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On lattice points where all nearby points are not relatively prime

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Abstract

Given positive integers a, b, let $M = M(a, b)$ and $N = N(a, b)$ be a minimal pair of positive integers such that we always have $\gcd(M - i, N - j) > 1$ for all $1 \le i \le a$ and $1 \le j \le b$. We give upper and lower bounds for M, N.

1 Introduction

In [5], Pighizzini and Shallit defined for a positive integer n the function $S(n)$ which is the least positive integer r such that there exists $m \in \{0, 1, \ldots, r\}$ with $\gcd(r - i, m - j) > 1$ for $0 \le i, j < n$. The above greatest common divisor condition is equivalent to the fact that a lattice point $(0, 0) \ne (x, y) \in \mathbb{Z}^2$ with $\gcd(x, y) > 1$ is *nonvisible* from the origin (see [3]). They showed that

$$S(n) < e^{(2+o(1))n^2 \log n} \qquad \text{as} \qquad n \to \infty, \tag{1}$$

and computed $S(n)$ and the corresponding m's for $n = 1, 2, 3$. This function was also studied in Wolfram's book [7, p. 1093] who computed $S(4)$.

Here, we generalize the function $S(n)$. Given positive integers a, b, let $(M(a,b), N(a,b))$ be a minimal pair of positive integers such that $\gcd(M-i, N-j) > 1$ for all $1 \le i \le a$ and $1 \le j \le b$. Here, by minimal, we mean that if both $(M(a,b), N(a,b))$ and $M'(a,b), N'(a,b))$ satisfy the requirements, then $M(a,b) < M'(a,b)$ and $N(a,b) > N'(a,b)$ (or vice-versa). Without loss of generality, we assume that $a \ge b$. In this note, we prove the following result. We always write p for a prime number.

Theorem 1. *If $a \ge b$, we then have*

(i) $\max\{M(a,b), N(a,b)\} \le \exp((6/\pi^2 + o(1))ab \log ab)$ *as* $b \to \infty$.

(ii) $\max\{M(a,b), N(a,b)\} \le \exp(0.721521 ab \log ab)$ *if* $b > 100$.

(iii) We have

$$M(a,b) \ge \exp((c_1 + o(1))b \log ab) \quad \text{and} \quad N(a,b) \ge \exp((c_1 + o(1))a \log ab),$$

where

$$c_1 = 1 - \sum_{p \ge 2} \frac{1}{p^2} = 0.547753\ldots$$

provided $b \to \infty$ *in such a way that* $\log \log a = o(b)$.

Taking $a = b = n$, (i) above shows that

$$S(n) \le \exp((12/\pi^2 + o(1))n^2 \log n) \qquad \text{as} \qquad n \to \infty,$$

which improves (1). We also give a lower bound for $S(n)$. We prove

Theorem 2. *For $n > 1$, we have*

$$S(n) \ge \exp(.82248 n \log n).$$

We also give an algorithm for computing M and N for a given a and b. This is stated in Section 3 and values of M and N are computed for some small values of a, b. The proof of Theorem 2 is given in Section 4.

2 Preliminaries

For a positive integer i, let p_i denote the i-th prime. Thus $p_1 = 2, p_2 = 3, \ldots$. For real $x > 1$, let

$$\pi(x) = \sum_{p \le x} 1 \quad \text{and} \quad \theta(x) = \sum_{p \le x} \log p.$$

From the prime number theorem, we have $\pi(x) \le s_1 x / \log x$ and $\theta(p_\ell) \le s_2 \ell \log \ell$ for positive constants s_1, s_2. The following results give explicit values of s_1 and s_2.

Lemma 3. *Let x be real and positive and ℓ be a positive integer. We have* (*i*) $\pi(x) \le \dfrac{x}{\log x}\left(1 + \dfrac{1.2762}{\log x}\right)$ *for* $x > 1$.

(*ii*) $p_\ell \ge \ell \log \ell$ *for* $\ell \ge 1$.

(*iii*) $\theta(p_\ell) \le \ell(\log \ell + \log\log \ell - .75)$ *for* $\ell \ge 8$.

(*iv*) $\theta(x) \ge x\left(1 - \dfrac{1}{\log x}\right)$ *for* $x \ge 41$.

(*v*) $\displaystyle\sum_{p \le x} \frac{1}{p} \le \log\log x + 0.2615 + \frac{1}{\log^2 x}$ *for* $x > 1$.

The estimates $(ii), (iv)$ and (v) are [6, (3.12), (3.16), (3.20)], respectively. The estimate (i) is due to Dusart [1] and (iii) is derived from estimates in [1]. See also [2].

For given integers $j \ge r \ge 1$, let

$$r' := r'(j) := \#\{i : 1 \le i \le r \text{ and } \gcd(i, j) = 1\}.$$

Let

$$R_j := \max\left\{r' - \frac{r\varphi(j)}{j} : 1 \le r < j\right\},$$

where $\varphi(j)$ is the Euler-phi function. It is easy to see that $R_p = 1 - 1/p$. For a real number x, let $\{x\}$ denote the fractional part of x; i.e., $\{x\} = x - \lfloor x \rfloor$. We prove the following estimate.

Lemma 4. *If* $n > 100$, *then*

$$\sum_{j=1}^{n} R_j \le .375 n \log n - .432 n - 10.$$

Proof. For $1 \le r < j$, we have

$$r'(j) \le r - \sum_{p|j} \left\lfloor \frac{r}{p} \right\rfloor + \sum_{pq|j} \left\lfloor \frac{r}{pq} \right\rfloor - \sum_{pqr|j} \left\lfloor \frac{r_j}{pqr} \right\rfloor + \cdots,$$

where $p, q, r, \ldots$ are primes dividing j. Since

$$\frac{\varphi(j)}{j} = 1 - \sum_{p|j} \frac{1}{p} + \sum_{pq|j} \frac{1}{pq} - \sum_{pqr|j} \frac{1}{pqr} + \cdots,$$

we get

$$r' - \frac{r\varphi(j)}{j} \le \sum_{p|j} \left\{\frac{r_j}{p}\right\} - \sum_{pq|j} \left\{\frac{r_j}{pq}\right\} + \sum_{pqr|j} \left\{\frac{r_j}{pqr}\right\} - \cdots.$$

Since $r/s \le \lfloor r/s \rfloor + 1 - 1/s$ holds for positive integers r, s, we get

$$R_j \le \sum_{p|j} \left(1 - \frac{1}{p}\right) + \sum_{pqr|j} \left(1 - \frac{1}{pqr}\right) + \cdots$$

Let $\omega(j)$ be the number of distinct prime divisors of j and put $\omega_t = \binom{j}{t}$. Then

$$R_j \le \sum_{t \text{ odd}} \omega_t - \sum_{p|j} \frac{1}{p} = 2^{\omega(j)-1} - \sum_{p|j} \frac{1}{p}.$$

Thus, for $n > 100$, we have

$$\begin{aligned}
\sum_{j=1}^{n} R_j &\le \sum_{j=1}^{100} R_j + \frac{1}{2} \sum_{j>100}^{n} 2^{\omega(j)} - \sum_{j>100}^{b} \sum_{p|j} \frac{1}{p} \\
&= \sum_{j=1}^{100} \left(R_j - 2^{\omega(j)-1} - \sum_{p|j} \frac{1}{p} \right) + \frac{1}{2} \sum_{j=1}^{n} 2^{\omega(j)} - \sum_{j=2}^{n} \sum_{p|j} \frac{1}{p} \\
&\le -130.4778 + \frac{1}{2} \sum_{j=1}^{n} 2^{\omega(j)} - \sum_{p \le n} \left\lfloor \frac{n}{p} \right\rfloor \frac{1}{p}. \qquad (2)
\end{aligned}$$

Assuming $n > 100$, we have

$$\begin{aligned}
\sum_{p \le n} \left\lfloor \frac{n}{p} \right\rfloor \frac{1}{p} &\ge \sum_{p \le n} \left(\frac{n+1}{p^2} - \frac{1}{p} \right) \ge (n+1) \sum_{p \le b} \left(\frac{1}{p^2} - \frac{1}{p(n+1)} \right) \\
&\ge (n+1) \sum_{p \le 101} \left(\frac{1}{p^2} - \frac{1}{101p} \right) \ge .432(n+1). \qquad (3)
\end{aligned}$$

As in the proof of [4, Lemma 9] for $n \ge 248$, and using exact computations for $n \in [101, 247]$, we obtain

$$\sum_{j=2}^{n} 2^{\omega(j)} - 120 \le .375 n \log n \qquad \text{for all} \qquad n > 100. \qquad (4)$$

Combining above estimates (2), (3) and (4), we get the assertion of the lemma. □

Lemma 5. *For a positive integer n, we have*

$$\sum_{j=1}^{n} \frac{\varphi(j)}{j} \le \frac{6n}{\pi^2} + \log n + 1. \qquad (5)$$

Proof. We have

$$\sum_{j=1}^{n} \frac{\varphi(j)}{j} = \sum_{j=1}^{b} \frac{\mu(j)}{j} \left\lfloor \frac{n}{j} \right\rfloor = \sum_{j=1}^{n} \frac{\mu(j)}{j} \left(\frac{n}{j} - \left\{ \frac{n}{j} \right\} \right) = n \sum_{j=1}^{b} \frac{\mu(j)}{j^2} - \sum_{j=1}^{n} \frac{\mu(j)}{j} \left\{ \frac{n}{j} \right\}.$$

Hence, inequality (5) follows from

$$\sum_{j=1}^{n}\frac{\mu(j)}{j^2}=\sum_{j=1}^{\infty}\frac{\mu(j)}{j^2}-\sum_{j>n}\frac{\mu(j)}{j^2}<\frac{6}{\pi^2}+\sum_{j>n}\frac{1}{j^2}\le\frac{6}{\pi^2}+\int_n^\infty\frac{du}{u^2}=\frac{6}{\pi^2}+\frac{1}{n},$$

and

$$-\sum_{j=1}^{n}\frac{\mu(j)}{j}\left\{\frac{n}{j}\right\}\le\sum_{j=2}^{n}\frac{1}{j}<\int_1^n\frac{du}{u}=\log n.$$

□

We now define two functions f and g on $\mathbb{N}$ with values in the positive real numbers given by

$$f(n)=\begin{cases}\sum_{j=1}^{n}\varphi(j)/j & \text{if } n\le 100,\\ 6n/\pi^2+\log n+1 & \text{if } n>100,\end{cases}$$

and

$$g(n)=\begin{cases}\sum_{j=1}^{n}R_j & \text{if } n\le 100,\\ .375n\log n-.432n-10 & \text{if } n>100.\end{cases}$$

We observe from Lemmas 4 and 5 that inequalities $f(n)\le 6n/\pi^2+\log n+1$ for $n\ge 1$ and $g(n)\le .375n\log n$ hold for all $n\ge 7$.

3 Proof of Theorem 1

3.1 Proof of the upper bounds (i) and (ii) in Theorem 1

Let a and b be positive integers with $a\ge b$. If $p\mid M$ and $p\mid N$ for each $p\le b$, then

$$\gcd(M-i,N-j)>1\quad\text{for}\quad 1\le i\le a, 1\le j\le b\quad\text{and}\quad\gcd(i,j)\ne 1.$$

If $p\mid M$ and $N\equiv 1 \pmod{p}$ for every $b<p\le a$, then

$$\gcd(M-i,N-1)>1\quad\text{for}\quad b<i\le a.$$

Let

$$T:=T(a,b):=\{(i,j):1\le i\le a,1\le j\le b,\gcd(i,j)=1\}\setminus\{(i,1):b<i\le a\},$$

and let $t=\#T$. We label the elements of $T(a,b)$ as

$$T(a,b)=\{(i_l,j_l):1\le l\le t\}$$

in lexicographic order. Hence, $(i_1,j_1)=(1,1),(i_2,j_2)=(1,2),\ldots$.

We consider the system of congruences

$$\begin{aligned} M,N &\equiv 0 \pmod p \quad \text{for} \quad p \le b;\\ M &\equiv 0 \pmod p \quad \text{and} \quad N \equiv 1 \pmod p \quad \text{for} \quad b < p \le a; \end{aligned}$$

and

$$M \equiv i_\ell \pmod{p_{\pi(b)+\ell}} \quad \text{and} \quad N \equiv j_l \quad \text{mod } p_{\pi(b)+\ell}) \quad \text{for} \quad 1 \le \ell \le t.$$

By the Chinese Remainder Theorem, we get

$$\max(M,N) \le \prod_{\ell \le \pi(a)+t} p_\ell. \tag{6}$$

We now estimate $\pi(a)+t$. For every $1 \le j \le b$, write $a = jq_j + r_j$ where $0 \le r_j < j$. By dividing a into intervals of length j, we obtain

$$\begin{aligned} t + a - b &= \sum_{j=1}^{b} (q_j \varphi(j) + r'_j) = a \sum_{j=1}^{b} \frac{\varphi(j)}{j} + \sum_{j=1}^{b} \left(r'_j - \frac{r_j \varphi(j)}{j} \right)\\ &\le a \sum_{j=1}^{b} \frac{\varphi(j)}{j} + \sum_{j=1}^{b} R_j, \end{aligned}$$

which gives

$$t + \pi(a) \le ab \left(\frac{\sum_{j=1}^{b} \varphi(j)/j - 1}{b} + \frac{b + \pi(a) + \sum_{j=1}^{b} R_j}{ab} \right).$$

Assume that $b > 100$. By Lemmas 4, 5, 3 (i) and the fact that $a \ge b$, we obtain

$$\begin{aligned} &\frac{\sum_{j=1}^{b} \varphi(j)/j - 1}{b} + \frac{b + \pi(a) + \sum_{j=1}^{b} R_j}{ab}\\ \le\ & \frac{6}{\pi^2} + \frac{\log b}{b} + \frac{b + .375 b \log b - .432 b - 10 + \pi(a)}{ab}\\ \le\ & \frac{6}{\pi^2} + \frac{\log b}{b} + \frac{.568 + \frac{3}{8} \log b}{a} + \frac{a(1 + 1.2762/\log a) - 10}{ab \log a}\\ \le\ & \frac{6}{\pi^2} + \frac{11 \log b}{8b} + \frac{1}{b \log b} \left(1 + \frac{1.2762}{\log b} \right) - \frac{10}{b^2}. \end{aligned} \tag{7}$$

In particular,

$$t + \pi(a) \le \left(\frac{6}{\pi^2} + o(1) \right) ab \qquad \text{when} \qquad b \to \infty. \tag{8}$$

Additionally, since the last expression (7) is a decreasing function of b, we obtain

$$t + \pi(a) \le .67252 ab \qquad \text{for} \qquad b > 100.$$

Define $h_0(b) = .67252$ if $b > 100$ and for $b \le 100$ let this function be defined in the following way:

$$\begin{aligned} h_0(b) &:= \frac{\sum_{j=1}^b \varphi(j)/j - 1}{b} \\ &+ \max_{b \le a \le 100} \left\{ \frac{b + \sum_{j=1}^b R_j + \pi(a)}{ab}, \frac{b + \sum_{j=1}^b R_j}{101b} + \frac{1}{b \log 101}\left(1 + \frac{1.2762}{\log 101}\right)\right\}. \end{aligned}$$

We then obtain from $a \ge b$ and Lemma 3 (i) that $t + \pi(a) \le h_0(b)ab$.

If $\pi(a) + t \le 7$, then $\max(M, N) \le 510510$. In fact, $b \le a \le 4$ in that case. Hence, we now assume that $\pi(a) + t \ge 8$. By Lemma 3 (i) and (iii) and from the fact that $a \ge b$, we have

$$\begin{aligned} \prod_{\ell \le \pi(a)+t} p_\ell &\le \exp\left(abh_0(b)(\log h_0(b)ab + \log\log h_0(b)ab - .75\right) \\ &\le \exp\left(abh_0(b)\log ab\left(1 + \frac{\log h_0(b) + \log\log h_0(b)ab - .75}{\log ab}\right)\right) \\ &\le \exp\left(abh_0(b)\log ab\left(1 + \frac{\log h_0(b) + \log\log h_0(b)b^2 - .75}{\log b^2}\right)\right) \\ &:= \exp(h_1(b)ab\log b). \end{aligned}$$

Here,

$$h_1(b) = h_0(b)\left(1 + \frac{\log h_0(b) + \log\log h_0(b)b^2 - .75}{\log b^2}\right).$$

Making $b \to \infty$, we get (i) of Theorem 1 from (8). For $b > 100$, since $h_0(b) = .67252$, we get

$$h_1(b) \le h_0(b)\left(1 + \frac{\log h_0(b) + \log\log h_0(b) \cdot 101^2 - .75}{\log 101^2}\right) \le .721521 := c_1,$$

which proves (ii) of Theorem 1. Our arguments give upper bounds for $M(a, b)$ and $N(a, b)$ in smaller ranges of b as well. That is, for $b \le 100$, we get $h_1(b) \le c_1(b)$, where the values of c_1 are given by:

b	c_1	b	c_1	b	c_1	b	c_1	b	c_1
2	9432	3	1.1429	4	.9344	5	.99964	6	.8587
7	.9074	8	.8448	9	.8279	10	.7813	11	.8186
12	.7718	13	.8034	14	.7752	15	.7608	16	.7435
17	.7689	18	.7419	19	.7646	20	.7454	≥ 21	.7463

3.2 Proof of the lower bound (iii) of Theorem 1

Let M, N satisfy the conditions of Theorem 1. For each pair (i, j) with $1 \le i \le a$ and $1 \le j \le b$, let $p_{i,j}$ be the least prime dividing $\gcd(M - i, N - j)$. We consider the set

$$\mathcal{P} = \{p_{i,j} : \ 1 \le i \le a, 1 \le j \le b\}.$$

Suppose that $p \in \mathcal{P}$. If $p \mid \gcd(M-i, N-j)$ and $p \mid \gcd(M-i', N-j')$ for some $1 \le i, i' \le a$ and $1 \le j, j' \le b$ with $(i,j) \ne (i',j')$. Then $p \mid (i-i')$ and $p \mid (j-j')$. In particular, $p \le a$. Thus, given $p \in \mathcal{P}$, let (i_0, j_0) be the least pair with $1 \le i_0 \le a$ and $1 \le j_0 \le b$ such that $p \mid \gcd(M-i, N-j)$. Then every other pair (i,j) with $1 \le i \le a$ and $1 \le j \le b$ such that $p \mid \gcd(M-i, N-j)$ has the property that $i = i_0 + up$ and $j = j_0 + vp$ for some nonnegative integers u, v with $0 \le u \le \lfloor (a-1)/p \rfloor$ and $0 \le v \le \lfloor (b-1)/p \rfloor$. Thus, for a fixed p, the number of pairs (i,j) for which $p = p_{i,j}$ is at most

$$\left(1 + \left\lfloor \frac{a-1}{p} \right\rfloor\right)\left(1 + \left\lfloor \frac{b-1}{p} \right\rfloor\right) = 1 + \left\lfloor \frac{a-1}{p} \right\rfloor + \left\lfloor \frac{b-1}{p} \right\rfloor + \left\lfloor \frac{a-1}{p} \right\rfloor \left\lfloor \frac{b-1}{p} \right\rfloor. \tag{9}$$

Putting also

$$T = T(a,b) = \{(i,j) : 1 \le i \le a,\ 1 \le j \le b\},$$

and summing up the above inequality (9) over all the possible primes $p \in \mathcal{P}$, we get that

$$\#T = ab \le \sum_{p \in \mathcal{P}} \left(1 + \frac{a+b}{p} + \frac{ab}{p^2}\right) \le \#\mathcal{P} + (a+b)\sum_{p \le a} \frac{1}{p} + ab \sum_{p \le a} \frac{1}{p^2}. \tag{10}$$

By the prime number theorem, in the right, the second sum is

$$(a+b)\left(\log\log a + O(1)\right) = o(ab)$$

because of the assumption that $\log\log t = o(b)$ as $b \to \infty$. Put

$$c_2 = \sum_{p \ge 2} \frac{1}{p^2} = 1 - c_1$$

and $P = \#\mathcal{P}$. We then get that

$$ab \le P + (c_2 + o(1))ab \quad \text{or} \quad P \ge (c_1 + o(1))ab \qquad (b \to \infty).$$

Now it is clear that

$$\begin{aligned} M^a &> \prod_{1 \le i \le a} (M-i) \ge \prod_{p \in \mathcal{P}} p \\ &\ge \prod_{k \le P} p_k = \exp((1+o(1))P\log P) = \exp\left((c_1 + o(1))ab\log ab\right), \end{aligned}$$

implying the desired inequality (iii) on M. A similar argument proves the inequality for N. Hence, (iii) of Theorem 1 is proved. □

4 Proof of Theorem 2

We now prove Theorem 2 by computing $M(a,a)$ for $a > 1$. We follow the same arguments as in Section 3.2 with $a = b$ and arrive at

$$\#T = a^2 \le \#\mathcal{P} + 2\sum_{p \le a} \left\lfloor \frac{a-1}{p} \right\rfloor + \sum_{p \le a} \left\lfloor \frac{a-1}{p} \right\rfloor^2,$$

giving

$$\#\mathcal{P} \geq a^2 - 2\sum_{p\leq a}\left\lfloor\frac{a-1}{p}\right\rfloor - \sum_{p\leq a}\left\lfloor\frac{a-1}{p}\right\rfloor^2 \geq a^2 - 2a\sum_{p\leq a}\frac{1}{p} - a^2\sum_{p\leq a}\frac{1}{p^2}, \tag{11}$$

and

$$M^a > \prod_{p\in\mathcal{P}} p \geq \prod_{i=1}^{\#\mathcal{P}} p_i = \exp(\theta(p_{\#\mathcal{P}})). \tag{12}$$

Let $a \leq 100$. We explicitly compute the integral part of the middle term of (11), which we call it P_a, and compute $(\prod_{i=1}^{P_a} p_i)^{\frac{1}{a}}$ to get a lower bound of M giving the assertion for $a \leq 100$. In fact we get $M \geq \exp(a\log a)$ for $a \geq 2$. Now we take $a \geq 101$. Then from Lemma 3 (v) and

$$\sum_{p\geq a}\frac{1}{p^2} \leq \zeta(2) - \sum_{i=1}^{100}\frac{1}{i^2} + \sum_{p\leq 100}\frac{1}{p^2} \leq .4604,$$

we get

$$\begin{aligned}\#\mathcal{P} \geq& a^2 - .4604a^2 - 2a\left(\log\log a + .2615 + \frac{1}{\log^2 a}\right)\\ \geq& a^2\left\{.5396 - \frac{2\log\log a + .523 + \frac{2}{\log^2 a}}{a}\right\} \geq .5032a^2\end{aligned}$$

since $a \geq 101$. This together with (12) and Lemma 3 (ii) and (iv) gives

$$\begin{aligned}M^a >& \exp\left(.5032a^2\log(.5032a^2)(1 - \frac{1}{\log(.5032a^2)})\right)\\ >& \exp\left(.5032a^2(\log a)(2 + \frac{\log .5032}{\log a})(1 - \frac{1}{\log(.5032a^2)})\right)\\ >& \exp(.82248a^2\log a)\end{aligned}$$

since $a \geq 101$. Hence, the proof. □

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