \left[\frac{\kappa\{\xi \cos(\theta - \eta) - 2r\cos\nu\}}{1+r^2-2r\cos(\theta-\gamma)} \right] \frac{1-r^2}{1+r^2-2r\cos(\theta-\gamma)} \tag{1.10}$$

where, $\gamma = \mu + \nu, \xi = \sqrt{r^4 + 2r^2\cos(2\nu) + 1}$ and $\eta = \mu + \arg\{r^2\cos(2\nu) + 1 + ir^2\sin(2\nu)\}$. The distribution obtained using the Möbius transformation on the Cardioid distribution (1.5) (with $\mu = 0$) has been studied by Wang and Shimizu (2012). The corresponding probability density is given by

$$f_{WS}(\theta) = \frac{1}{2\pi} \left[1 + 2\rho \frac{\{\xi \cos(\theta - \eta) - 2r\cos\nu\}}{1+r^2-2r\cos(\theta-\gamma)} \right] \frac{1-r^2}{1+r^2-2r\cos(\theta-\gamma)} \tag{1.11}$$

The basic objective of this paper is to generate tractable families of skewed distributions using Möbius transformation of skewed wrapped distributions. Such a possibility avoids the numerical inconveniences in using wrapped normal and von Mises distributions, the former involving the sum of an infinite series and the latter one involving computation of a Bessel function. We also note that the wrapped Cauchy family is closed under the Möbius transformation (see Kato and Jones (2010)). In this article we propose new extended versions of wrapped exponential distribution and wrapped Laplace distribution via Möbius transformation. These classes of distributions are capable of modeling symmetry as well as asymmetry.

The organization of the paper is as follows. Section 2 gives the details about the family of distributions resulting from applying the Möbius transformation on a wrapped exponential random variable where as Section 3 presents such results for the family obtained from a wrapped Laplace random variable. In Section 4, we consider two real data examples and check the adequacy of the models developed above in contrast to other families.

2 Möbius transformation of wrapped exponential distribution

Suppose that $\tilde{\Theta}$ follows the wrapped exponential distribution(Jammalamadaka and Kozubowski (2004)) with density function given by

$$f_{WE}(\tilde{\theta}) = \frac{\lambda e^{-\lambda\tilde{\theta}}}{1-e^{-2\pi\lambda}}, \ 0 \leq \tilde{\theta} < 2\pi, \lambda > 0. \tag{2.1}$$

Suppose $g_1(\theta)$ is defined as

$$g_1(\theta) = \nu + 2\arctan\left[\frac{1+r}{1-r}\tan\left(\frac{\theta-\mu-\nu}{2}\right)\right], \tag{2.2}$$

then inverse function $g^{-1}(.)$ is given by

$$g^{-1}(\theta) = g_1(\theta) \mod 2\pi = \begin{cases} g_1(\theta)+2\pi & \text{if } \nu < \pi \wedge \theta \geq \mu+\nu+\pi, \\ g_1(\theta)-2\pi & \text{if } \nu > \pi \wedge \theta \leq \mu+\nu+\pi, \\ g_1(\theta) & \text{otherwise.} \end{cases} \tag{2.3}$$

Then the distribution of Θ is given by

$$f_{MWE}(\theta) = f_{WE}(g^{-1}(\theta)) \ \frac{1-r^2}{1+r^2-2r\cos(\theta-\mu-\nu)} \tag{2.4}$$

which can be written as

$$f_{MWE}(\theta) = \begin{cases} \frac{\lambda\left(1-r^2\right)e^{-\lambda\left(\nu+2\arctan\left[\frac{1+r}{1-r}\tan\left(\frac{\theta-\mu-\nu}{2}\right)\right]+2\pi\right)}}{\left(1-e^{-2\pi\lambda}\right)(1+r^2-2r\cos(\theta-\mu-\nu))} & \text{if } \nu < \pi \wedge \theta-\mu-\nu \geq \pi \\ \frac{\lambda\left(1-r^2\right)e^{-\lambda\left(\nu+2\arctan\left[\frac{1+r}{1-r}\tan\left(\frac{\theta-\mu-\nu}{2}\right)\right]-2\pi\right)}}{\left(1-e^{-2\pi\lambda}\right)(1+r^2-2r\cos(\theta-\mu-\nu))} & \text{if } \nu > \pi \wedge \theta-\mu-\nu \leq \pi\ \cdot \\ \frac{\lambda\left(1-r^2\right)e^{-\lambda\left(\nu+2\arctan\left[\frac{1+r}{1-r}\tan\left(\frac{\theta-\mu-\nu}{2}\right)\right]\right)}}{\left(1-e^{-2\pi\lambda}\right)(1+r^2-2r\cos(\theta-\mu-\nu))} & \text{otherwise} \end{cases} \tag{2.5}$$

The class of distributions in (2.5) becomes that of wrapped exponential distributions when $r = 0$.

Remark 1. *There is an oddity in the resulting distribution in that there may be a jump in the interior of the interval* $[0, 2\pi)$ *(see Figure 1). If we consider the periods of intervals* $[2(n-1)\pi, 2n\pi), n = 1, 2, ...$ *the original pdf is periodic, however there is a jump at the end points of the intervals. This does not look that odd in the original distribution as the jump is at the end of the interval, however this jump point may be transferred to an interior point of the interval* $[0, 2\pi)$. *This may look odd, especially where an explanation of sudden jump may not be explained. Since the wrapped exponential has jumps at* $\tilde{\theta} = 2n\pi, n = 1, 2, ...,$ *it is clear from* (1.8) *that the Möbius transformation will have jumps at* $\theta = \nu + \mu + 2\arctan\left[\frac{r-1}{r+1}\tan\left(\frac{\tilde{\nu}}{2}\right)\right] + 2n\pi, n = 1, 2....$ *In order to eliminate jump in the interior of* $[0, 2\pi)$, *we may consider* $\nu = 0, \mu = 2\pi$, *that is a 2-parameter distribution that may be appropriate in some situations, see the example in the Section 4.1.*

2.1 Asymmetry of the distribution

If $f_{MWE}(\theta)$ is symmetric then, $f_{MWE}(\mu+\nu+a+\theta) - f_{MWE}(\mu+\nu+a-\theta) = 0$ for at least one $a \in [0, \pi)$. Now in order for this to hold we must have

$$\begin{aligned} & f_{MWE}(\mu+\nu+a+\theta) - f_{MWE}(\mu+\nu+a-\theta) \\ & = \frac{(\lambda\left(1-r^2\right))}{1-e^{-2\pi\lambda}} \times \left(\frac{\exp\left(-\lambda g_1^{-1}(a+\theta)\right)}{1+r^2-2r\cos(a+\theta)} - \frac{\exp\left(-\lambda g_1^{-1}(a-\theta)\right)}{1+r^2-2r\cos(a-\theta)}\right) \\ & = 0. \end{aligned} \tag{2.6}$$

In (2.6), the first term cannot be zero. The second term becomes zero only if $a = 0$ or π and $\lambda = 0$. But since $\lambda > 0$, the class of distributions in (2.5) is always asymmetric. As $\lambda \to 0$, the distribution (2.5) tends to be symmetric.

2.2 Unimodality

In this section we discuss the modality of the proposed distribution (2.5). The first derivative of $f_{MWE}(\theta)$ with respect to θ is given by

$$f'_{MWE}(\theta) = \frac{\lambda\left(1-r^2\right)\left(\lambda\left(r^2-1\right)-2r\sin(\theta-\mu-\nu)\right)e^{-\lambda\left(\nu-2\tan^{-1}\left(\frac{(r+1)\tan\left(\frac{1}{2}(\theta-\mu-\nu)\right)}{r-1}\right)\right)}}{\left(1-e^{-2\pi\lambda}\right)\left(1+r^2-2r\cos(\theta-\mu-\nu)\right)^2}.$$

Since $f'_{MWE}(\theta)=0$ gives the solution

$$\theta_0 = \mu+\nu-\arcsin\left(\frac{\lambda\left(1-r^2\right)}{2r}\right),$$

the distribution possesses an angle mode at $\theta=\theta_0$, when $0<\lambda\leq\frac{2r}{1-r^2}$ and decreasing density when $\lambda>\frac{2r}{1-r^2}$. In Figures 1, 2 and 3 we plot the probability density function (p.d.f.) of (2.5) for various parameter combinations that depicts the features of the transformed density discussed above. Note that the original wrapped exponential distribution gives only a decreasing density.

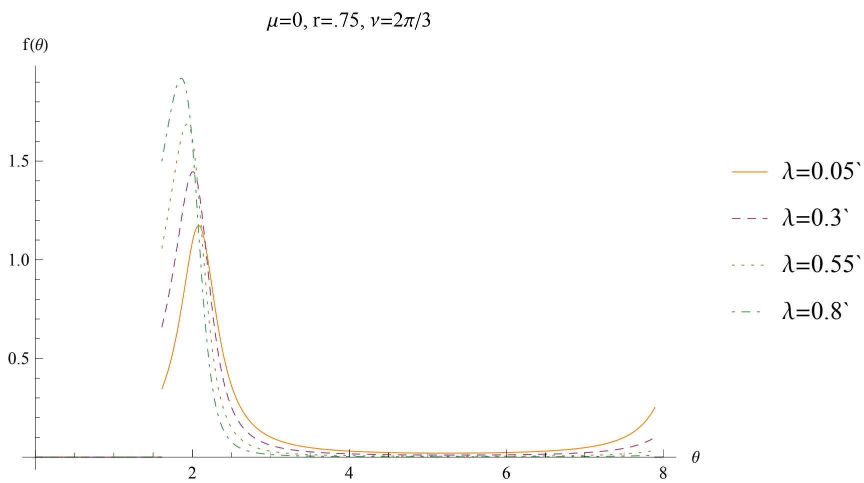

Figure 1: Plot of p.d.f. of Möbius transformed Wrapped Exponential

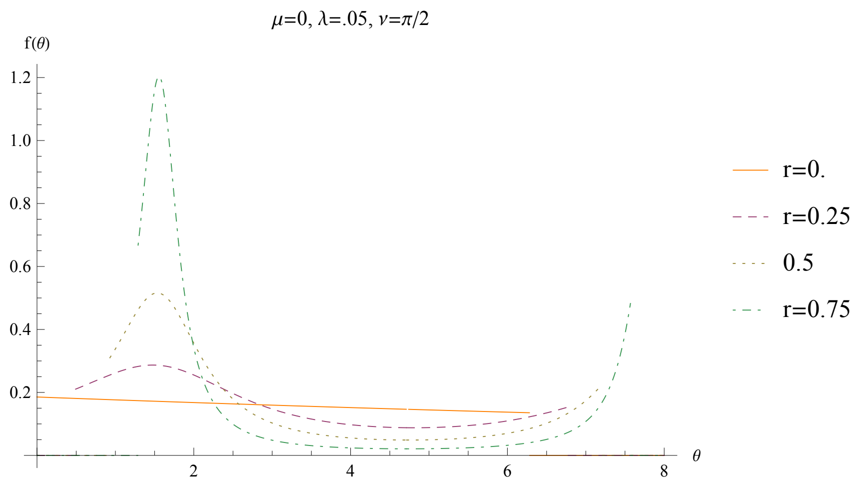

Figure 2: Plot of p.d.f of Möbius transformed Wrapped Exponential

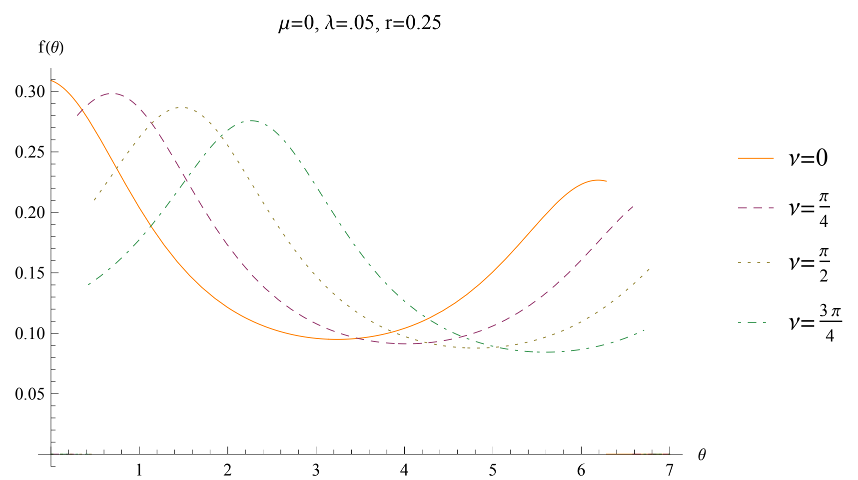

Figure 3: Plot of p.d.f of Möbius transformed Wrapped Exponential

2.3 Trigonometric moments

For the Möbius transformed wrapped exponential distribution (2.5), the n^{th} trigonometric moment is obtained from the characteristic function $\Phi_\theta(n)$ given by

$$\Phi_\theta(n) = E_\theta(e^{\mathrm{i}\, n\theta}). \tag{2.7}$$

The first trigonometric moment is given by

$$\Phi_\theta(1) = e^{i\mu}\left(\frac{\lambda\,{}_2F_1\left(1, i\lambda+1; i\lambda+2; -e^{-i\nu}\right)}{\lambda - i} + e^{i\nu} r\,{}_2F_1\left(1, i\lambda; i\lambda+1; -e^{-i\nu} r\right)\right) \tag{2.8}$$

where ${}_2F_1\left(a, b; c; d\right)$ is the generalized hyper-geometric function (Slater (1966)) . The second trigonometric moment can be expressed as

$$\Phi_\theta(2) = \frac{e^{2i(\mu+\nu)}\left(\left(r^2-1\right)\left(-i\lambda\left(r^2-1\right)+r^2+1\right)\,{}_2F_1\left(1, i\lambda; i\lambda+1; -e^{-i\nu} r\right) + \frac{r+ie^{i\nu}\left(\lambda\left(r^2-1\right)^2-i\right)}{r+e^{i\nu}}\right)}{r^2}. \tag{2.9}$$

Mean direction of the distribution (2.5) is given by $\alpha = \arg(\Phi_\theta(1))$. A contour plot of mean directions with $\mu = 0$ and $\nu = \pi$ is shown in Figure 4. Now we discuss the skewness of the distribution (2.5). A measure of skewness for circular distributions, s is defined by

$$s = \frac{E\left[\sin\left(2(\Theta - \alpha)\right)\right]}{(1-\rho)^{3/2}}, \tag{2.10}$$

where α is the mean direction and $\rho = |\Phi_\theta(1)|$ is the mean resultant length of the distribution of Θ. We plotted s against ν in Figures 5 and 6 for $\mu = 0$ and different values of r and λ. From these figures it can be seen that skewness changes significantly with the change on ν. Both figures suggest that skewness attains its maximum near $\nu = \pi$.

2.4 Maximum likelihood estimation

Now we discuss the parameter estimation of the distribution (2.5) using maximum likelihood method. Suppose $\underline{\theta} = \{\theta_1, \theta_2 \ldots \theta_n\}$ is an iid sample of size n from the distribution (2.5), then the log-likelihood function is given by

$$\log L(\lambda, \mu, r, \nu; \underline{\theta}) = \sum_{i=1}^{n} \log f_{MWE}(\theta_i).$$

Thus

$$\frac{d\log L}{d\lambda} = \frac{n}{\lambda} - \nu n - n\pi\coth(\lambda\pi) + n\pi(3I_2(\theta_i) - I_1(\theta_i) + I_3(\theta_i)) + 2\sum_{i=1}^{n}\arctan\left(\frac{1+r}{1-r}\tan\left(\frac{(\theta_i-\mu-\nu)}{2}\right)\right), \tag{2.11}$$

$$\frac{d\log L}{d\mu} = \sum_{i=1}^{n}\frac{\lambda - \lambda r^2 - 2r\sin(\mu+\nu-\theta_i)}{1+r^2-2r\cos(\mu+\nu-\theta_i)}, \tag{2.12}$$

$$\frac{d\log L}{dr} = \sum_{i=1}^{n}\frac{2\lambda\,(r^2-1)\sin(\mu+\nu-\theta_i) - 2\,(r^2+1)\cos(\mu+\nu-\theta_i) + 4r}{(r^2-1)\,(1+r^2-2r\cos(\mu+\nu-\theta_i))}, \tag{2.13}$$

and

$$\frac{d\log L}{d\nu} = \sum_{i=1}^{n} -\frac{2r(\lambda r - \lambda\cos(\mu+\nu-\theta_i) + \sin(\mu+\nu-\theta_i))}{1+r^2-2r\cos(\mu+\nu-\theta_i} \tag{2.14}$$

Where $I_1(\theta)$ denotes the indicator function of $\theta \in \nu < \pi \wedge \theta - \mu - \nu \geq \pi$, $I_2(\theta)$ denotes the indicator function of $\theta \in \nu > \pi \wedge \theta - \mu - \nu \leq \pi$ and $I_3(\theta) = (1 - I_1(\theta))(1 - I_2(\theta))$. Solving (2.11) to (2.14) simultaneously we can find the maximum likelihood estimates for the parameters. No explicit forms for the estimates are available but we can find the estimates for a particular sample directional data using numerical methods.

3 Möbius transformation of wrapped Laplace distribution

Wrapped version of asymmetric Laplace distribution as proposed by Jammalamadaka and Kozubowski (2004) that has the form given by

$$f_{WL}(\tilde{\theta}) = p\frac{\lambda_1 e^{-\lambda_1\tilde{\theta}}}{1-e^{-2\pi\lambda_1}} + (1-p)\frac{\lambda_2 e^{\lambda_2\tilde{\theta}}}{e^{2\pi\lambda_2}-1} \tag{3.1}$$

with $\lambda_1, \lambda_2 > 0$ and $0 \leq p \leq 1$. The wrapped Laplace distribution can be viewed as a mixture of a wrapped exponential and wrapped negative exponential distributions. Jammalamadaka and Kozubowski (2004) re-parametrized the distribution in the following manner, given by

$$f_{WL}(\tilde{\theta}) = \frac{\kappa\lambda}{\kappa^2+1}\left(\frac{e^{-\kappa\lambda\theta}}{1-e^{-2\pi\kappa\lambda}} + \frac{e^{\frac{\lambda}{\kappa}\theta}}{e^{\frac{2\pi\lambda}{\kappa}}-1}\right), 0 \leq \theta < 2\pi, \kappa, \lambda > 0. \tag{3.2}$$

But it can be seen that $f_{WL}(\theta)$ is a valid distribution for negative values of κ and λ. And also $f_{WL}(\theta; \kappa, \lambda) = f_{WL}(\theta; -\kappa, -\lambda)$ and $f_{WL}(\theta; -\kappa, \lambda) = f_{WL}(\theta; \kappa, -\lambda)$.

Similar to the wrapped exponential density, the density of wrapped Laplace distribution is also defined only in $[0, 2\pi)$. Let $\tilde{\Theta}$ be distributed as wrapped Laplace distribution, then the distribution of Möbius transformation of $\tilde{\Theta}$ is

$$\begin{aligned} f_{MWL}(\theta) &= f_{WL}(g^{-1}(\theta)) \frac{1 - r^2}{1 + r^2 - 2r\cos(\theta - \mu - \nu)} \\ &= \frac{\kappa\lambda(1 - r^2)}{(\kappa^2 + 1)(1 + r^2 - 2r\cos(\theta - \gamma))} \left(\frac{e^{-\kappa\lambda g^{-1}(\theta)}}{1 - e^{-2\pi\kappa\lambda}} + \frac{e^{\frac{\lambda}{\kappa} g^{-1}(\theta)}}{e^{\frac{2\pi\lambda}{\kappa}} - 1} \right), \end{aligned} \tag{3.3}$$

where $g^{-1}(\theta)$ is as defined in (2.3).

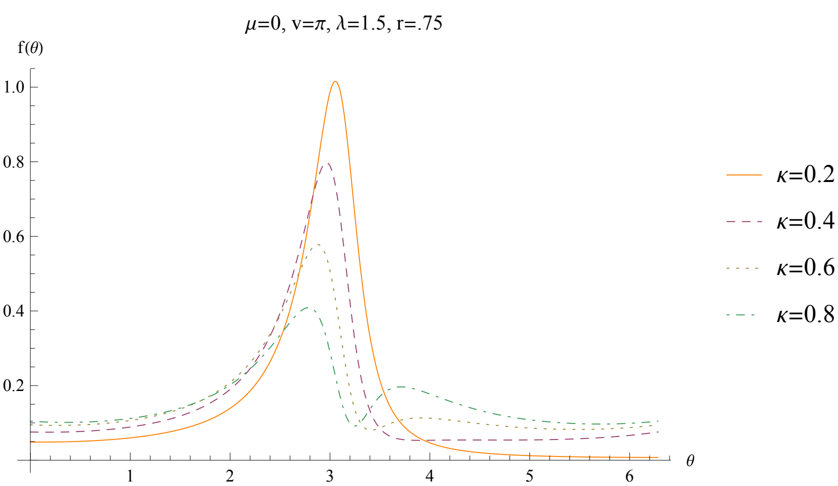

Figure 9: Plot of p.d.f of Möbius transformed Wrapped Laplace

3.1 Symmetry of the distribution

It is easily checked that the Möbius transformation of wrapped Laplace distribution (3.3) is symmetric only if $\kappa = 1$ and $\nu = \pi$.

3.2 Trigonometric moments

For the distribution in (3.3), the first trigonometric moment is given by

$$\Phi_\theta(1) = \frac{1}{(\kappa^2+1)\,r(\kappa-i\lambda)} \times e^{i\mu}\left(e^{i\nu}(\kappa-i\lambda)\left(\kappa^2 r^2\,{}_2F_1\left(1,-\frac{i\lambda}{\kappa};1-\frac{i\lambda}{\kappa};-e^{-i\nu}r\right)+\left(r^2\right.\right.\right.$$
$$\left.\left.\left.-1\right)\,{}_2F_1\left(1,i\kappa\lambda;i\kappa\lambda+1;-e^{-i\nu}r\right)+1\right)-i\kappa^2\lambda r\,{}_2F_1\left(1,1-\frac{i\lambda}{\kappa};2-\frac{i\lambda}{\kappa};-e^{-i\nu}r\right)\right), \tag{3.4}$$

and the second trigonometric moment can be expressed as

$$\begin{aligned}\Phi_\theta(2) = {} & \frac{\kappa\lambda e^{2i(\mu+\nu)}}{(\kappa^2+1)\left(r+e^{i\nu}\right)} \times \left(\frac{r^2\left((\kappa+i\lambda)\left(r+e^{i\nu}\right)\,{}_2F_1\left(1,-\frac{i\lambda}{\kappa};1-\frac{i\lambda}{\kappa};-e^{-i\nu}r\right)-i\lambda e^{i\nu}\right)}{\lambda}+\right. \\
& \frac{r^2\left((1-i\kappa\lambda)\left(r+e^{i\nu}\right)\,{}_2F_1\left(1,i\kappa\lambda;i\kappa\lambda+1;-e^{-i\nu}r\right)+i\kappa\lambda e^{i\nu}\right)}{\kappa\lambda} \\
& +\frac{2e^{-i\nu}r\left(\lambda\left(r+e^{i\nu}\right)\,{}_2F_1\left(1,1-\frac{i\lambda}{\kappa};2-\frac{i\lambda}{\kappa};-e^{-i\nu}r\right)-ie^{i\nu}(\kappa-i\lambda)\right)}{\kappa-i\lambda} \\
& +\frac{2e^{-i\nu}r\left(e^{i\nu}(1+i\kappa\lambda)-i\kappa\lambda\left(r+e^{i\nu}\right)\,{}_2F_1\left(1,i\kappa\lambda+1;i\kappa\lambda+2;-e^{-i\nu}r\right)\right)}{\kappa\lambda-i} \\
& -\frac{e^{-2i\nu}\left((\kappa-i\lambda)\left(r+e^{i\nu}\right)\,{}_2F_1\left(1,2-\frac{i\lambda}{\kappa};3-\frac{i\lambda}{\kappa};-e^{-i\nu}r\right)+e^{i\nu}(-2\kappa+i\lambda)\right)}{\lambda+2i\kappa} \\
& \left.+\frac{e^{-2i\nu}\left(e^{i\nu}(2+i\kappa\lambda)-i(\kappa\lambda-i)\left(r+e^{i\nu}\right)\,{}_2F_1\left(1,i\kappa\lambda+2;i\kappa\lambda+3;-e^{-i\nu}r\right)\right)}{\kappa\lambda-2i}\right)\end{aligned} \tag{3.5}$$

Mean direction of the distribution (3.3) is given by $\alpha = \arg(\Phi_\theta(1))$. A contour plot of mean direction with $r = 0.25$, $\mu = \pi/2$ and $\nu = \pi/2$ is shown in Figure 11. For distribution (3.3), the skewness s against ν plot in Figures 12, 13 and 14 for $\mu = 0$ and different values of r, κ and λ. In Figure 12 skewness changes sign 3 times for every values of r. But in Figure 14 skewness changes sign 5 times as λ increases. Figure 13 suggests that skewness has less variation with respect to κ.

4 Data analysis

To check the adequacy of the models discussed here, we consider two data sets that have been in the literature, the first is known as '*Ants Data*' and the second one known as the '*Wind Data*'. (see Fisher (1993) and Kato and Jones (2010) respectively) . We fit our proposed models along with those due to Kato and Jones (2010) and Wang and Shimizu (2012) to the data for visual inspection. We use maximum likelihood estimation method to find the estimates of the parameters. This is for illustration purpose only; our aim is not rigorous testing of models.

4.1 Ants data

The Ants data consist of directions chosen by 100 ants in response to an evenly illuminated black target as described in Fisher (1993), p.243. Tables 1 and 2 shows the parameter estimates, maximized log-likelihood, AIC and BIC values for different distributions. We also plot the histogram of the data with estimated densities in Figure 15. According to Table 2, Möbius transformed wrapped exponential model gives better fit compared to other distributions, according to the likelihood as well as AIC and BIC values. However, it is evident from Figure 15 that, this will not be preferred in practice due the jump discontinuity. Among other distributions MVM has an edge over the others according to the likelihood criteria, but WC has an edge over the others according to the AIC and BIC. MWL and MCd are close competitors and may be preferred over MVM due to there simpler forms.

As mentioned earlier, in order to eliminate jump in the interior of $[0, 2\pi)$, we may consider a sub distribution of (2.5), with $\nu = 0, \mu = 2\pi$. In this case the density becomes,

$$f_{MWE}(\theta) = \begin{cases} \dfrac{\lambda\left(1-r^2\right)e^{2\lambda \arctan\left(\frac{(r+1)\tan\left(\frac{\theta}{2}\right)}{r-1}\right)}}{\left(e^{2\pi\lambda}-1\right)(1+r^2-2r\cos(\theta))} & \pi \le \theta < 2\pi \\ \dfrac{\lambda\left(1-r^2\right)e^{2\lambda\left(\arctan\left(\frac{(r+1)\tan\left(\frac{\theta}{2}\right)}{r-1}\right)+\pi\right)}}{\left(e^{2\pi\lambda}-1\right)(1+r^2-2r\cos(\theta))} & 0 \le \theta < \pi \end{cases} \tag{4.1}$$

where r is redefined in $-1 < r < 1$ and $\lambda > 0$. We fit Ants data with (4.1). The parameter estimates are $\hat{\lambda} = 0.03213, \hat{r} = -0.6502$. For the fit we got, maximized log-likelihood=-131.41, AIC=266.81 and BIC=270.02. Thus (4.1) also gives better fit compared other distributions according to AIC and BIC values. We also plot histogram with fitted density in Figure 16. From Figure 16 it can be noted that, fitted distribution has no jump in the interior of $[0, 2\pi)$.

Table 1: Parameter estimated for Ants data

Distribution	ML Estimates
Mob Wr Exp	$\hat{\lambda} = 0.1446, \hat{\mu} = 4.63, \hat{\nu} = 4.86, \hat{r} = 0.72$
Mob Wr Lap	$\hat{\kappa} = 2.93, \hat{\lambda} = 0.635, \hat{\mu} = 3.987, \nu = 5.55, \hat{r} = 0.53$
Wr Cauchy	$\rho = 0.650205, \mu = 3.242$
Mob VonMis	$\kappa = 1.45966, \mu = 1.796, \nu = 2.07205, r = 0.596061$
Mob Cardioid	$r = 0.6502, \rho = 2.57 \times 10^{-7}, \nu = 0.415, \eta = 3.2415$

Table 2: Comparison of fit for Ants data

Distribution	Max Log Likelihood	AIC	BIC
Wr Cauchy	-131.57	267.15	272.36
Mob WrExp	-125.99	259.98	270.40
Mob WrLap	-130.23	270.46	283.48
Mob VonMis(Kato and Jones (2010))	-129.99	267.99	278.41
Mob Cardioid(Wang and Shimizu (2012))	-131.57	271.15	281.57

4.2 Wind data

The second example consists of n = 711 wind directions at Neuglobsow, Germany (Kato and Jones (2010)). These wind data are part of a larger data set that contains atmospheric observations from the Umweltbundesamt (German Federal Environment Agency). Table 3 shows the parameter estimates, maximized log-likelihood, AIC and BIC values for the distribution in (2.5). We also plotted the histogram of the data with estimated densities in Figure 17. From this Table 4, Möbius transformed wrapped Laplace model seems to give better fit compared to other distributions. The visual inspection of Figure 17 shows the sharp edge at the mode that is the characteristic of the Laplace distribution.

Table 3: Parameters estimated for Wind data

Distribution	ML Estimates
Mob Wr Exp	$\hat{\lambda} = 8.9 \times 10^{-9}, \hat{\mu} = 4.609, \hat{\nu} = 4.79, \hat{r} = 0.56$
Mob Wr Lap	$\hat{\kappa} = -0.457, \hat{\lambda} = 1.527, \hat{\mu} = 4.856, \nu = 0.658, \hat{r} = 0.087$
Wr Cauchy	$\rho = 0.56119, \mu = 4.60226$
Mob VonMis	$\kappa = 1.82, \mu = 3.313, \nu = 2.21, r = 0.50524$
Mob Cardioid	$r = 0.439, \rho = 0.4306, \nu = 4.713, \eta = 5.11$

Table 4: Comparison of fit for Wind data

Distribution	Max Log Likelihood	AIC	BIC
Wr Cauchy	-1049.4	2102.81	2108.02
Mob WrExp	-1050.09	2108.18	2126.44
Mob WrLap	-1002.44	2014.89	2037.72
Mob VonMis(Kato and Jones (2010))	-1006.72	2021.44	2039.71
Mob Cardioid(Wang and Shimizu (2012))	-1009.82	2027.65	2045.92

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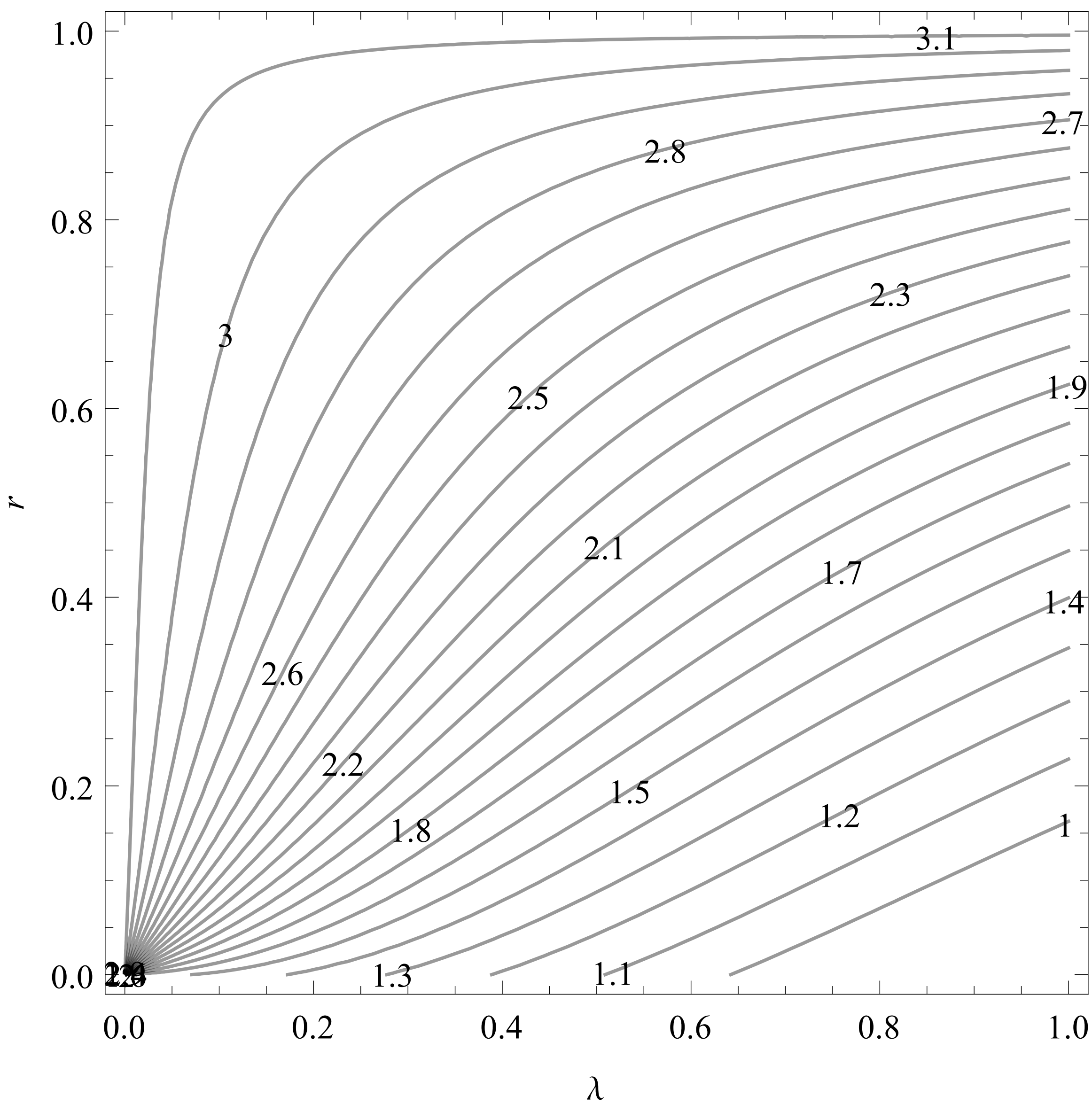

Figure 4: Contour plot of mean direction for Mob WrExp with $\mu = 0$ and $\nu = \pi$ as function of r and λ

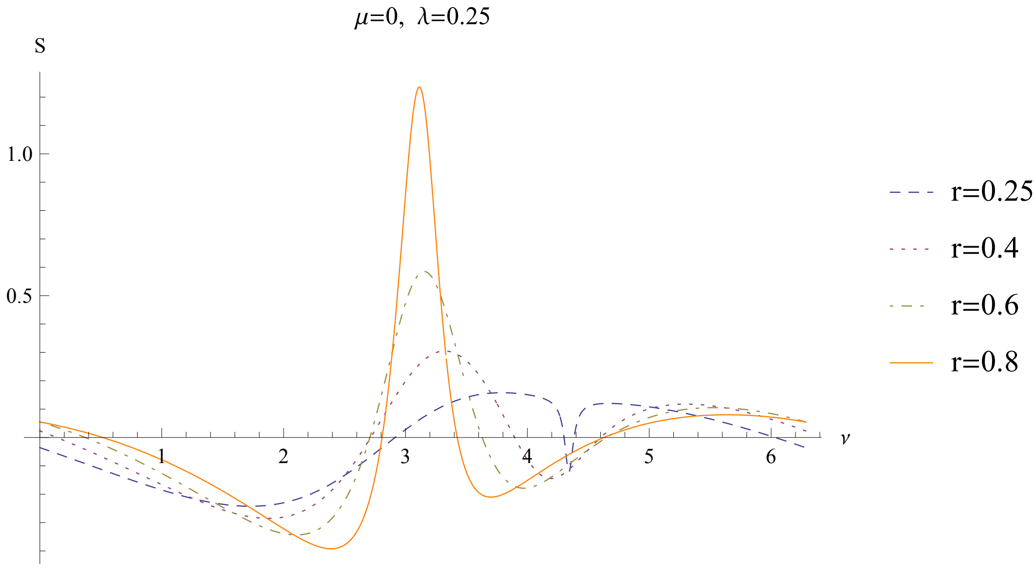

Figure 5: Skewness, s for Mob WrExp as a function of ν

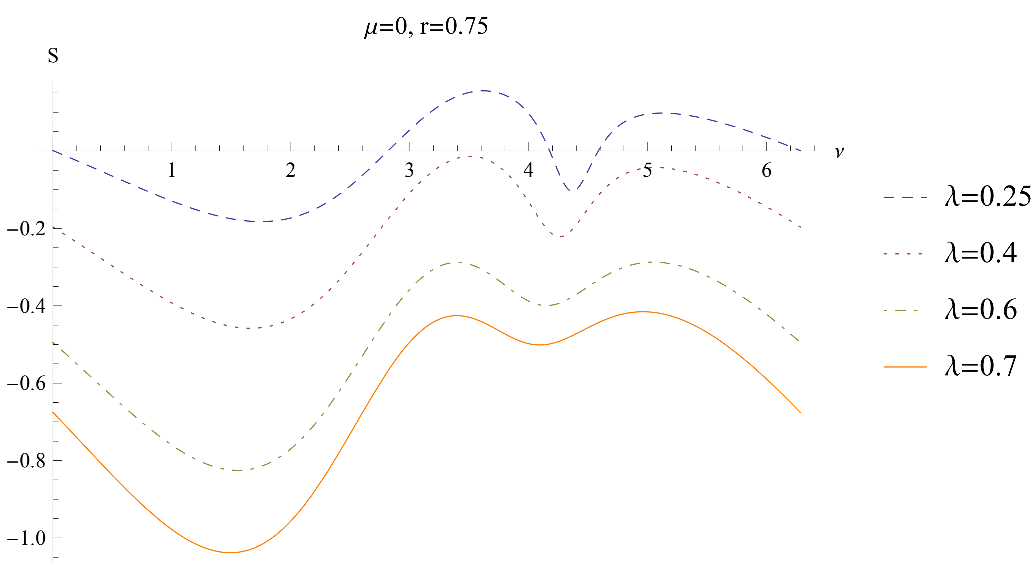

Figure 6: Skewness, s for Mob WrExp as a function of ν

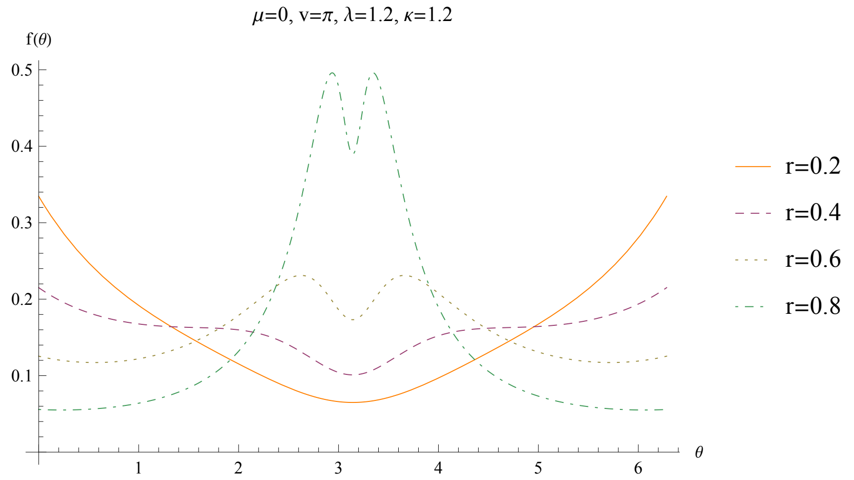

Figure 7: Plot of p.d.f of Möbius transformed Wrapped Laplace

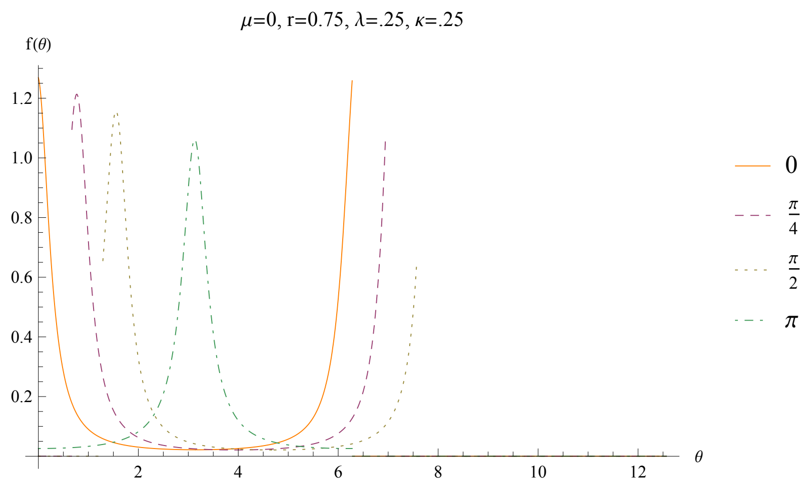

Figure 8: Plot of p.d.f of Möbius transformed Wrapped Laplace

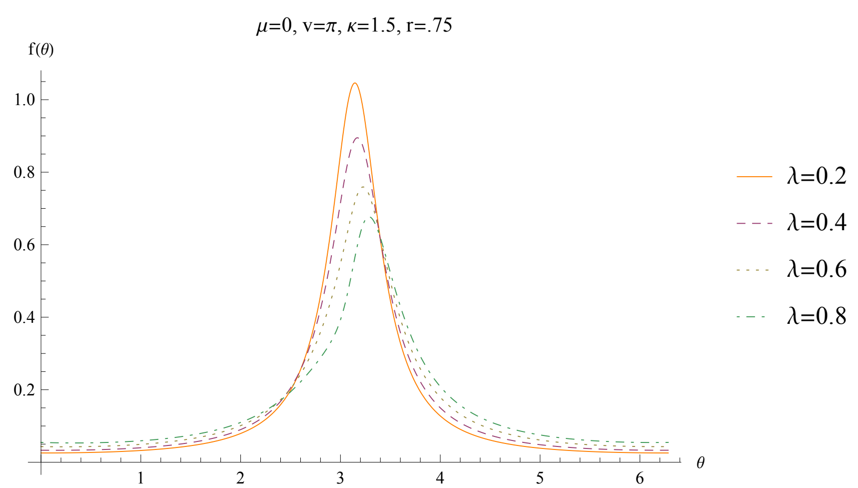

Figure 10: Plot of p.d.f of Möbius transformed Wrapped Laplace

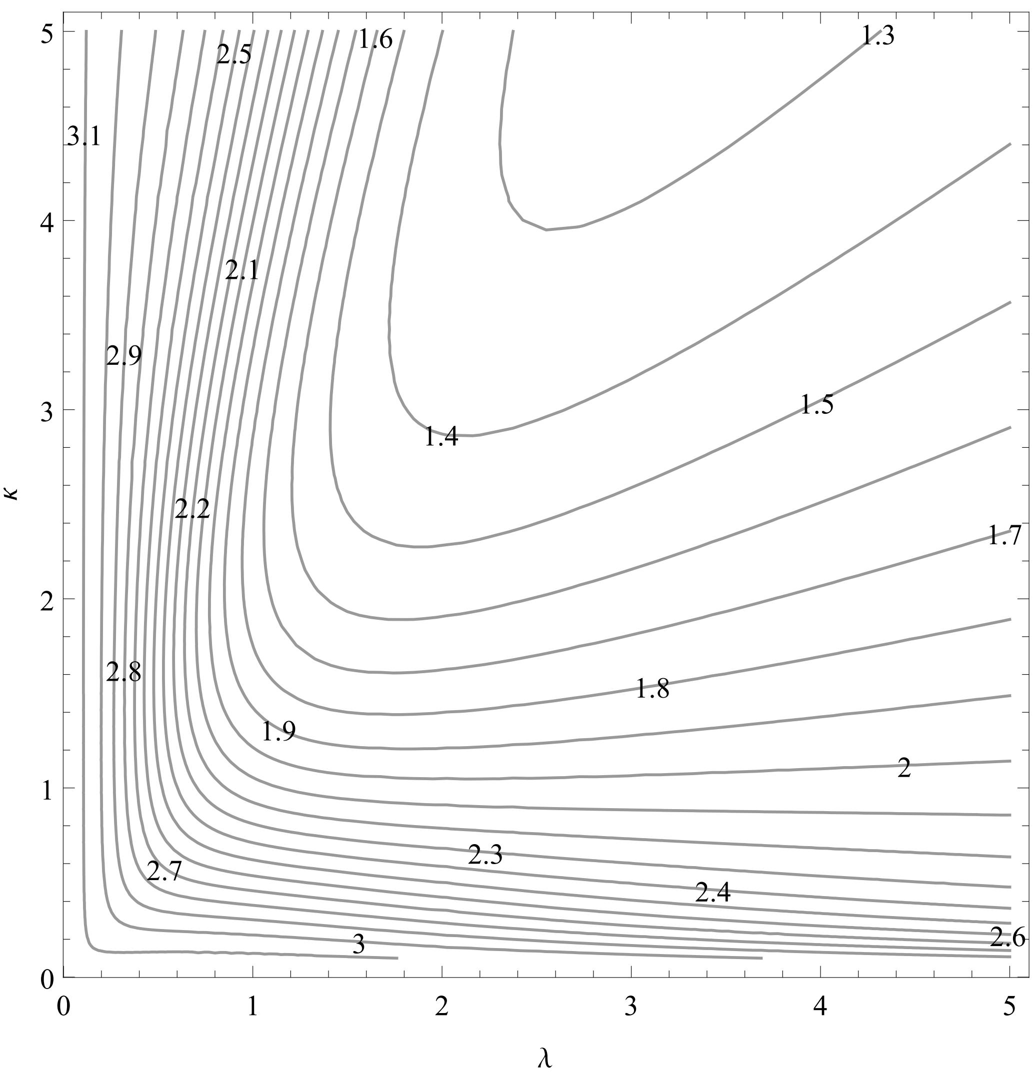

Figure 11: Contour plot of mean direction for Mob WrExp with $r = 0.25$, $\mu = \pi/2$ and $\nu = \pi/2i$ as function of κ and λ

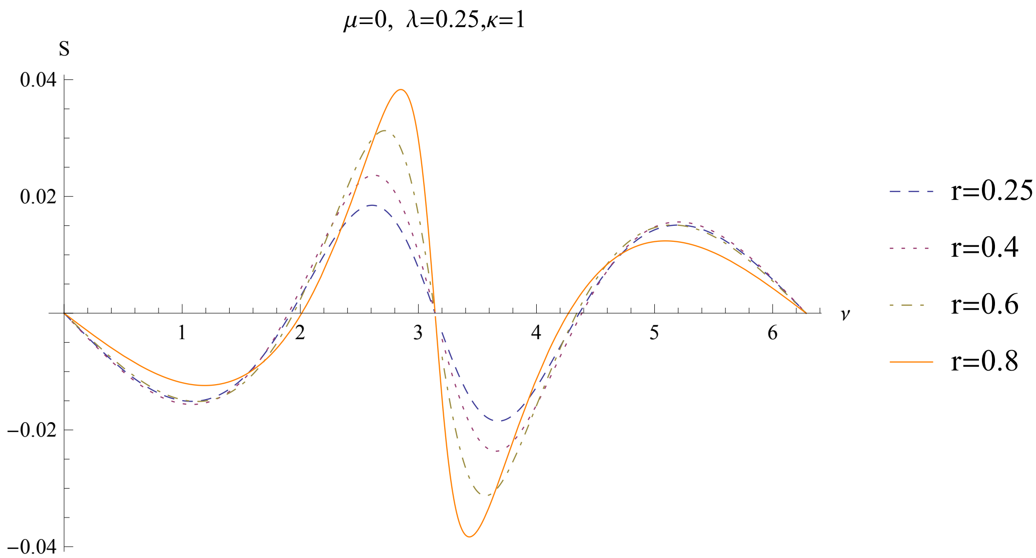

Figure 12: Skewness, s for Mob WrLap as a function of ν

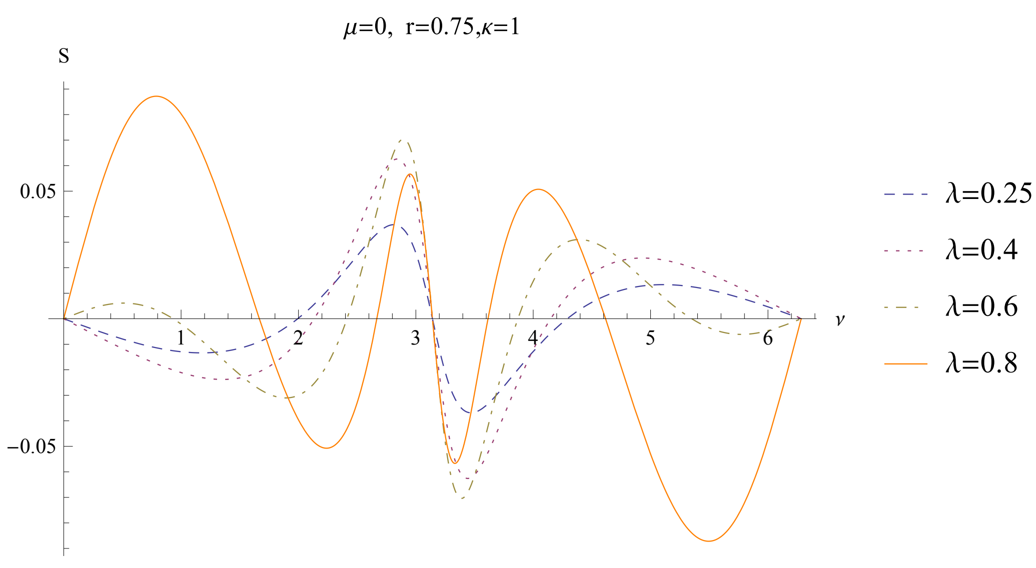

Figure 13: Skewness, s for Mob WrExp as a function of ν

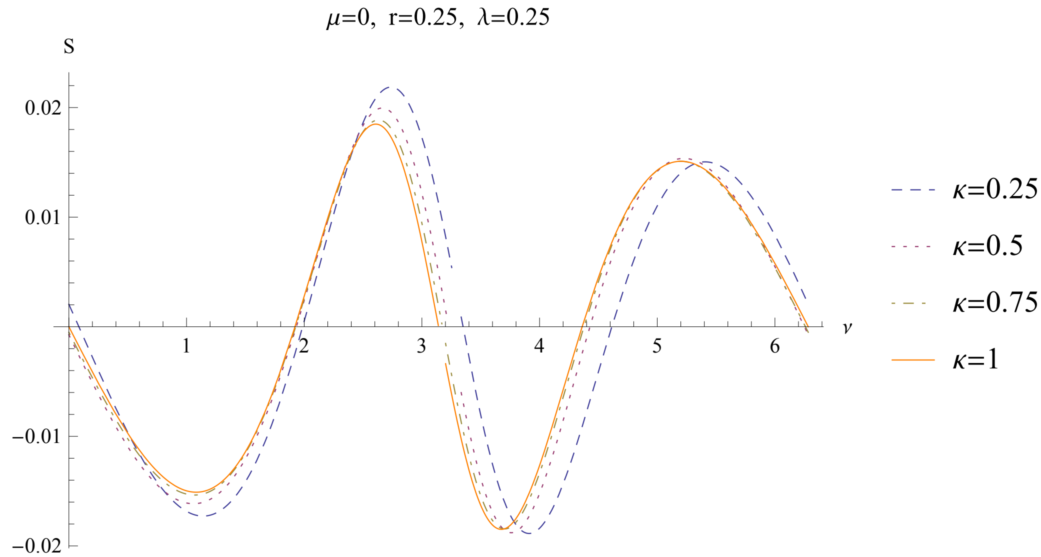

Figure 14: Skewness, s for Mob WrExp as a function of ν

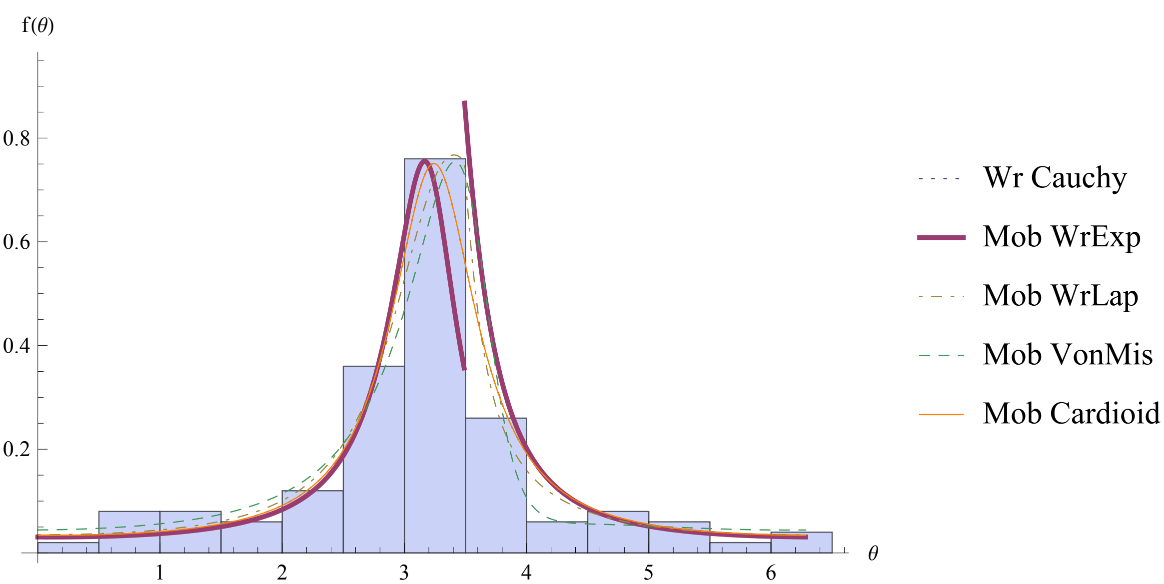

Figure 15: Histogram with estimated densities

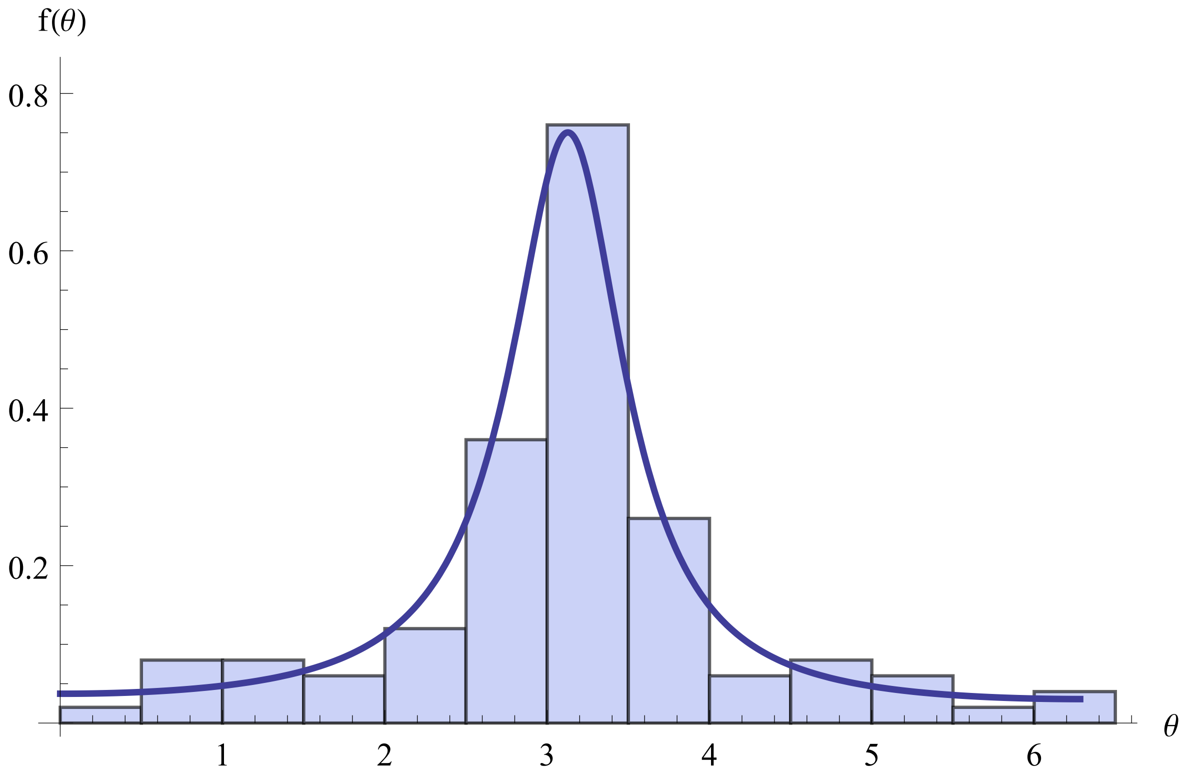

Figure 16: Histogram with estimated density(4.1)

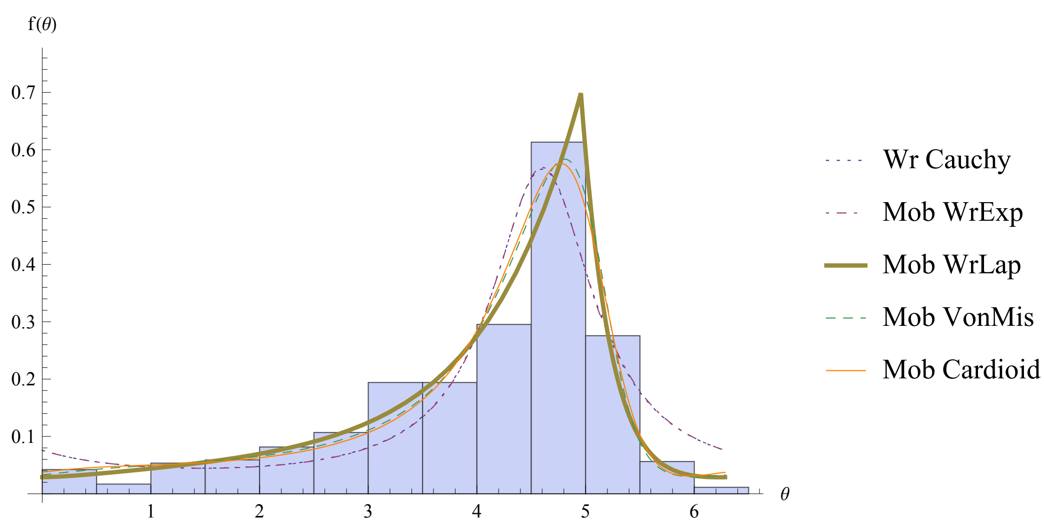

Figure 17: Histogram with estimated densities

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