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Positivity properties of the matrix $\left[(i+j)^{i+j}\right]$

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POSITIVITY PROPERTIES OF THE MATRIX $[(i+j)^{i+j}]$

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Abstract. Let $p_1 < p_2 < \cdots < p_n$ be positive real numbers. It is shown that the matrix whose i, j entry is $(p_i + p_j)^{p_i+p_j}$ is infinitely divisible, nonsingular and totally positive.

1. Introduction

Matrices whose entries are obtained by assembling natural numbers in special ways often possess interesting properties. The most famous example of such a matrix is the Hilbert matrix $H = \left[\frac{1}{i+j-1}\right]$ which has inspired a lot of work in diverse areas. Some others are the min matrix $M = \left[\min(i,j)\right]$, and the Pascal matrix $P = \left[\binom{i+j}{i}\right]$. There is a considerable body of literature around each of these matrices, a sample of which can be found in [3], [5] and [7].

In this note we initiate the study of one more matrix of this type. Let A be the $n \times n$ matrix with its (i,j) entry equal to $(i+j-1)^{i+j-1}$. Thus

$$A = \begin{bmatrix} 1 & 2^2 & 3^3 & \cdots & n^n \\ 2^2 & 3^3 & 4^4 & \cdots & (n+1)^{n+1} \\ 3^3 & 4^4 & 5^5 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ n^n & \cdots & \cdots & \cdots & (2n-1)^{2n-1} \end{bmatrix}. \tag{1}$$

More generally, let $p_1 < p_2 < \cdots < p_n$ be positive real numbers, and consider the $n \times n$ matrix

$$B = \left[(p_i + p_j)^{p_i+p_j}\right]. \tag{2}$$

The special choice $p_i = i - 1/2$ in (2) gives us the matrix (1). We investigate the behaviour of these matrices with respect to different

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kinds of positivity.

A real symmetric matrix S is said to be *positive semidefinite* (psd) if for every vector x, we have $\langle x, Sx\rangle \geq 0$. Further if $\langle x, Sx\rangle = 0$ only when $x = 0$, then we say S is *positive definite*. This is equivalent to saying that S is psd and nonsingular. If S is a psd matrix, then for every positive integer m, the mth *Hadamard power* (entrywise power) $S^{\circ m} = \left[s_{ij}^m\right]$ is also psd. Now suppose $s_{ij} \geq 0$. We say that S is *infinitely divisible* if for every real number $r > 0$, the matrix $S^{\circ r} = \left[s_{ij}^r\right]$ is psd. (See [3], Chapter 5 of [4], and Chapter 7 of [9] for expositions of this topic.) The principal minors of a psd matrix are nonnegative. This may not be so for other minors. A matrix with nonnegative entries is called *totally positive* if all its minors are nonnegative. It is called *strictly totally positive* if all its minors are positive. We recommend the books [8, 10, 11] and the survey article [1] for an account of totally positive matrices.
Our main result is the following:

Theorem. *Let $p_1 < p_2 < \cdots < p_n$ be positive real numbers. Then the matrix B defined in* (2) *is infinitely divisible, nonsingular and totally positive.*

There is another way of stating this. Let X be a subset of $\mathbb{R}$. A continuous function $K : X \times X \to \mathbb{R}$ is said to be a *positive definite kernel* if for every n and for every choice $x_1 < \cdots < x_n$ in X, the matrix $\left[K(x_i, x_j)\right]$ is positive definite. In the same way we can define infinitely divisible and totally positive kernels. Our theorem says that the kernel $K(x, y) = (x + y)^{x+y}$ on $(0, \infty) \times (0, \infty)$ is infinitely divisible and totally positive. This is an addition to the examples given in [3, 8, 10, 11]. The three matrices in the first paragraph also have the properties mentioned in the theorem.

2. Proof

Let H_1 be the space of all vectors $x = (x_1, ..., x_n)$ with $\sum x_i = 0$. A real symmetric matrix S is said to be *conditionally positive definite* (cpd) if $\langle x, Sx\rangle \geq 0$ for all $x \in H_1$. If $-S$ is cpd, then S is said to be *conditionally negative definite* (cnd). According to a theorem of C. Loewner, a matrix $S = [s_{ij}]$ is infinitely divisible if and only if the matrix $[\log s_{ij}]$ is cpd. See Exercise 5.6.15 in [4].

By Loewner's theorem cited above, in order to prove that the matrix B defined in (2) is infinitely divisible it is enough to show that the matrix

$$C = [(p_i + p_j)\log(p_i + p_j)] \tag{3}$$

is cpd. It is convenient to use the formula

$$\log x = \int_0^\infty \left(\frac{1}{1+\lambda} - \frac{1}{x+\lambda}\right) d\lambda, \quad x > 0,$$

which can be easily verified. Using this we can write our matrix C as

$$C = \left[\int_0^\infty \left(\frac{p_i+p_j}{1+\lambda} - \frac{p_i+p_j}{p_i+p_j+\lambda}\right) d\lambda\right].$$

We will show that the matrix $[p_i+p_j]$ is cpd, and the matrix $\left[\frac{p_i+p_j}{p_i+p_j+\lambda}\right]$ is cnd for each $\lambda > 0$. From this it follows that C is a cpd matrix.

Let D be the diagonal matrix $D = \mathrm{diag}(p_1, ..., p_n)$ and E the matrix with all its entries equal to 1. Then $[p_i+p_j] = DE+ED$. Every vector x in H_1 is annihilated by E. Hence $\langle x, (DE+ED)x\rangle = \langle x, DEx\rangle + \langle Ex, Dx\rangle = 0$. So, the matrix $[p_i+p_j]$ is cpd. Using the identity

$$\frac{p_i+p_j}{p_i+p_j+\lambda} = 1 - \frac{\lambda}{p_i+p_j+\lambda},$$

we can write

$$\left[\frac{p_i+p_j}{p_i+p_j+\lambda}\right] = E - \lambda C_\lambda,$$

where $C_\lambda = \left[\frac{1}{p_i+p_j+\lambda}\right]$. This is a Cauchy matrix (see [4]) and is positive definite. Hence it is also cpd. The matrix E annihilates H_1, and therefore is cnd. Hence $E - \lambda C_\lambda$ is cnd for every $\lambda > 0$. This completes the proof of the assertion that C is cpd, and B infinitely divisible.

Since C_λ is positive definite, $\langle x, C_\lambda x\rangle > 0$ for every non zero vector x. If $x \in H_1$, then $Ex = 0$, and $\langle x, (DE+ED)x\rangle = 0$. So, the arguments given above also show that $\langle x, Cx\rangle > 0$ for every non zero vector x in H_1. By Lemma 4.3.5 in [2], this condition is necessary and sufficient for C to be nonsingular. Using the next proposition, we can conclude that B is nonsingular.

Proposition. *If C is a nonsingular conditionally positive definite matrix, then the matrix $[e^{c_{ij}}]$ is positive definite.*

Proof. By Proposition 5.6.13 of [4] we can express C as

$$C = P + YE + E\overline{Y},$$

where P is a psd matrix and Y is a diagonal matrix. By Problem 7.5.P.25 in [9] the matrix $[e^{c_{ij}}]$ is positive definite unless P has two equal columns. Suppose the ith column of P is equal to its jth column. Let x be any vector with coordinates $x_i = -x_j \neq 0$, and all other coordinates zero. Then $x \in H_1$ and $Px = 0$. Hence $\langle x, Cx\rangle = 0$. This is not possible since C is a nonsingular cpd matrix. ■

We have proved that the matrix B is infinitely divisible and nonsingular. These properties are inherited by the matrix A defined in (1). This is, moreover, a *Hankel matrix*; i.e, each of its antidiagonals has the same entry. Theorem 4.4 of [11] gives a simple criterion for strict total positivity of such a matrix. According to this a Hankel matrix A is strictly totally positive if and only if A is positive definite and so is the matrix $\tilde{A}$ obtained from A by deleting its first column and last row. For the matrix A in (1), $\tilde{A}$ is the $(n-1)\times(n-1)$ matrix whose (i,j) entry is $(i+j)^{i+j}$. Both A and $\tilde{A}$ are positive definite. Hence A is strictly totally positive. In fact we have shown that for every $r > 0$, the matrix $A^{\circ r}$ is strictly totally positive.

Now let $k_1 < k_2 < \cdots < k_n$ be positive integers. The matrix K with entries $k_{ij} = (k_i+k_j)^{k_i+k_j}$ is principal submatrix of A. Hence it is infinitely divisible and strictly totally positive. The same holds for $K^{\circ r}$ for every $r > 0$. Next let $0 < q_1 < q_2 < \cdots < q_n$ be rational numbers. Let $q_j = l_j/m_j$, where l_j and m_j are positive integers. Let m be the LCM of $m_1, \ldots, m_n$ and $k_j = mq_j$. Then $k_1 < k_2 < \cdots < k_n$, and as seen above, the matrix $K = \left[(k_i + k_j)^{k_i+k_j}\right]$ is infinitely divisible and strictly totally positive. Now consider the matrix $Q = \left[(q_i + q_j)^{q_i+q_j}\right]$. Then for each $r > 0$

$$\begin{aligned} Q^{\circ r} &= \left[(q_i+q_j)^{(q_i+q_j)r}\right] \\ &= \left[\frac{(k_i+k_j)^{(k_i+k_j)r/m}}{m^{q_i r} m^{q_j r}}\right] \\ &= XK^{\circ r/m}X^*, \end{aligned}$$

where X is the positive diagonal matrix with entries $\frac{1}{m^{q_1 r}}, \ldots, \frac{1}{m^{q_n r}}$ on its diagonal. We have seen that the matrix $K^{\circ r/m}$ is positive definite and strictly totally positive. Hence, so is the matrix $Q^{\circ r}$. A continuity argument completes the proof of the theorem. ■

We believe that the matrix B in (2) is strictly totally positive. However the continuity argument that we have invoked at the last step only shows that it is a limit of such matrices.

Like for the other matrices mentioned in the opening paragraph, it would be interesting to have formulas for the determinant of A.

In a recent work [6] of ours, we have studied spectral properties of the matrices $[(p_i + p_j)^r]$, where r is any positive real number.

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