isid/ms/2016/01
January 18, 2016
http://www.isid.ac.in/~statmath/index.php?module=Preprint

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K. R. Parthasarathy and Ritabrata Sengupta

Indian Statistical Institute, Delhi Centre
7, SJSS Marg, New Delhi–110 016, India

EXTENDABILITY AND COMPLETE EXTENDABILITY OF GAUSSIAN STATES

K. R. PARTHASARATHY AND RITABRATA SENGUPTA

ABSTRACT. Motivated by the notions of k-extendability and complete extendability of the state of a finite level quantum system as described by Doherty et al (Phys. Rev. A, 69:022308), we introduce parallel definitions in the context of Gaussian states and derive necessary and sufficient conditions for their extendability. It is shown that every separable Gaussian state is completely extendable but the converse is still a conjecture. However, the converse is proved for two mode Gaussian states.

Keywords. Gaussian state, exchangeable Gaussian state, extendability, entanglement.

Mathematics Subject Classification (2010): 81P40, 81P99, 94A15.

1. Introduction

One of the most important problems in quantum mechanics as well as quantum information theory is to determine whether a given bipartite state is separable or entangled [NC10]. There are several methods in tackling this problem leading to a long list of important publications. A detailed discussion on this topic is available in the survey articles by Horodecki et al [HHHH09], and Gühne and Tóth [GT09]. One such condition which is both necessary and sufficient for separability in finite dimensional product spaces is complete extendability [DPS04].

Definition 1.1. *Let $k \in \mathbb{N}$. A state $\rho \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$ is said to be k-extendable with respect to system B if there is a state $\tilde{\rho} \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B^{\otimes k})$ which is invariant under any permutation in $\mathcal{H}_B^{\otimes k}$ and $\rho = \operatorname{Tr}_{\mathcal{H}_B^{\otimes (k-1)}} \tilde{\rho}$, $k \geq 2$.*

A state $\rho \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$ is said to be completely extendable if it is k-extendable for all $k \in \mathbb{N}$.

The following theorem of Doherty, Parrilo, and Spedalieri [DPS04] emphasizes the importance of the notion of complete extendability.

Theorem A. [DPS04] *A bipartite state $\rho \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$ is separable if and only if it is completely extendable with respect to one of its subsystems.*

It is fairly simple to see that separability implies complete extendability. However, the proof of the converse depends upon an application of quantum de Finetti theorem [HM76]. The link between separability and extendability has found applications in quantum information theory [BCY11, BH13]. Here we study the same link in the context of quantum Gaussian states.

The importance of finite mode Gaussian states and their covariance matrices in general quantum theory as well as quantum information has been highlighted extensively in the literature. A comprehensive survey of Gaussian states and their properties can be found in the book of

The authors thank Professor Ajit Iqbal Singh for useful comments and suggestions. RS acknowledges financial support from the National Board for Higher Mathematics, Govt. of India.

Holevo [Hol11]. For their applications to quantum information theory the reader is referred to the survey article by Weedbrook et al [WPGP+12] as well as Holevo's book [Hol12]. For our reference we use [ADMS95, Par10, Par13] for Gaussian states and for notations in the following sections we use [PS15b] and [PS15a].

If ρ is a state of a quantum system and X_i, $i = 1, 2$ are two real-valued observables, or equivalently, self-adjoint operators with finite second moments in the state ρ then the covariance between X_1 and X_2 in the state ρ is the scalar quantity

$$\mathrm{Tr}\left(\frac{1}{2}(X_1X_2 + X_2X_1)\rho\right) - (\mathrm{Tr}\, X_1\rho)\cdot(\mathrm{Tr}\, X_2\rho),$$

which is denoted by $\mathrm{Cov}_\rho(X_1, X_2)$. Suppose q_1, p_1; q_2, p_2; $\cdots$; q_n, p_n are the position - momentum pairs of observables of a quantum system with n degrees of freedom obeying the canonical commutation relations. Then we express

$$(X_1, X_2, \cdots, X_{2n}) = (q_1, -p_1, q_2, -p_2, \cdots, q_n, -p_n).$$

If ρ is a state in which all the X_j's have finite second moments we write

$$S_\rho = [[\mathrm{Cov}_\rho(X_i, X_j)]], \quad i, j \in \{1, 2, \cdots, 2n\}. \tag{1.1}$$

We call S_ρ the covariance matrix of the position momentum observables. If we write

$$J_{2n} = \begin{bmatrix} 0 & 1 & & & & & \\ -1 & 0 & & & & & \\ & & 0 & 1 & & & \\ & & -1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & 1 \\ & & & & & -1 & 0 \end{bmatrix} \tag{1.2}$$

or equivalently $\bigoplus_1^n \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ for the $2n \times 2n$ block diagonal matrix, the complete Heisenberg uncertainty relations for all the position and momentum observables assume the form of the following matrix inequality

$$S_\rho + \frac{\imath}{2} J_{2n} \geq 0. \tag{1.3}$$

Conversely, if S is any real $2n \times 2n$ symmetric matrix obeying the inequality $S + \frac{\imath}{2} J_{2n} \geq 0$, then there exists a state ρ such that S is the covariance matrix S_ρ of the observables $q_1, -p_1$; $q_2, -p_2$; $\cdots$; $q_n, -p_n$. In such a case ρ can be chosen to be a Gaussian state with mean zero. Recall [Par10], a state ρ in $\Gamma(\mathcal{H})$ with $\mathcal{H} = \mathbb{C}^n$ is an n-mode Gaussian state if its Fourier transform $\hat{\rho}$ is given by

$$\hat{\rho}(\boldsymbol{x} + \imath\boldsymbol{y}) = \exp\left[-\imath\sqrt{2}(\boldsymbol{l}^T\boldsymbol{x} - \boldsymbol{m}^T\boldsymbol{y}) - \begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{pmatrix}^T S \begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{pmatrix}\right]. \tag{1.4}$$

for all $\boldsymbol{x}$, $\boldsymbol{y} \in \mathbb{R}^n$ where $\boldsymbol{l}$, $\boldsymbol{m}$ are the momentum-position mean vectors and S their covariance matrix.

2. Gaussian extendability

Definition 2.1 (Gaussian extendability)**.** *Let $k \in \mathbb{N}$. A Gaussian state ρ_g in $\Gamma(\mathbb{C}^m) \otimes \Gamma(\mathbb{C}^n)$ is said to be Gaussian k-extendable with respect to the second system if there is a Gaussian state $\tilde{\rho}_g$ in $\Gamma(\mathbb{C}^m) \otimes \Gamma(\mathbb{C}^n)^{\otimes k}$ which is invariant under any permutation in $\Gamma(\mathbb{C}^n)^{\otimes k}$ and $\rho_g = \operatorname{Tr}_{\Gamma(\mathbb{C}^n)^{\otimes(k-1)}} \tilde{\rho}_g$, $k \geq 2$.*

A Gaussian state ρ_g in $\Gamma(\mathbb{C}^m) \otimes \Gamma(\mathbb{C}^n)$ is said to be Gaussian completely extendable if it is Gaussian k-extendable for every $k \in \mathbb{N}$.

Remark 2.1. *Since we confine our attention to Gaussian states only, throughout this paper, we use the terms k-extendability and complete extendability to mean Gaussian k-extendability and Gaussian complete extendability respectively, unless stated otherwise.*

We shall use the following result.

Theorem B. *Let*

$$X = \left[\begin{array}{c|c} A & B \\ \hline B^\dagger & C \end{array}\right]$$

be a Hermitian block matrix with real or complex entries, A and C being strictly positive matrices of order $m \times m$ and $n \times n$ respectively. Then $X \geq 0$ if and only if

$$A \geq BC^{-1}B^\dagger.$$

Proof. For a proof, see Theorem 1.3.3 in the book of Bhatia [Bha07]. □

Entanglement property of a Gaussian state depends only on its covariance matrix. Hence without loss of generality, we can confine our attention to the Gaussian states with mean zero. Thus an $(m+n)$-mode mean zero Gaussian state in $\Gamma(\mathbb{C}^m) \otimes \Gamma(\mathbb{C}^n)$ is uniquely determined by a $2(m+n) \times 2(m+n)$ covariance matrix

$$S = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}.$$

Here A and C are covariance matrices of the m and n-mode marginal states respectively.

If $\rho(\mathbf{0}, \mathbf{0}; S)$, written in short as $\rho(S)$ in $\Gamma(\mathbb{C}^m) \otimes \Gamma(\mathbb{C}^n)$ is k-extendable with respect to the second system, then there exists a real matrix θ of order $2n \times 2n$ such that the extended matrix

$$S_k = \left[\begin{array}{c|cccc} A & B & B & \cdots & B \\ \hline B^T & C & \theta & \cdots & \theta \\ B^T & \theta^T & C & \cdots & \theta \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B^T & \theta^T & \theta^T & \cdots & C \end{array}\right] \tag{2.1}$$

is the covariance matrix of a Gaussian state in $\Gamma(\mathbb{C}^m) \otimes \Gamma(\mathbb{C}^n)^{\otimes k}$. Then it satisfies inequality (1.3) in the form

$$S_k + \frac{\imath}{2} J_{2(m+kn)} \geq 0. \tag{2.2}$$

Hence by Definition 1.1, $\rho(S)$ is completely extendable if the inequality (2.2) holds for every $k = 1, 2, \cdots$.

Let us denote the marginal covariance matrix corresponding to $\Gamma(\mathbb{C}^n)^{\otimes k}$ by

$$\Sigma_k(C,\theta) = \begin{bmatrix} C & \theta & \cdots & \theta \\ \theta^T & C & \cdots & \theta \\ \vdots & \vdots & \ddots & \vdots \\ \theta^T & \theta^T & \cdots & C \end{bmatrix}.$$

If ρ is completely extendable, S_k is a covariance matrix for each k, and hence $\Sigma_k(C,\theta)$ is a covariance matrix for each k as well. Using Theorem 1 of [PS15a] (see also [KW09]), such a pair (C,θ) defines a covariance matrix $\Sigma_k(C,\theta)$ for each $k = 1, 2, 3, \cdots$ if and only if

(i) θ is a real symmetric positive semidefinite matrix, and
(ii) $C - \theta + \frac{\imath}{2} J_{2n} \geq 0$.

In particular, S_k is of the form

$$S_k = \left[\begin{array}{c|cccc} A & B & B & \cdots & B \\ \hline B^T & C & \theta & \cdots & \theta \\ B^T & \theta & C & \cdots & \theta \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B^T & \theta & \theta & \cdots & C \end{array}\right], \tag{2.3}$$

where θ is a real positive semidefinite matrix.

Our first theorem gives a necessary and sufficient condition for complete extendability of Gaussian states.

Lemma 2.1. *Let ρ be a bipartite Gaussian state in $\Gamma(\mathbb{C}^m) \otimes \Gamma(\mathbb{C}^n)$ with covariance matrix $S = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$, where A and C are marginal covariance matrices of the first and second system respectively. Further, let no pure state be a marginal of the state ρ. Then ρ is completely extendable with respect to the second system if and only if there exists a real positive matrix θ such that*

$$C + \frac{\imath}{2} J_{2n} \geq \theta \geq B^T \left(A + \frac{\imath}{2} J_{2m}\right)^{-1} B. \tag{2.4}$$

Proof. Without loss of generality, we may assume that A and C are written in their Williamson normal forms. Since no pure state is a marginal of ρ, $\frac{1}{2} I_2$ is not a sub-matrix of ether A or C. This implies $\left(A + \frac{\imath}{2} J_{2m}\right)$ and $\left(C + \frac{\imath}{2} J_{2n}\right)$ are invertible, and hence we can apply Theorem B, when A and C are replaced respectively by $\left(A + \frac{\imath}{2} J_{2m}\right)$ and $\left(C + \frac{\imath}{2} J_{2n}\right)$. Thus,

$$C + \frac{\imath}{2} J_{2n} \geq B^T \left(A + \frac{\imath}{2} J_{2m}\right)^{-1} B.$$

The necessity of the left part of inequality 2.4 is already contained in the discussion above (2.3). Hence, all we need to prove is the right part the same inequality starting from (2.2).

Setting $|\psi_k\rangle = \frac{1}{\sqrt{k}}[1, 1, \cdots, 1]^T \in \mathbb{C}^k$ and $\sqrt{k}\mathcal{B}_k = B \otimes \langle\psi_k|$, the left hand side of (2.2) can be expressed as

$$\begin{bmatrix} A + \frac{\imath}{2} J_{2m} & \sqrt{k}\mathcal{B}_k \\ \sqrt{k}\mathcal{B}_k^T & \Sigma_k + \frac{\imath}{2} J_{2nk} \end{bmatrix}.$$

By Theorem B this matrix is positive if and only if

$$\Sigma_k + \frac{\imath}{2} J_{2nk} \geq k \mathcal{B}_k^T \left(A + \frac{\imath}{2} J_{2m}\right)^{-1} \mathcal{B}_k.$$

By elementary algebra, this is equivalent to

$$\left(C - \theta + \frac{\imath}{2} J_{2n}\right) \otimes (I_k - |\psi_k\rangle\langle\psi_k|) + \left(C + \overline{k-1}\theta + \frac{\imath}{2} J_{2n}\right) \otimes |\psi_k\rangle\langle\psi_k|$$
$$\geq k B^T \left(A + \frac{\imath}{2} J_{2m}\right)^{-1} B \otimes |\psi_k\rangle\langle\psi_k| .$$

Since $|\psi_k\rangle\langle\psi_k|$ and $I_k - |\psi_k\rangle\langle\psi_k|$ are mutually orthogonal projections, it follows that the inequality above is equivalent to

$$\left(C + \overline{k-1}\theta + \frac{\imath}{2} J_{2n}\right) \geq k B^T \left(A + \frac{\imath}{2} J_{2m}\right)^{-1} B,$$

which can be rewritten as

$$\frac{1}{k}\left(C - \theta + \frac{\imath}{2} J_{2n}\right) + \theta \geq B^T \left(A + \frac{\imath}{2} J_{2m}\right)^{-1} B, \quad \text{for every } k \in \mathbb{N}. \tag{2.5}$$

Since $\left(C - \theta + \frac{\imath}{2} J_{2n}\right)$ is positive and the left hand side decreases monotonically to θ as $k \to \infty$, it follows that (2.5) is equivalent to

$$\theta \geq B^T \left(A + \frac{\imath}{2} J_{2m}\right)^{-1} B.$$

□

We now consider the case when the Gaussian state ρ_g admits a pure marginal state.

Proposition 2.1. *If* $X = \begin{bmatrix} A & B \\ B^T & \frac{1}{2} I_{2s} \end{bmatrix}$ *is a Gaussian covariance matrix, then* $B = 0$.

Proof. Without loss of generality, we may assume that A is written in its Williamson normal form $A = \oplus_{j=1}^n \kappa_j I_2$, where $\kappa_j \geq \frac{1}{2}$ for each j. Then X can be written as

$$\left[\begin{array}{ccc|ccc} & & & B_{11} & \cdots & B_{1s} \\ & \bigoplus_{j=1}^n \kappa_j I_2 & & \vdots & \ddots & \vdots \\ & & & B_{n1} & \cdots & B_{ns} \\ \hline B_{11}^T & \cdots & B_{n1}^T & & & \\ \vdots & \ddots & \vdots & & \bigoplus_{s\text{-copies}} \frac{1}{2} I_2 & \\ B_{1s}^T & \cdots & B_{ns}^T & & & \end{array}\right];$$

where each B_{jl} is a 2×2 block real matrix. Thus, to prove the above proposition it is sufficient to show that corresponding to each j, $1 \leq j \leq s$, and each l, $1 \leq l \leq s$, if $\begin{bmatrix} \kappa_j I_2 & B_{jl} \\ B_{jl}^T & \frac{1}{2} I_2 \end{bmatrix}$ is a Gaussian covariance matrix, then $B_{jl} = 0$.

By local transformations, this last 2×2 block matrix can be brought to the form

$$\left[\begin{array}{c|c} \kappa_j I_2 & \begin{matrix} e_{jl} & 0 \\ 0 & f_{jl} \end{matrix} \\ \hline \begin{matrix} e_{jl} & 0 \\ 0 & f_{jl} \end{matrix} & \frac{1}{2} I_2 \end{array}\right].$$

The Gaussian property described by inequality (1.3) implies that

$$\begin{bmatrix} \kappa_j & \frac{\imath}{2} & e_{jl} & 0 \\ -\frac{\imath}{2} & \kappa_j & 0 & f_{jl} \\ e_{jl} & 0 & \frac{1}{2} & \frac{\imath}{2} \\ 0 & f_{jl} & -\frac{\imath}{2} & \frac{1}{2} \end{bmatrix} \geq 0. \tag{2.6}$$

Observe that

$$\det \begin{bmatrix} \kappa_j & 0 & f_{jl} \\ 0 & \frac{1}{2} & \frac{\imath}{2} \\ f_{jl} & -\frac{\imath}{2} & \frac{1}{2} \end{bmatrix} = -\frac{f_{jl}^2}{2} \leq 0,$$

which contradicts (2.6) unless $f_{jl} = 0$. Similarly $e_{jl} = 0$. Hence the block matrix $B_{jl} = 0$ for each j, l. □

Theorem 2.1. *Let ρ be a bipartite Gaussian state in $\Gamma(\mathbb{C}^m) \otimes \Gamma(\mathbb{C}^n)$ with covariance matrix $S = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$, where A and C are marginal covariance matrices of the first and second system respectively. Then ρ is completely extendable with respect to the second system if and only if there exists a real positive matrix θ such that*

$$C + \frac{\imath}{2} J_{2n} \geq \theta \geq B^T \left(A + \frac{\imath}{2} J_{2m}\right)^{-} B, \tag{2.7}$$

where $\left(A + \frac{\imath}{2} J_{2m}\right)^{-}$ is the Moore-Penrose inverse of $A + \frac{\imath}{2} J_{2m}$.

Proof. Since the case where both $A + \frac{\imath}{2} J_{2m}$ and $C + \frac{\imath}{2} J_{2n}$ are invertible has already been dealt with in Lemma 2.1, we only need to prove in the case when ρ admits pure marginal states.

Without loss of generality let us assume that A and C are written in their Williamson normal forms. Let $A = \left(\oplus_1^k \kappa_j I_2\right) \bigoplus \left(\oplus_{k+1}^m \frac{1}{2} I_2\right) = A' \bigoplus \frac{1}{2} I_{2(m-k)}$ and $C = \left(\oplus_1^s \mu_l I_2\right) \bigoplus \left(\oplus_{s+1}^n \frac{1}{2} I_2\right) = C' \bigoplus \frac{1}{2} I_{2(n-s)}$, where κ_j, $\mu_l > \frac{1}{2}$ for every j, l. By Proposition 2.1, B has the form

$$B = \left[\begin{array}{c|c} B' & \\ \hline & \end{array}\right],$$

where B' is a real matrix of order $2k \times 2s$ and rest of the entries are zero matrices of appropriate order.

Consider the marginal Gaussian state, whose covariance matrix is

$$\begin{bmatrix} A' & B' \\ B'^T & C' \end{bmatrix}. \tag{2.8}$$

Since A' and C' do not have any principal sub-matrix of the form $\frac{1}{2} I_2$, by Lemma 2.1, the marginal Gaussian state with covariance matrix given by (2.8) is completely extendable if and only if there is a real $2s \times 2s$ matrix θ' such that

$$C' + \frac{\imath}{2} J_{2s} \geq \theta' \geq B'^T \left(A' + \frac{\imath}{2} J_{2k} \right)^{-1} B'.$$

Observe that

$$\begin{aligned} B^T \left(A + \frac{\imath}{2} J_{2n} \right)^{-} B &= \left[\begin{array}{c|c} B'^T & \\ \hline & \end{array} \right] \left(\left(A' + \frac{\imath}{2} J_{2k} \right)^{-1} \bigoplus \left(\bigoplus_{(m-k)\text{-copies}} \begin{bmatrix} \frac{1}{2} & \frac{\imath}{2} \\ -\frac{\imath}{2} & \frac{1}{2} \end{bmatrix} \right) \right) \left[\begin{array}{c|c} B' & \\ \hline & \end{array} \right] \\ &= \left[\begin{array}{c|c} B'^T \left(A' + \frac{\imath}{2} J_{2k} \right)^{-1} B' & \mathbf{0}_{2s \times 2(n-s)} \\ \hline \mathbf{0}_{2(n-s) \times 2s} & \mathbf{0}_{2(n-s) \times 2(n-s)} \end{array} \right], \end{aligned}$$

$\mathbf{0}$ with indices denoting zero matrices. Set $\theta = \theta' \bigoplus \mathbf{0}_{2(n-s)\times 2(n-s)}$. It is easy to see that such a real matrix θ satisfies the conditions of inequality (2.7). Hence the theorem is proved. □

Theorem 2.2. *Any separable Gaussian state in a bipartite system is completely extendable.*

Proof. Let ρ be an $(m+n)$ mode Gaussian state with covariance matrix $\begin{bmatrix} A & B^T \\ B & C \end{bmatrix}$ with A and C being the m and n-mode marginal covariance matrices. By a theorem of Werner and Wolf [WW01], ρ is separable if and only if there exist m-mode and n-mode Gaussian states with covariance matrices X and Y respectively such that

$$\begin{bmatrix} A & B^T \\ B & C \end{bmatrix} \geq \begin{bmatrix} X & \\ & Y \end{bmatrix}.$$

Set $E = A - X$, $G = C - Y$, and $F = B$. Then the above inequality can be expressed as

$$\begin{bmatrix} E & F^T \\ F & G \end{bmatrix} \geq 0.$$

By the previous discussions and Theorem 2.1, we need to construct a real, symmetric, $n \times n$ matrix φ such that for every k-extension, the matrix

$$\begin{bmatrix} E & F^T & F^T & F^T & \cdots & F^T \\ F & G & \varphi & \varphi & \cdots & \varphi \\ F & \varphi & G & \varphi & \cdots & \varphi \\ F & \varphi & \varphi & G & \cdots & \varphi \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ F & \varphi & \varphi & \varphi & \cdots & G \end{bmatrix} \geq 0.$$

Calculations similar to those in Theorem 2.1, show that this is possible if and only if

$$G \geq \varphi \geq F E^{-} F^T. \tag{2.9}$$

We choose

$$\varphi = tG + (1-t) F E^{-} F^T, \quad t \in [0, 1]. \tag{2.10}$$

Notice that for every $k = 1, 2, \cdots$,

$$\begin{bmatrix} X & & & \\ & Y & & \\ & & \ddots & \\ & & & Y \end{bmatrix} + \begin{bmatrix} E & F^T & F^T & F^T & \cdots & F^T \\ F & G & \varphi & \varphi & \cdots & \varphi \\ F & \varphi & G & \varphi & \cdots & \varphi \\ F & \varphi & \varphi & G & \cdots & \varphi \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ F & \varphi & \varphi & \varphi & \cdots & G \end{bmatrix} = \begin{bmatrix} A & B^T & B^T & B^T & \cdots & B^T \\ B & C & \varphi & \varphi & \cdots & \varphi \\ B & \varphi & C & \varphi & \cdots & \varphi \\ B & \varphi & \varphi & C & \cdots & \varphi \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ B & \varphi & \varphi & \varphi & \cdots & C \end{bmatrix},$$

where the first term in the left hand side, $X \oplus (\oplus_k Y)$, is a Gaussian covariance matrix, and the second one is a positive matrix thanks to the construction above. Thus the right hand side is also a Gaussian covariance matrix. Hence the theorem is proved with the extension matrix φ satisfying equation (2.10). □

We conjecture that the converse is also true, i.e. any completely extendable Gaussian state is separable. This is same as saying that separability is equivalent to the existence of a real matrix in the convex matrix interval $\left[B^T\left(A + \frac{\imath}{2}J_{2m}\right)^{-}B, C + \frac{\imath}{2}J_{2n}\right]$ whose end points are complex matrices.

Though we do not have a proof of this statement in general, we prove it for states in two mode systems.

Theorem 2.3. *Any two-mode quantum Gaussian state ρ is completely extendable if and only if it is separable.*

Proof. Since entanglement of a state is invariant under local unitary transformations, we may assume the covariance matrix S_ρ has the following form

$$S_\rho = \left[\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{λI_2}} & \alpha & 0 \\ & & 0 & -\beta \\ \hline \alpha & 0 & \multicolumn{2}{c}{\multirow{2}{*}{μI_2}} \\ 0 & -\beta & & \end{array}\right], \quad \text{where } \lambda,\, \mu \geq \frac{1}{2}, \quad \alpha,\, \beta \geq 0. \tag{2.11}$$

Furthermore, we may assume both λ, $\mu > \frac{1}{2}$, because, otherwise the off-diagonal blocks would be zero and the state would be both separable and completely extendable. Using inequality (1.3) and Theorem B, we conclude that S_ρ is a Gaussian covariance matrix if and only if

$$\begin{aligned} \left(\mu I_2 + \frac{\imath}{2}J_2\right) &\geq \begin{bmatrix} \alpha & 0 \\ 0 & -\beta \end{bmatrix} \left(\lambda I_2 + \frac{\imath}{2}J_2\right)^{-1} \begin{bmatrix} \alpha & 0 \\ 0 & -\beta \end{bmatrix} \\ &= \frac{1}{\lambda^2 - \frac{1}{4}} \begin{bmatrix} \lambda\alpha^2 & \frac{\imath}{2}\alpha\beta \\ -\frac{\imath}{2}\alpha\beta & \lambda\beta^2 \end{bmatrix} \\ &= \begin{bmatrix} \alpha^2\delta & \frac{\imath}{2}\frac{\alpha\beta\delta}{\lambda} \\ -\frac{\imath}{2}\frac{\alpha\beta\delta}{\lambda} & \beta^2\delta \end{bmatrix} \qquad \text{where } \delta = \frac{\delta}{\lambda^2 - \frac{1}{4}} \end{aligned}$$

$$\Leftrightarrow \begin{bmatrix} \mu - \alpha^2\delta & \frac{\imath}{2}\left(1 - \frac{\alpha\beta\delta}{\lambda}\right) \\ -\frac{\imath}{2}\left(1 - \frac{\alpha\beta\delta}{\lambda}\right) & \mu - \beta^2\delta \end{bmatrix} \geq 0.$$

Simplifying we get conditions for Gaussianity as:

$$\begin{cases} \mu \geq \alpha^2\delta \quad \text{and} \quad \mu \geq \beta^2\delta, \\ \\ (\mu-\alpha^2\delta)(\mu-\beta^2\delta) \geq \frac{1}{4}\left(1-\frac{\alpha\beta\delta}{\lambda}\right)^2. \end{cases} \tag{2.12}$$

By Simon's theorem [Sim00], a two mode Gaussian state is entangled if and only if it is not positive under partial transpose. S_ρ is entangled if and only if

$$\left[\begin{array}{cc|cc} \lambda & \frac{\imath}{2} & \alpha & 0 \\ -\frac{\imath}{2} & \lambda & 0 & -\beta \\ \hline \alpha & 0 & \mu & -\frac{\imath}{2} \\ 0 & -\beta & \frac{\imath}{2} & \mu \end{array}\right] \ngeq 0.$$

Using Ando's theorem and calculations similar to Gaussianity, the above matrix condition holds if and only if

$$\begin{bmatrix} \mu-\alpha^2\delta & -\frac{\imath}{2}\left(1+\frac{\alpha\beta\delta}{\lambda}\right) \\ \frac{\imath}{2}\left(1+\frac{\alpha\beta\delta}{\lambda}\right) & \mu-\beta^2\delta \end{bmatrix} \ngeq 0.$$

The condition for entanglement becomes

$$(\mu-\alpha^2\delta)(\mu-\beta^2\delta) < \frac{1}{4}\left(1+\frac{\alpha\beta\delta}{\lambda}\right)^2. \tag{2.13}$$

Combining (2.12) and (2.13) together, a two-mode Gaussian state as in (2.11) is entangled if and only if

$$\begin{cases} \mu \geq \alpha^2\delta \quad \text{and} \quad \mu \geq \beta^2\delta, \\ \\ \frac{1}{4}\left(1-\frac{\alpha\beta\delta}{\lambda}\right)^2 \leq (\mu-\alpha^2\delta)(\mu-\beta^2\delta) < \frac{1}{4}\left(1+\frac{\alpha\beta\delta}{\lambda}\right)^2. \end{cases} \tag{2.14}$$

If in addition, the entangled state is also completely extendable, then by conditions (i) and (ii) of theorem 2.1, there exists a real symmetric positive matrix $\theta = \begin{bmatrix} p & r \\ r & q \end{bmatrix}$ such that

$$\begin{bmatrix} \mu & \frac{\imath}{2} \\ -\frac{\imath}{2} & \mu \end{bmatrix} \geq \begin{bmatrix} p & r \\ r & q \end{bmatrix} \geq \begin{bmatrix} \alpha^2\delta & \frac{\imath}{2}\frac{\alpha\beta\delta}{\lambda} \\ -\frac{\imath}{2}\frac{\alpha\beta\delta}{\lambda} & \beta^2\delta \end{bmatrix}.$$

Taking real parts entry wise in this inequality we may assume that p and q are of the form;

$$\begin{aligned} p &= (1-x)\mu + x\alpha^2\delta, \quad 0 \leq x \leq 1, \\ q &= (1-y)\mu + y\beta^2\delta, \quad 0 \leq y \leq 1. \end{aligned}$$

Using the determinant conditions for positivity in the left and left and right parts of the matrix inequalities, we get:

$$xy(\mu-\alpha^2\delta)(\mu-\beta^2\delta) \geq \frac{1}{4} + r^2, \tag{2.15}$$

$$(1-x)(1-y)(\mu-\alpha^2\delta)(\mu-\beta^2\delta) \geq \frac{1}{4}\frac{\alpha^2\beta^2\delta^2}{\lambda^2} + r^2. \tag{2.16}$$

Set $D^2 = 4(\mu - \alpha^2\delta)(\mu - \beta^2\delta)$. By inequalities (2.14) – (2.16) we get the following:

$$\left(1 - \frac{\alpha\beta\delta}{\lambda}\right)^2 \leq D^2 < \left(1 + \frac{\alpha\beta\delta}{\lambda}\right)^2 \tag{2.17}$$

$$xyD^2 \geq 1 + 4r^2 \tag{2.18}$$

$$(1-x)(1-y)D^2 \geq \frac{\alpha^2\beta^2\delta^2}{\lambda^2} + 4r^2. \tag{2.19}$$

From (2.17) we get

$$D < 1 + \frac{\alpha\beta\delta}{\lambda}. \tag{2.20}$$

From (2.18) and (2.19) we get

$$\left(\sqrt{xy} + \sqrt{(1-x)(1-y)}\right) D \geq \sqrt{1+4r^2} + \sqrt{\frac{\alpha^2\beta^2\delta^2}{\lambda^2} + 4r^2} \geq 1 + \frac{\alpha\beta\delta}{\lambda} > D. \tag{2.21}$$

a contradiction, because the coefficient of D on the left hand side of (2.21) is in $[0, 1]$.

Hence a two mode quantum Gaussian state is completely entangled if and only if it is separable. $\square$

3. Conclusion

Motivated by the notions of extendability and complete extendability of finite level states as described by Doherty et al [DPS04] we introduce similar definitions for Gaussian states. A necessary and sufficient condition is obtained for the complete extendability of a bipartite Gaussian state in terms of its covariance matrix. A bipartite separable Gaussian state turns out to be completely extendable. If a 2-mode Gaussian state is completely extendable then it is separable.

It is conjectured that the complete extendability of a bipartite Gaussian state with arbitrary modes is separable.

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Theoretical Statistics and Mathematics Unit, Indian Statistical Institute, Delhi Centre, 7 S J S Sansanwal Marg, New Delhi 110 016, India
E-mail address, K R Parthasarathy: krp@isid.ac.in

Theoretical Statistics and Mathematics Unit, Indian Statistical Institute, Delhi Centre, 7 S J S Sansanwal Marg, New Delhi 110 016, India
E-mail address, Ritabrata Sengupta: rb@isid.ac.in