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ON SOME POSITIVE DEFINITE FUNCTIONS

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Abstract. We study the function $(1-\|x\|)/(1-\|x\|^r)$, and its reciprocal, on the Euclidean space $\mathbb{R}^n$, with respect to properties like being positive definite, conditionally positive definite, and infinitely divisible.

1. Introduction

For each $n \geq 1$, consider the space $\mathbb{R}^n$ with the Euclidean norm $\|\cdot\|$. According to a classical theorem going back to Schoenberg [11] and much used in interpolation theory (see, e.g., [8]), the function $\varphi(x) = \|x\|^r$ on $\mathbb{R}^n$, for any n, is conditionally negative definite if and only if $0 \leq r \leq 2$. It follows that if r_j, $1 \leq j \leq m$, are real numbers with $0 \leq r_j \leq 2$, then the function

$$g(x) = 1 + \|x\|^{r_1} + \cdots + \|x\|^{r_m} \tag{1}$$

is conditionally negative definite, and by another theorem of Schoenberg, (see the statement **S5** in Section 2 below), the function

$$f(x) = \frac{1}{1 + \|x\|^{r_1} + \cdots + \|x\|^{r_m}} \tag{2}$$

is infinitely divisible. (A nonnegative function f is called infinitely divisible if for each $\alpha > 0$ the function $f(x)^\alpha$ is positive definite.) We also know that for any $r > 2$, the function $\varphi(x) = 1/(1+\|x\|^r)$ cannot be positive definite. (See, e.g., Corollary 5.5.6 of [2].)

With this motivation we consider the function

$$f(x) = \frac{1}{1 + \|x\| + \|x\|^2 + \cdots + \|x\|^m}, \quad m \geq 1, \tag{3}$$

and its reciprocal, and study their properties related to positivity. More generally, we study the function

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$$f(x) = \frac{1 - \|x\|}{1 - \|x\|^r}, \quad r > 0, \tag{4}$$

and its reciprocal. As usual, when $\|x\| = 1$ the right-hand side of (4) is interpreted as the limiting value $1/r$. This convention will be followed throughout the paper. The function (3) is the special case of (4) when $r = m + 1$.

Our main results are the following.

Theorem 1.1. *Let $0 < r \leq 1$. Then for each n, the function $f(x) = \frac{1-\|x\|}{1-\|x\|^r}$ on $\mathbb{R}^n$ is conditionally negative definite. As a consequence, the function $g(x) = \frac{1-\|x\|^r}{1-\|x\|}$ is infinitely divisible.*

The case $r \geq 1$ turns out to be more intricate.

Theorem 1.2. *Let n be any natural number. Then the function $g(x) = \frac{1-\|x\|^r}{1-\|x\|}$ on $\mathbb{R}^n$ is conditionally negative definite if and only if $1 \leq r \leq 3$. As a consequence the function $f(x) = \frac{1-\|x\|}{1-\|x\|^r}$ is infinitely divisible for $1 \leq r \leq 3$.*

In the second part of Theorem 1.2 the condition $1 \leq r \leq 3$ is sufficient but not necessary. We will show that the function f is infinitely divisible for $1 \leq r \leq 4$. On the other hand we show that when $r = 9$, f need not even be positive definite for all n.

In the case $n = 1$ we can prove the following theorem.

Theorem 1.3. *For every $1 \leq r < \infty$ the function $f(x) = \frac{1-|x|}{1-|x|^r}$ on $\mathbb{R}$ is positive definite.*

2. Some classes of matrices and functions

Let $A = [a_{ij}]$ be an $n \times n$ real symmetric matrix. Then A is said to be *positive semidefinite (psd)* if $\langle x, Ax \rangle \geq 0$ for all $x \in \mathbb{R}^n$, *conditionally positive definite (cpd)* if $\langle x, Ax \rangle \geq 0$ for all $x \in \mathbb{R}^n$ for which $\sum x_j = 0$, and *conditionally negative definite (cnd)* if $-A$ is cpd. If $a_{ij} \geq 0$, then for any real number r, we denote by $A^{\circ r}$ the rth *Hadamard power* of A; i.e., $A^{\circ r} = [a_{ij}^r]$. If $A^{\circ r}$ is psd for all $r \geq 0$, we say that A is *infinitely divisible.*

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. We say f is *positive definite* if for every n, and for every choice of real numbers $x_1, x_2, \ldots, x_n$, the $n \times n$ matrix $[f(x_i - x_j)]$ is psd. In the same way, f is called cpd, cnd, or infinitely divisible if the matrices $[f(x_i - x_j)]$ have the corresponding property.

Next, let f be a nonnegative C^∞ function on the positive half line $(0, \infty)$. Then f is called *completely monotone* if

$$(-1)^n f^{(n)}(x) \geq 0 \quad \text{for all} \quad n \geq 0. \tag{5}$$

According to a theorem of Bernstein and Widder, f is completely monotone if and only if it can be represented as

$$f(x) = \int_0^\infty e^{-tx}\, d\mu(t),$$

where μ is a positive measure. f is called a *Bernstein function* if its derivative f' is completely monotone; i.e., if

$$(-1)^{n-1} f^{(n)}(x) \geq 0 \quad \text{for all} \quad n \geq 1. \tag{6}$$

Every such function can be expressed as

$$f(x) = a + bx + \int_0^\infty (1 - e^{-tx}) d\mu(t), \tag{7}$$

where $a, b \geq 0$ and μ is a measure satisfying the condition $\int_0^\infty (1 \wedge t)\, d\mu(t) < \infty$. If this measure μ is absolutely continuous with respect to the Lebesgue measure, and the associated density $m(t)$ is a completely monotone function, then we say that f is a *complete Bernstein function.*

The class of complete Bernstein functions coincides with the class of *Pick functions* (or *operator monotone functions*). Such a function has an analytic continuation to the upper half-plane $\mathbb{H}$ with the property that $\operatorname{Im} f(z) \geq 0$ for all $z \in \mathbb{H}$. See Theorem 6.2 in [10].

For convenience we record here some basic facts used in our proofs.These can be found in the comprehensive monograph [10], or in the survey paper [1].

S1. A function φ on $(0, \infty)$ is completely monotone, if and only if the function $f(x) = \varphi(\|x\|^2)$ is continuous and positive definite on $\mathbb{R}^n$ for every $n \geq 1$.

S2. A function φ on $(0, \infty)$ is a Bernstein function if and only if the function $f(x) = \varphi(\|x\|^2)$ is continuous and cnd on $\mathbb{R}^n$ for every $n \geq 1$.

S3. If f is a Bernstein function, then $1/f$ is completely monotone.

S4. If f is a Bernstein function, then for each $0 < \alpha < 1$, the functions $f(x)^\alpha$ and $f(x^\alpha)$ are also Bernstein. If f is completely monotone, then $f(x^\alpha)$ has the same property for $0 < \alpha < 1$.

S5. A function f on $\mathbb{R}$ is cnd if and only if e^{-tf} is positive definite for every $t > 0$. Combining this with the Bernstein-Widder theorem, we see that if f is a nonnegative cnd function and φ is completely monotone, then the composite function $\varphi \circ f$ is positive definite. In particular, if $r > 0$, and we choose $\varphi(x) = x^{-r}$, we see that the function $f(x)^{-r}$ is positive definite. In other words $1/f$ is infinitely divisible.

3. Proofs and Remarks

Our proof of Theorems 1.1 and 1.2 relies on the following proposition. This is an extension of results of T. Furuta [5] and F. Hansen [6].

Proposition 3.1. *Let p, q be positive numbers with $0 < p \leq 1$, and $p \leq q \leq p+1$. Then the function $f(x) = (1-x^q)/(1-x^p)$ on the positive half-line is operator monotone.*

Proof. The case $p = q$ is trivial; so assume $p < q$. It is convenient to use the formula

$$\frac{1-x^q}{1-x^p} = \frac{q}{p} \int_0^1 \left(\lambda\, x^p + 1 - \lambda\right)^{\frac{q-p}{p}} \, d\lambda, \tag{8}$$

which can be easily verified. If z is a complex number with $\operatorname{Im} z > 0$, then for $0 < \lambda < 1$, the number $\lambda z^p + 1 - \lambda$ lies in the sector $\{w : 0 < \operatorname{Arg} w < p\pi\}$. Since $0 < \frac{q-p}{p} \leq \frac{1}{p}$, we see that $(\lambda z^p + 1 - \lambda)^{\frac{q-p}{p}}$ lies in the upper half-plane. This shows that the function represented by (8) is a Pick function. ∎

Now let $0 < r \leq 1$. Choosing $p = r/2$ and $q = 1/2$, we see from Proposition 3.1 that the function $\varphi(x) = \frac{1-x^{1/2}}{1-x^{r/2}}$ is operator monotone. Appealing to fact **S2** we obtain Theorem 1.1.

Next let $1 \leq r \leq 3$. Choosing $p = 1/2$ and $q = r/2$, we see from Proposition 3.1 that the function $\varphi(x) = \frac{1-x^{r/2}}{1-x^{1/2}}$ is operator monotone. Again appealing to **S2** we see that the function $g(x) = \frac{1-\|x\|^r}{1-\|x\|}$ is cnd on the Euclidean space $\mathbb{R}^n$ for every n.

The necessity of the condition $1 \leq r \leq 3$ is brought out by the Lévy-Khinchine formula. A continuous function $g : \mathbb{R} \to \mathbb{C}$ is cnd if and only it can be represented as

$$g(x) = a + ibx + c^2x^2 + \int_{\mathbb{R}\setminus\{0\}} \left(1 - e^{itx} + \frac{itx}{1+t^2}\right) d\nu(t),$$

where a, b, c are real numbers, and ν is a positive measure on $\mathbb{R}\setminus\{0\}$ such that $\int (t^2/(1+t^2))d\nu(t) < \infty$. See [10]. It is clear then that $g(x) = O(x^2)$ at ∞. So, if $r > 3$, the function $g(x)$ of Theorem 1.2 cannot be cnd on $\mathbb{R}$. This proves Theorem 1.2 completely.

Now we show that $f(x) = \frac{1-\|x\|}{1-\|x\|^r}$ is infinitely divisible for $1 \le r \le 4$. The special case $r = 4$ is easy. We have

$$\frac{1-\|x\|}{1-\|x\|^4} = \frac{1}{1+\|x\|+\|x\|^2+\|x\|^3} = \frac{1}{1+\|x\|}\ \frac{1}{1+\|x\|^2},$$

and we know that both $\frac{1}{1+\|x\|}$ and $\frac{1}{1+\|x\|^2}$ are infinitely divisible, and therefore so is their product. The general case is handled as follows.

By Proposition 3.1, the function $\frac{1-x^r}{1-x}$ is operator monotone for $1 \le r \le 2$. Repeating our arguments above, we see that $\frac{1-\|x\|^2}{1-\|x\|^{2r}}$ is an infinitely divisible function for $1 \le r \le 2$. We know that $\frac{1}{1+\|x\|}$ is infinitely divisible; hence so is the product

$$\frac{1-\|x\|^2}{1-\|x\|^{2r}}\ \frac{1}{1+\|x\|} = \frac{1-\|x\|}{1-\|x\|^{2r}}, \quad 1 \le r \le 2.$$

In other words $\frac{1-\|x\|}{1-\|x\|^r}$ is infinitely divisible for $2 \le r \le 4$.

We now consider what happens for $r > 4$. In the special case $n = 1$, Theorem 1.3 says that this function is at least positive definite for all $r > 4$. By a theorem of Pólya (see [2], p.151) any continuous, nonnegative, even function on $\mathbb{R}$ which is convex and monotonically decreasing on $[0, \infty)$ is positive definite. So Theorem 1.3 follows from the following proposition.

Proposition 3.2. *The function*

$$f(x) = \frac{1-x}{1-x^r}, \quad 1 < r < \infty, \tag{9}$$

on the positive half-line $(0, \infty)$ *is monotonically decreasing and convex.*

Proof. A calculation shows that

$$f'(x) = \frac{(1-r)x^r + rx^{r-1} - 1}{(1-x^r)^2}, \tag{10}$$

and

$$\begin{aligned} f''(x) &= \frac{1}{(1-x^r)^3}\left\{r(1-r)x^{2r-1} + r(1+r)x^{2r-2}\right. \\ &\quad \left. -r(1+r)x^{r-1} - r(1-r)x^{r-2}\right\}. \\ &= \frac{1}{(1-x^r)^3}\ \varphi(x), \quad \text{say.} \end{aligned} \tag{11}$$

Since $f''(x)$ is well-defined at 1, the function φ must have a zero of order at least three at 1. On the other hand, by the Descartes rule of signs, (see [9],p.46), $\varphi(x)$ can have at most three positive zeros. Thus the only zero of φ in $(0, \infty)$ is at the point $x = 1$.

Next note that when x is small, the last term of $\varphi(x)$ is dominant, and therefore $\varphi(x) > 0$. On the other hand, when x is large, the first term of $\varphi(x)$ is dominant, and therefore $\varphi(x) < 0$. Thus $\varphi(x)$ is positive if $x < 1$, and negative if $x > 1$. This shows that $f''(x) \geq 0$. Hence f is convex. Since $f(0) = 1$, and $\lim_{x\to\infty} f(x) = 0$, this also shows that f is monotonically decreasing, a fact which can be easily seen otherwise too. ■

Does the function f in (9) have any stronger convexity properties? We have seen that if $1 \leq r \leq 2$, then the reciprocal of f is operator monotone. Hence by fact **S3**, f is completely monotone for $1 \leq r \leq 2$. For $r > 2$, however f is not even log-convex.

Recall that a nonnegative function f on $(0, \infty)$ is called log-*convex* if $\log f$ is convex. If f', f'' exist, this condition is equivalent to

$$(f'(x))^2 \leq f(x)\ f''(x) \quad \text{for all} \quad x. \tag{12}$$

(See [12],p.485). A completely monotone function is log-convex.

Proposition 3.3. *The function* $f(x) = \frac{1-x}{1-x^r}$ *on* $(0, \infty)$ *is* log-*convex if and only if* $1 \leq r \leq 2$.

Proof. From the expressions (9), (10) and (11) we see that

$$f(x)f''(x) - (f'(x))^2 = \frac{\psi(x)}{(1-x^r)^4}, \tag{13}$$

where

$$\begin{aligned} \psi(x) &= (r-1)x^{2r} - 2rx^{2r-1} + rx^{2r-2} + (r^2 - r + 2)x^r \\ &\quad -2r(r-1)x^{r-1} - 1 + r(r-1)x^{r-2}. \end{aligned} \tag{14}$$

Using condition (12) we see from (13) that f is log-convex if and only if $\psi(x) \geq 0$ for all x. If $r > 2$, it is clear from (14) that $\psi(0) = -1$, and ψ is negative in a neighbourhood of 0. So f is not log-convex.

We have already proved that when $1 < r < 2$, f is completely monotone, and hence log-convex. It is instructive to see how the latter property can be derived easily using the condition (12). It is clear from (13) that ψ must have a zero of order at least 4 at 1. On the other hand, there are just four sign changes in the coefficients on the right-hand side of (14). So by the Descartes rule of signs ([9],p.46) ψ has at most four positive zeros. Thus ψ has only one zero, it is at 1 and has multiplicity four. The coefficients of both x^{2r} and x^{r-2} in (14) are positive. Hence ψ is always nonnegative. ∎

Because of **S1**, the function $f(x) = \frac{1-\|x\|}{1-\|x\|^r}$ would be positive definite on $\mathbb{R}^n$ for every n, if and only if the function

$$h(x) = \frac{1 - x^{1/2}}{1 - x^{r/2}}, \tag{15}$$

on $(0, \infty)$ were completely monotone. From **S4** we see that this would be a consequence of the complete monotonicity of the function $f(x) = \frac{1-x}{1-x^r}$; but the latter holds if and only if $1 \leq r \leq 2$. We now show that when $r = 9$, the function h in (15) is not even log convex.

For this we use the fact that h is log convex if and only if

$$h\left(\frac{x+y}{2}\right)^2 \leq h(x)h(y) \quad \text{for all } x, y. \tag{16}$$

Choose $x = 9/25$, $y = 16/25$. Then $\frac{x+y}{2} = 1/2$. When $r = 9$, the function h in (15) reduces to

$$h(x) = \left(\sum_{j=0}^{8} x^{j/2}\right)^{-1}.$$

So, the inequality (16) would be true for the chosen values of x and y, if we have

$$\sum_{j=0}^{8} \left(\frac{3}{5}\right)^j \sum_{j=0}^{8} \left(\frac{4}{5}\right)^j \leq \left(\sum_{j=0}^{8} \left(\frac{1}{\sqrt{2}}\right)^j\right)^2.$$

A calculation shows that this is not true as, up to the first decimal place, the left-hand side is 10.7 and the right-hand side is 10.6.

We are left with some natural questions:

1. What is the smallest r_0 for which the function f of Theorem 1.2 is not infinitely divisible (or positive definite) for all $\mathbb{R}^n$? Our analysis shows that $4 < r_0 < 9$.
2. What is the smallest n_0 for which there exists some $r > 4$, such that this function f is not positive definite on $\mathbb{R}^{n_0}$?
3. Is the function f in Theorem 1.3 infinitely divisible on $\mathbb{R}$? By Theorem 10.4 in [12] a sufficient condition for this to be true is log convexity of the function $\frac{1-x}{1-x^r}$ on $(0, \infty)$. We have seen that this latter condition holds if and only if $1 \le r \le 2$. Note that we have shown by other arguments that f is infinitely divisible for $1 \le r \le 4$.

Several examples of infinitely divisible functions arising in probability theory are listed in [12]. Many more with origins in our study of operator inequalities can be found in [3] and [7]. It was observed already in [4] that the function defined in (2) is infinitely divisible.

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