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Abstract: A replacement procedure to construct orthogonal arrays of strength *three* was proposed by Suen, Das and Dey (2001). This method was later extended by Suen and Dey (2003). In this paper, we further explore the replacement procedure to obtain some new families of orthogonal arrays of strength *three*.
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1 Introduction and Preliminaries

Asymmetric orthogonal arrays introduced by Rao (1973) have received wide attention in recent years. Such arrays are useful in experimental designs as universally optimal fractional factorial plans and their use in industrial experiments for quality improvement has also been widespread. The construction of asymmetric orthogonal arrays of strength *two* have been studied extensively, and one may refer to Hedayat, Sloane and Stufken (1999) for a comprehensive account of these. More recent work on orthogonal arrays of strength two include those by Suen and Kuhfeld (2005) and Chen, Ji and Lei (2014). Methods of constructing asymmetric orthogonal arrays of strength greater than *two* have not been studied as extensively as those of strength two. Some of these methods can be found e.g., in Hedayat *et al.* (1999), Dey and Mukerjee (1999), Suen, Das and Dey (2001), Suen and Dey (2003), Jiang and Yin (2013) and Zhang, Zong and Dey (2016).

Suen *et al.* (2001) proposed a replacement procedure for replacing a column with 2^k-symbols in an orthogonal array of strength *three* by several 2-symbol columns to obtain new families of tight asymmetric orthogonal arrays of strength *three*. The replacement procedure of Suen *et al.* (2001) was extended by Suen and Dey (2003). In this paper, we further explore the replacement procedure to obtain several new families of orthogonal arrays of strength *three*. Some of the constructed arrays are *tight*.

Recall that an orthogonal array $OA(N, n, m_1 \times \cdots \times m_n, g)$ of strength g is an $N \times n$ matrix with elements in the ith $(1 \leq i \leq n)$ column from a finite set of $m_i(\geq 2)$ distinct elements, such that in every $N \times g$ subarray, all possible combinations of elements appear equally often as rows. When $m_1 = m_2 = \cdots = m_n = m$, say, we have a symmetric orthogonal array, denoted by $OA(N, n, m, g)$; otherwise the array called asymmetric. It is well-known that in an $OA(N, n, m_1 \times \cdots \times m_n, 3)$ of strength *three*,

$$N \geq 1 + \sum_{i=1}^{n}(m_i - 1) + (m^* - 1)\left\{\sum_{i=1}^{n}(m_i - 1) - (m^* - 1)\right\}, \qquad (1)$$

where $m^* = \max\limits_{1\leq i\leq n} m_i$. Arrays of strength *three* attaining the above bound are called *tight*.

Throughout, following the terminology in factorial experiments, we call the columns of an $OA(N, n, m_1 \times \cdots \times m_n, g)$ *factors*, and denote these factors by $L_1, L_2, \cdots, L_n$. Let $GF(s)$ denote a Galois field of order s. We can then write the elements of $GF(s)$ as $\{0, 1, w, w^2, \cdots, w^{s-2}\}$, where 0, 1 are the identity elements of $GF(s)$ with respect to the operations of 'addition' and 'multiplication', respectively and w is a primitive element of $GF(s)$. Throughout, for a matrix A, A^T denotes its transpose. We shall need the following result.

Lemma 1. *Let α, β be two elements of $GF(s)$ such that $\alpha^2 = \beta^2$. Then (i) $\alpha = \beta$, if s is even, (ii) either $\alpha = \beta$ or $\alpha = -\beta$, if s is odd.*

If $\alpha_0, \alpha_1, \cdots, \alpha_{s-1}$ are the elements of $GF(s)$, then the set $S = \{\alpha_0^2, \alpha_1^2, \cdots, \alpha_{s-1}^2\}$ contains all the elements of $GF(s)$, if s is even. If s is odd, then the elements of the set S are 0 and $(s-1)/2$ non-zero elements of $GF(s)$, each non-zero element appearing twice in S.

For the factor L_i $(1 \leq i \leq n)$, define u_i columns, each of order $t \times 1$ over $GF(s)$, say d_{i1}, d_{i2}, $\cdots$, d_{iu_i}. Thus, for the n factors, we have in all $\sum_{i=1}^{n} u_i$ columns. Also, let B be a $s^t \times t$ matrix whose rows are all possible t-tuples over $GF(s)$. Suen *et al.* (2001) proved the following result.

Theorem 1 (Suen *et al.*, 2001). *Consider a* $t \times \sum_{i=1}^{n} u_i$ *matrix* $H = [A_1, A_2, \cdots, A_n]$, $A_i = [d_{i1}, d_{i2}, \cdots, d_{iu_i}]$, $1 \leq i \leq n$, *such that for any choice of* g *matrices* A_{i_1}, A_{i_2}, $\cdots$, A_{i_g} *from* A_1, A_2, $\cdots$, A_n, *the* $t \times \sum_{j=1}^{g} u_{i_j}$ *matrix* $[A_{i_1}, A_{i_2}, \cdots, A_{i_g}]$ *has full column rank over* $GF(s)$. *Then an* $OA(s^t, n, (s^{u_1}) \times (s^{u_2}) \times \cdots \times (s^{u_n}), g)$ *can be constructed.*

2 A Replacement Method

Let $s = p^k$, where p is a prime and $k(\geq 2)$ is an integer. We start with an asymmetric orthogonal array $OA(s^m, n, (s^r) \times s^{n-1}, 3)$ (note that when $r = 1$, the starting orthogonal array is a symmetric orthogonal array $OA(s^m, n, s, 3)$). Following a replacement procedure, an asymmetric orthogonal array $OA(ps^m, t+u+1, (ps^r) \times s^t \times p^u, 3)$ can be constructed, where t, u are integers. Our method of construction involves the following steps.

Step 1: Construct an asymmetric orthogonal array $OA(s^m, n, (s^r) \times s^{n-1}, 3)$. This array can be constructed using Theorem 1 by selecting n matrices over $GF(s)$, namely $A_1, A_2, \cdots, A_n$, such that for any choice of 3 distinct matrices A_i, A_j, A_k, $i, j, k \in \{1, 2, \cdots, n\}$, the matrix $[A_i, A_j, A_k]$ has full column rank. For example, an array $OA(s^4, s+2, (s^2) \times s^{s+1}, 3)$ $(s = 2^k)$ can be constructed by taking these matrices as $A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}^T$, $A_2 = [0, 0, 0, 1]^T$, $A_3 = [0, 0, 1, 0]^T$, $A_{4+i} = [0, w^{2i}, 1, w^i]^T$, $0 \leq i \leq s-2$, where w is a primitive element of $GF(2^k)$.

Step 2: Note that for obtaining the $OA(s^m, n, (s^r) \times s^{n-1}, 3)$, we use the elements of $GF(s)$, $s = p^k$. However, we find it more convenient to use elements of $GF(p)$ than those of $GF(p^k)$. In order to replace the elements of $GF(p^k)$ by those of $GF(p)$, we need a matrix representation of the elements of $GF(p^k)$, where the entries of these matrices are the elements of $GF(p)$. Let the irreducible polynomial of $GF(p^k)$ be $w^k + \alpha_{k-1}w^{k-1} + \cdots + \alpha_1 w + \alpha_0$, where w is a primitive element of $GF(p^k)$, $\alpha_j \in GF(p)$, $0 \leq j \leq k-1$. Then the companion matrix of the irreducible polynomial is

$$W = \begin{bmatrix} 0 & 0 & \cdots & 0 & -\alpha_0 \\ 1 & 0 & \cdots & 0 & -\alpha_1 \\ 0 & 1 & \cdots & 0 & -\alpha_2 \\ \vdots & & & & \\ 0 & 0 & \cdots & 1 & -\alpha_{k-1} \end{bmatrix}.$$

A typical element w^i of $GF(p^k)$ corresponds to a $k \times k$ matrix W^i with entries from $GF(p)$, where 0 is represented by a null matrix of order k and 1 is represented by identity matrix of order k. Replacing the elements of $GF(p^k)$ in A_1, A_2, $\cdots$, A_n by those of $GF(p)$, we get matrices $A_1^*, A_2^*, \cdots, A_n^*$, where A_1^* is an $mk \times rk$ matrix with elements from $GF(p)$, and A_j^* $(2 \leq j \leq n)$ are of order $mk \times k$.

Next, define the following matrices:

$$P_1 = \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & A_1^* \end{bmatrix}, \quad P_i = \begin{bmatrix} \mathbf{0}^T \\ A_i^* \end{bmatrix}, i = 2, \cdots, n, \tag{2}$$

where $\mathbf{0}$ is a null vector. Let $L_1 = [P_1, P_2, \cdots, P_n]$ and B_1 be a $p^{mk+1} \times (mk+1)$ matrix with rows as all possible $(mk+1)$-tuples over $GF(p)$. Take the product $B_1 L_1$ and replace the $p^{rk+1} = ps^r$ distinct combinations under the $(rk+1)$ columns of $B_1 P_1$ by ps^r distinct levels of L_1 as well as the $p^k = s$ distinct combinations under the k columns of $B_1 P_j$ by s distinct levels of the factor L_j for each j, $2 \leq j \leq n$. The array $OA(ps^m, n, (ps^r) \times s^{n-1}, 3)$ can now be constructed via Theorem 1.

Step 3: Finally, we give a method to construct an orthogonal array of type $OA(ps^m, t+u+1, (ps^r) \times s^t \times p^u, 3)$. To obtain this family of orthogonal arrays, we replace a p^k-level column by several p-level columns in the array $OA(ps^m, n, (ps^r) \times s^{n-1}, 3)$. The replacement procedure is as follows.

Let B_2 be a matrix of order $k \times h$. Consider the matrices P_i $(i \in \{2, \cdots, n\})$ defined above. Let

$$M_i = P_i B_2 = \begin{bmatrix} \mathbf{0}^T \\ A_i^* B_2 \end{bmatrix}, 2 \leq i \leq n, \tag{3}$$

then the elements of the first row of M_i are all zeros. Substituting the first row of all zeros by a row of all ones, the matrices

$$Q_i = \begin{bmatrix} \mathbf{e} \\ A_i^* B_2 \end{bmatrix} \tag{4}$$

are obtained. This replacement procedure does not disturb the orthogonality and can be used for each factor P_i, $2 \leq i \leq n$. Then the array $OA(ps^m, t+u+1, (ps^r) \times s^t \times p^u, 3)$, $0 \leq t \leq n-1$ can be constructed by choosing the matrix $H = [P_1, P_2, \cdots, P_{t+1}, Q_{t+2}, \cdots, Q_n]$, where $Q_i = [q_{i1}, q_{i2}, \cdots, q_{i,h}]$ and q_{ij} is the column of the factor $P_{t+1+(i-t-2)h+j}$ with p symbols, $t+2 \leq i \leq n$, $1 \leq j \leq h$.

Thus, following the above steps, an orthogonal array $OA(ps^m, t+u+1, (ps^r) \times s^t \times p^u, 3)$ can be constructed. Since B_2 depends on whether s is even or odd, we deal with these two case separately. In the next section, we construct several families of asymmetric orthogonal arrays of strength three.

3 Construction of Orthogonal Arrays

When s is a power of two, several tight orthogonal arrays of strength three were obtained by Suen *et al.* (2001) by invoking Theorem 1 and also via a replacement procedure. In this section, we first obtain two *new* families of orthogonal arrays of strength three, when s is a power or of two. To begin with, we have the following results.

Lemma 2. *Let B be a $k \times (2^k - 1)$ matrix whose columns are all possible k-tuples over $GF(2)$, excluding the null column and G be defined as $G = \begin{bmatrix} \boldsymbol{e} \\ B \end{bmatrix}$, where $\boldsymbol{e}$ is a row of all ones of order $2^k - 1$. Then any three columns of G are linearly independent.*

Proof. Since each of the matrices of the following three types

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

has full column rank, choosing any three columns g_1, g_2, g_3 of G, in the matrix $[g_1, g_2, g_3]$ there exits a 3×3 subarray with one of the above three types. This can be done through elementary row and column operations. So any three columns of G are linearly independent. ∎

Theorem 2. *Suppose A_1, A_2, $\cdots$, A_n are matrices over $GF(2^k)$ with element 1 at the same position, and any three of them are linearly independent. If the matrix $F = [P_1, P_2, \cdots, P_t, Q_{t+1}, \cdots, Q_n]$ is obtained by the replacement* *method described above for s even, then any three factors of F are linearly independent.*

A proof of this Theorem is given in Appendix.

Using Theorem 2, we have the following families of tight arrays.

Theorem 3. *If s is a power of two, then a tight array $OA(2s^4, s^2-(s-2)t, (2s^2)\times s^t \times 2^{(s+1-t)(s-1)}, 3)$ can be constructed for $0 \le t \le s+1$.*

Proof. First, we construct an array $OA(s^4, s+2, (s^2)\times s^{s+1}, 3)$. According to Suen *et al.* (2001), the matrices corresponding to the factors are as follows:

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad A_{4+i} = \begin{bmatrix} 0 \\ w^{2i} \\ 1 \\ w^i \end{bmatrix}, 0 \le i \le s-2,$$

where w is a primitive element of $GF(2^k)$. Then the array $OA(s^4, s+2, (s^2)\times s^{s+1}, 3)$ can be constructed by Theorem 1. For $k \ge 3$, each matrix A_k has element 1 at the same position, and by Theorem 2, we know that any three factors of $F = [P_3, P_4, \cdots, P_t, Q_{t+1}, \cdots, Q_{s+2}]$ are linearly independent. By the replacement steps, P_2 is obtained via the replacement of A_2. Next, we prove the orthogonality of P_2 and F, for which we need to consider six cases: $[P_2, P_a, P_b]$, $[P_2, P_a, Q_b]$, $[P_2, Q_a, Q_b]$, $[Q_2, P_a, P_b]$, $[Q_2, P_a, Q_b]$, $[Q_2, Q_a, Q_b]$, a, $b \ge 3$. To save space, we provide a proof of only the case $[Q_2, Q_a, Q_b]$; the other cases can be handled in a similar fashion.

Replacing the s-symbol column in P_2, P_a, P_b by 2^k-1 columns with 2 symbols each, we get

$$Q_2 = \begin{bmatrix} \mathbf{e} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ B \end{bmatrix}, \quad Q_a = \begin{bmatrix} \mathbf{e} \\ \mathbf{0} \\ W^{2i}B \\ B \\ W^iB \end{bmatrix}, \quad Q_b = \begin{bmatrix} \mathbf{e} \\ \mathbf{0} \\ W^{2j}B \\ B \\ W^jB \end{bmatrix}.$$

where $\mathbf{e}$ is a $1\times(2^k-1)$ vector of all ones, $\mathbf{0}$ is a null matrix of order $k\times(2^k-1)$ and, B is a matrix of order $k\times(2^k-1)$ whose columns are all possible k-tuples over $GF(2)$, excluding the null column. Choosing any three columns q_2, q_a, q_b of Q_2, Q_a, Q_b, the matrix $[q_2, q_a, q_b]$ exists a 3×3 subarray with one of the following types:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & \alpha \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix},$$

where α is any element of $GF(2)$. Hence an array $OA(2s^4, s^2-(s-2)t, (2s^2)\times s^t \times 2^{(s+1-t)(s-1)}, 3)$ can be constructed for $0 \le t \le s+1$. The tightness of the array follows from (1). ■

Theorem 4. *If s is a power of two, then a tight array $OA(2s^5, s^3-(s-2)t, (2s^2)\times s^t \times 2^{(s^2+s+1-t)(s-1)}, 3)$ can be constructed for $0 \le t \le s^2+s+1$.*

Proof. First, construct an array $OA(s^5, s^2+s+2, (s^2)\times s^{s^2+s+1}, 3)$. Following Suen *et el.* (2001), define the matrices corresponding to the factors as:

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}^T, \quad A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T,$$

$A_3, \cdots, A_{s+2}$ are of the form $[0, \alpha^2, 0, 1, \alpha]$, $\alpha \in GF(s)$, and $A_{s+3}, \cdots, A_{s^2+s+2}$ are of the form $[\beta^2, \gamma^2, 1, \beta, \gamma]$, $\beta, \gamma \in GF(s)$. Then the array $OA(s^5, s^2+s+2, (s^2)\times s^{s^2+s+1}, 3)$ can be constructed via Theorem 1.

For $3 \le k \le s+2$, each matrix A_k has element 1 at the same position, by Theorem 2, any three factors of $F_1 = [P_3, P_4, \cdots, P_{t_1}, Q_{t_1+1}, \cdots, Q_{s+2}]$ are linearly independent. In a similar way, any three factors of $F_2 = [P_{s+3}, P_{s+4}, \cdots, P_{t_2}, Q_{t_2+1}, \cdots, Q_{s^2+s+2}]$ are linearly independent. By the replacement steps, P_2 is obtained from A_2. The orthogonality of P_2, F_1 and F_2 can be handled as in the case of Theorem 3. Hence an array $OA(2s^5, s^3-(s-2)t, (2s^2)\times s^t \times 2^{(s^2+s+1-t)(s-1)}, 3)$ can be constructed for $0 \le t \le s^2+s+1$. The tightness of the array follows from (1). ■

We now consider the case when s is an odd prime power. We confine to the case when s is a power of 3 as, arrays for this case are not too large. For other values of s being an odd prime power, methods similar to $s = 3^k$ can be employed, but then the size of the arrays will be quite large. When s is a power of three, the elements of $GF(s)$ can be written as $\{-w^{\frac{s-3}{2}}, \cdots, -w, -w^0, 0, w^0, w, \cdots, w^{\frac{s-3}{2}}\}$. An orthogonal array $OA(s^m, n, (s^r)\times s^{n-1}, 3)$ ($s = 3^k$, k (≥ 2) is an integer) can be constructed by selecting n matrices over $GF(3^k)$, namely A_1, A_2, $\cdots$, A_n such that these matrices satisfy the rank condition of Theorem 1. Here also, for constructing an $OA(s^m, n, (s^r)\times s^{n-1}, 3)$, it is more convenient to use elements of $GF(3)$ than those of $GF(3^k)$. So an array $OA(3s^m, n, (3s^r) \times s^{n-1}, 3)$ can be constructed by the replacement steps of Section 2.

To obtain an orthogonal array of the type $OA(3s^m, t+u+1, (3s^r) \times s^t \times 3^u, 3)$, we replace a 3^k-symbol column by several 3-symbol columns. Similar to the case when s is even, we start by replacing a column with 3^k symbols by $3^k - 1$ columns with all possible k-tuples over $GF(3)$, excluding the null column. We replace a 3^k-symbol column by $2k$ columns each with 3 symbols, without disturbing the orthogonality. Let $B_3 = [I_k, 2I_k]$, I_k is identity matrix of order k. From (2), (3) and (4), we get $M_i = P_i B_3 = \begin{bmatrix} \mathbf{0}^T & \mathbf{0}^T \\ A_i^* & 2A_i^* \end{bmatrix}$, $Q_i = \begin{bmatrix} \mathbf{e} & \mathbf{e} \\ A_i^* & 2A_i^* \end{bmatrix}$, $2 \leq i \leq n$, where, as before, $\mathbf{0}$ is a null vector and $\mathbf{e}$ is a row of all ones. It can be seen that the factors L_i $(2 \leq i \leq n)$ corresponding to 3^k symbols can be replaced by $2k$ factors with 3 symbols each, denoted by the matrix Q_i, without disturbing the rank condition. Then the array $OA(3s^m, (n-1-t)2k+t+1, (3s^r) \times s^t \times 3^{2k(n-1-t)}, 3)$ $(0 \leq t \leq n-1)$ can be constructed by choosing the matrix $H = [P_1, P_2, \cdots, P_{t+1}, Q_{t+2}, \cdots, Q_n]$, where $Q_i = [q_{i1}, q_{i2}, \cdots, q_{(i,2k)}]$ and q_{ij} is the column of the factor $L_{t+1+(i-t-2)k+j}$ with 3 symbols, $t+2 \leq i \leq n$, $1 \leq j \leq 2k$.

Here is an example to illustrate the above steps of construction.

Example. Suppose $s = 9 = 3^2$ so that $k = 2$. We start from constructing a symmetric orthogonal array $OA(9^3, 10, 9, 3)$. Define the following matrices correspond to the factors: $A_1 = [1, 0, 0]^T$, $A_2 = [1, 1, 1]^T$, $A_3 = [1, w, w^2]^T$, $A_4 = [1, w^2, w^4]^T$, $A_5 = [1, w^3, w^6]^T$, $A_6 = [1, w^4, w^0]^T$, $A_7 = [1, w^5, w^2]^T$, $A_8 = [1, w^6, w^4]^T$, $A_9 = [1, w^7, w^6]^T$, $A_{10} = [0, 0, 1]^T$, where w is a primitive element of $GF(3^2)$. Theorem 1 can now be used to construct an orthogonal array $OA(9^3, 10, 9, 3)$. An irreducible polynomial of $GF(3^2)$ is taken as $w^2 + w + 2$. Then the companion matrix is

$$W = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix},$$

and the elements of $GF(3^2)$ can be represented by 2×2 matrices

$$0 \equiv \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad 1 \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad w \equiv \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}, \quad w^2 \equiv \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}, \quad w^3 \equiv \begin{bmatrix} 2 & 2 \\ 2 & 0 \end{bmatrix},$$

$$w^4 \equiv \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad w^5 \equiv \begin{bmatrix} 0 & 2 \\ 2 & 1 \end{bmatrix}, \quad w^6 \equiv \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \quad w^7 \equiv \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

Replacing the elements of $GF(3^2)$ in A_1, A_2, $\cdots$, A_{10} by the above matrices, we get matrices A_i^* $(1 \le i \le 10)$ with the elements from $GF(3)$ as

$$A_1^* = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}^T, \quad \cdots, \quad A_{10}^* = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T.$$

From the matrices A_i^* $(1 \le i \le 10)$, we get the matrices P_i $(1 \le i \le 10)$, where

$$P_1 = \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & A_1^* \end{bmatrix}, \quad P_j = \begin{bmatrix} \mathbf{0}^T \\ A_j^* \end{bmatrix}, \quad 2 \le j \le 10.$$

According to the previous analysis and using the matrices P_i $(1 \le i \le 10)$, the array $OA(3 \times 9^3, 10, 27 \times 9^9, 3)$ can be constructed. We next illustrate the replacement procedure for replacing a 9-symbol column by 4 columns, each with 3-symbols. For example, the first 9-symbol column denoted by P_2, can be written as

$$P_2 = \begin{array}{ll} 00 & \to 1111 \\ 10 & \to 1020 \\ 01 & \to 0102 \\ 10 & \to 1020 \\ 01 & \to 0102 \\ 10 & \to 1020 \\ 01 & \to 0102 \end{array} = Q_2.$$

The other 9-symbol columns can be handled in a similar way. Hence the array $OA(3 \times 9^3, 37 - 3t, 27 \times 9^t \times 3^{36-4t}, 3)$ $(0 \le t \le 9)$ can be constructed by choosing the matrix $H = [P_1, P_2, \cdots, P_{t+1}, Q_{t+2}, \cdots, Q_{10}]$, where the matrix Q_i has four columns, each having 3 symbols.

For obtaining more orthogonal arrays, we need the following results.

Lemma 3. *Let $B = [I_k, 2I_k]$, where I_k is an identity matrix of order k. Define the matrix G as $G = \begin{bmatrix} \mathbf{e} \\ B \end{bmatrix}$, where $\mathbf{e}$ is a row of all ones of order $2k$. Then any three columns of G are linearly independent.*

Proof. Choose any three columns g_1, g_2, g_3 of G. Suppose the three columns g_1, g_2, g_3 are from $\begin{bmatrix} \mathbf{e} \\ I_k \end{bmatrix}$ or $\begin{bmatrix} \mathbf{e} \\ 2I_k \end{bmatrix}$, then clearly, the matrix $[g_1, g_2, g_3]$ has full column rank. Suppose the three columns g_1, g_2, g_3 are from $\begin{bmatrix} \mathbf{e} \\ I_k \end{bmatrix}$ and $\begin{bmatrix} \mathbf{e} \\ 2I_k \end{bmatrix}$, respectively. Then the matrix $[g_1, g_2, g_3]$ exists a 3×3 subarray with one of the following types by conducting row and column transformation:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}.$$

Hence any three columns of G are linearly independent. ■

Theorem 5. *Suppose A_1, A_2, $\cdots$, A_n are matrices with element 1 at the same position over $GF(3^k)$, and any three of them are linearly independent. If the matrix $F = [P_1, P_2, \cdots, P_t, Q_{t+1}, \cdots, Q_n]$ is obtained by the replacement method described earlier, then any three factors of F are linearly independent.* A proof of this Theorem is given in the Appendix.

By Theorem 5, we have the following results.

Theorem 6. *If s is a power of three, then an array $OA(3s^4, 2sk - (2k-1)t + 2k + 1, (3s^2) \times s^t \times 2^{2k(s+1-t)}, 3)$ can be constructed for $0 \leq t \leq s+1$.*

Proof. First, construct an array $OA(s^4, s+2, (s^2) \times s^{s+1}, 3)$. Following Suen *et al.* (2001), the matrices corresponding to the factors are

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad A_{4+2i} = \begin{bmatrix} 0 \\ w^{2i} \\ 1 \\ w^i \end{bmatrix},$$

$$A_{5+2i} = \begin{bmatrix} 1 \\ w^{2i} \\ 1 \\ -w^i \end{bmatrix},$$

$0 \leq i \leq \frac{s-3}{2}$, where w is a primitive element of $GF(3^k)$. Then the array $OA(s^4, s+2, (s^2) \times s^{s+1}, 3)$ can be constructed by Theorem 1. For $k \geq 3$, each matrix A_k has element 1 at the same position, and by Theorem 5, we know that any three factors of $F = [P_3, P_4, \cdots, P_t, Q_{t+1}, \cdots, Q_{s+2}]$ are linearly independent. By the replacement steps, P_2 is obtained from A_2. Let $N_1 = \{3, 4, 6, 8, \cdots, s+1\}$, $N_2 = \{5, 7, 9, \cdots, s+2\}$. Next, we prove the orthogonality of P_2 and F, which has 18 cases. To save space, we prove only the case $[Q_2, Q_a, Q_b]$, $a \in N_1$, $b \in N_2$. Replacing the s-symbol column in P_2, P_a, P_b by $2k$ columns with 3-symbols each, we get

$$Q_2 = \begin{bmatrix} \mathbf{e} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ B \end{bmatrix}, \quad Q_a = \begin{bmatrix} \mathbf{e} \\ \mathbf{0} \\ W^{2i}B \\ B \\ W^iB \end{bmatrix}, \quad Q_b = \begin{bmatrix} \mathbf{e} \\ B \\ W^{2j}B \\ B \\ -W^jB \end{bmatrix},$$

where $\mathbf{e}$ is a row of ones of order $2k$, $\mathbf{0}$ is a null matrix of order $k \times 2k$ and $B = [I_k, 2I_k]$. Choosing three columns q_2, q_a, q_b of Q_2, Q_a, Q_b, in the matrix $[q_2, q_a, q_b]$ there exists a 3×3 subarray of one of the following types:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & \alpha \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 2 & \alpha \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 1 & \alpha \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 2 & \alpha \end{bmatrix},$$

where α is any element of $GF(3)$. Other cases can be handled in a similar fashion. Hence an array $OA(3s^4, 2sk - (2k-1)t + 2k + 1, (3s^2) \times s^t \times 2^{2k(s+1-t)}, 3)$ can be constructed for $s = 3^k$, $0 \le t \le s + 1$. ∎

Theorem 7. *If s is a power of three, then an array $OA(3s^5, 2(s+1)ks - (2k-1)t + 1, (3s^2) \times s^t \times 3^{2k(s^2+s-t)}, 3)$ can be constructed for $0 \le t \le s^2 + s$.*

Proof. First, construct an $OA(s^5, s^2+s+1, (s^2) \times s^{s^2+s}, 3)$. Following Zhang *et al.* (2016), define the matrices corresponding to the factors:

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}^T, \quad A_2 = \begin{bmatrix} 0 & 2 & 0 & 0 & 1 \end{bmatrix}^T,$$

$A_3, \cdots, A_{s+1}$ of the form $[0, \alpha^2, 0, 1, \alpha]$, $\alpha \in GF(s)$, $\alpha \neq 0$, and $A_{s+2}, \cdots, A_{s^2+s+1}$ of the form $[\beta^2, \gamma^2, 1, \beta, \gamma]$, $\beta, \gamma \in GF(s)$. Then an array $OA(s^5, s^2 + s + 1, (s^2) \times s^{s^2+s}, 3)$ can be constructed by Theorem 1. For $3 \le k \le s+1$, each matrix A_k has element 1 at the same position, and by Theorem 5, any three factors of $F_1 = [P_3, P_4, \cdots, P_{t_1}, Q_{t_1+1}, \cdots, Q_{s+1}]$ are linearly independent. In a similar way, any three factors of $F_2 = [P_{s+2}, P_{s+3}, \cdots, P_{t_2}, Q_{t_2+1}, \cdots, Q_{s^2+s+1}]$ are linearly independent. By the replacement steps, P_2 is obtained from A_2. The orthogonality of P_2, F_1 and F_2 can be handled in a similar way as in Theorem 6. Hence the array $OA(3s^5, 2(s+1)ks - (2k-1)t + 1, (3s^2) \times s^t \times 3^{2k(s^2+s-t)}, 3)$ can be constructed for $0 \le t \le s^2 + s$. ∎

Appendix

Proof of Theorem 2.

Without loss of generality, set $A_i = \begin{bmatrix} 1 \\ D_i \end{bmatrix}$, $1 \le i \le n$. By the replacement method, we obtain $A_i^* = \begin{bmatrix} I_k \\ D_i^* \end{bmatrix}$, $P_i = \begin{bmatrix} \mathbf{0}^T \\ I_k \\ D_i^* \end{bmatrix}$, $Q_i = \begin{bmatrix} \mathbf{e} \\ B \\ D_i^* B \end{bmatrix}$, where I_k is identity matrix of order k, $\mathbf{0}$ is a null vector of order $k \times 1$, $\mathbf{e}$ is a row of all ones of order $2^k - 1$, B is a matrix of order $k \times (2^k - 1)$ whose columns are all possible k-tuples over $GF(2)$, excluding the null column. Choosing any three factors of F have the following cases:

Case 1 : Let the three columns f_1, f_2, f_3 be from Q_i. This case then follows from Lemma 2.

Case 2 : Let the three columns f_1, f_2, f_3 be from Q_i, Q_j, Q_k, i, j, k are all distinct. f_1 is the i_1-th column of Q_i, f_2 is the j_1-th column of Q_j, f_3 is the k_1-th column of Q_k.

(a) Suppose i_1, j_1, k_1 are all distinct. This case follows from Lemma 2.

(b) Suppose $i_1 = j_1 \ne k_1$, we can choose a row in B and $D_i^* B$, respectively. Then in the matrix $[f_1, f_2, f_3]$ there exists a 3×3 subarray is one of the following types:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & \alpha \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & \alpha \end{bmatrix},$$

where α is any element of $GF(2)$.

(c) Suppose $i_1 = j_1 = k_1$. We can choose two rows in $D_i^* B$. Then in the matrix $[f_1, f_2, f_3]$ there exists 3×3 subarray with the same types as in case (b).

Case 3 : Let two columns f_1, f_2 be from Q_i, f_3 is from Q_j, $i \ne j$. f_1, f_2 are the i_1-th, j_1-th columns of Q_i, f_3 is the k_1-th column of Q_j. Suppose $i_1 \ne j_1 = k_1$. Then this case is the same as Case $(2b)$. Suppose $i_1 \ne j_1 \ne k_1$, then this case is same as Case $(2a)$.

Case 4 : Choosing the matrix P_i, two columns f_1, f_2 are from Q_j. In the matrix $[P_i, f_1, f_2]$ there exists a $(k+2)\times(k+2)$ subarray of one of the following types:

$$\begin{bmatrix} \mathbf{0}^T & 1 & 1 \\ I_k & \mathbf{0} & \mathbf{0} \\ \alpha & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} \mathbf{0}^T & 1 & 1 \\ I_k & \mathbf{0} & \mathbf{0} \\ \alpha & 0 & 1 \end{bmatrix},$$

where α is any element of $GF(2)$. Then this matrix $[P_i, f_1, f_2]$ has rank $k+2$.

Case 5 : Choosing the matrices P_i, P_j, and a column f_1 is from Q_k. The matrix

$$[P_i, P_j, f_1] = \begin{bmatrix} \mathbf{0}^T & \mathbf{0}^T & 1 \\ I_k & I_k & b \\ D_i^* & D_j^* & D_k^* b \end{bmatrix},$$

where b is a column of B, and this matrix has rank $2k+1$.

Hence any three factors of F are linearly independent.

Proof of Theorem 5

Without loss of generality, set $A_i = \begin{bmatrix} 1 \\ D_i \end{bmatrix}$, $1 \le i \le n$. By the replacement method for s odd, we obtain $Q_i = \begin{bmatrix} \mathbf{e} & \mathbf{e} \\ I_k & 2I_k \\ D_i^* & 2D_i^* \end{bmatrix}$, where $\mathbf{e}$ is a row of all ones with order k, I_k is identity matrix of order k. Choosing any three factors of F, we have the following cases:

Case 1 : Let three columns f_1, f_2, f_3 be from Q_i. This case follows from Lemma 3.

Case 2 : Let three columns f_1, f_2, f_3 be from Q_i, Q_j, Q_k, i, j, k are all distinct. f_1 is the i_1-th column of Q_i, f_2 is the j_1-th column of Q_j, f_3 is the k_1-th column of Q_k.

(a) Suppose i_1, j_1, k_1 are all distinct. This case follows from Lemma 3.

(b) Suppose $i_1 = j_1 \ne k_1$. We can choose a row in $[I_k, 2I_k]$ and $[D_i^*, 2D_i^*]$, respectively. A row in $[I_k, 2I_k]$ corresponds to the columns of f_1, f_2, f_3 has two same elements and one different element, another row corresponds to the columns of f_1, f_2, f_3 have two different elements in the corresponding position of $[I_k, 2I_k]$ with two same elements. For example, the matrix $[f_1, f_2, f_3]$ exists a 3×3 subarray $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 0 \\ 1 & 0 & \alpha \end{bmatrix}$, where α is any element of $GF(3)$.

(c) Suppose $i_1 = j_1 = k_1$. We can choose two rows in $[D_i^*, 2D_i^*]$. Then in the matrix $[f_1, f_2, f_3]$ there exists 3×3 subarrays with the same types as in Case (b).

Case 3 : Let two columns f_1, f_2 be from Q_i, f_3 is from Q_j, $i \neq j$. f_1, f_2 are the i_1-th, j_1-th column of Q_i, f_3 is the k_1-th column of Q_j. Then in the matrix $[f_1, f_2, f_3]$ exists a 3×3 subarrays with the same types of case (b).

Case 4 : Choosing the matrix P_i and two columns f_1, f_2 from Q_j, the matrix $[P_i, f_1, f_2]$ has a $(k+2) \times (k+2)$ subarray with one of the following types:

$$\begin{bmatrix} \mathbf{0}^T & 1 & 1 \\ I_k & \mathbf{0} & \mathbf{0} \\ \alpha & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} \mathbf{0}^T & 1 & 1 \\ I_k & \mathbf{0} & \mathbf{0} \\ \alpha & 0 & 2 \end{bmatrix}, \quad \begin{bmatrix} \mathbf{0}^T & 1 & 1 \\ I_k & \mathbf{0} & \mathbf{0} \\ \alpha & 1 & 2 \end{bmatrix},$$

where α is any element of $GF(3)$, $\mathbf{0}$ is a null vector with order $k \times 1$. This matrix has rank $k+2$.

Case 5 : Choosing the matrices P_i, P_j, a column f_1 from Q_k, the matrix

$$[P_i, P_j, f_1] = \begin{bmatrix} \mathbf{0}^T & \mathbf{0}^T & 1 \\ I_k & I_k & b \\ D_i^* & D_j^* & D_k^* b \end{bmatrix},$$

where b is a column of $[I_k, 2I_k]$, has rank $2k+1$.

Hence any three factors of F are linearly independent.

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