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Asymptotic behaviour of Gaussian minima

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ASYMPTOTIC BEHAVIOUR OF GAUSSIAN MINIMA

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Abstract. We investigate what happens when an entire sample path of a smooth Gaussian process on a compact interval lies above a high level. Specifically, we determine the precise asymptotic probability of such an event, the extent to which the high level is exceeded, the conditional shape of the process above the high level, and the location of the minimum of the process given that the sample path is above a high level.

1. Introduction

Extremal behaviour of Gaussian processes has been the subject of numerous studies. It is of interest from the point of view of the extreme value theory, or large deviations theory and of the theory of sample path properties of stochastic processes. The asymptotic distribution of the supremum of bounded Gaussian processes has been very thoroughly studied; highlights include Dudley (1973), Berman and Kôno (1989) and Talagrand (1987), and the books of Piterbarg (1996), Adler and Taylor (2007) and Azaïs and Wschebor (2009). In this paper we are interested in another type of the asymptotic behaviour of Gaussian processes: the situation when an entire sample path of the process is above a high level. Such situations are important for understanding the structure of the high level excursion sets of Gaussian random processes and fields.

Very loosely speaking, we are interested in the asymptotics of the Gaussian minima when these minima are high. Dealing with high Gaussian minima is not easy. A finite-dimensional situation (in the language of dependent lognormal random variables) is considered in Guliashvili and Tankov (2016). We, on the other hand, consider minima of zero mean sample continuous Gaussian processes. The processes we consider are often stationary, but some nonstationary processes fall within our framework as well.

We now describe the questions of interest to us more concretely. Let $\mathbf{X} := (X_t : t \in \mathbb{R})$ be a centered Gaussian process with continuous paths,

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defined on some probability space $(\Omega, \mathcal{F}, P)$. Let $[a, b]$ be a compact interval, and let $u > 0$ be a high level. We study a number of problems related to the situation described above, i.e. the situation when the entire sample path $(X_t : t \in [a, b])$ lies above the level u. Specifically, we are interested in the following questions.

Question 1. What is the precise asymptotic behaviour of the probability

$$P\left(\min_{a\le t\le b} X_t > u\right)$$

as $u \to \infty$?

Question 2. Given the event

$$B_u := \left\{\min_{a\le t\le b} X_t > u\right\}, \tag{1.1}$$

how does the conditional distribution of $(X_t : t \in [a, b])$ behave as $u \to \infty$?

Question 3. Conditionally on B_u, what can be said about the asymptotics of the overshoot

$$\min_{a\le t\le b} X_t - u\,,$$

as $u \to \infty$?

Question 4. Consider the location of the minimum of the process,

$$\arg\min_{a\le t\le b} X_t$$

taken to be the leftmost location of the minimum in case there are ties (it is elementary that this location is a well defined random variable). What is the asymptotic distribution of the location of the minimum given B_u, as $u \to \infty$?

Some information on Questions 1 and 2 is contained in Adler et al. (2014) . Regarding **Question 1**, the latter paper describes the probabilities of the type $P\left(\min_{a\le t\le b} X_t > u\right)$ on the logarithmic level, while in the present paper we are interested in precise asymptotics of that probability. Regarding **Question 2**, the latter paper studies the asymptotic behaviour of the ratio

$$\frac{1}{u}X_t,\ a \le t \le b$$

given B_u, as $u \to \infty$, while in the present paper we would like to know the deviations of the sample path from this linear in u behaviour. Furthermore, the paper of Adler et al. (2014) provides no information on **Question 3** and **Question 4** above.

For stationary (not necessarily Gaussian) processes a general theory of the location of the supremum (or infimum) of the process is developed in Samorodnitsky and Shen (2013). However, the limiting behaviour of the minimum location in **Question 4** even in the stationary case is outside of that theory.

We obtain fairly precise answers to the above questions. However, in order to achieve this level of precision, we will impose much stricter smoothness

assumptions on the process $\mathbf{X}$ then those imposed in Adler et al. (2014). We describe the precise assumptions on the process in Section 2. Section 3 contains preliminary results, while the main results of the paper with answers to Questions 1-4 are stated in Section 4. These results are proved in Section 5. Finally, Section 6 presents two examples illustrating the main results of the paper.

2. Assumptions on the process $\mathbf{X}$

In this section we will state and discuss the assumptions on the Gaussian process $\mathbf{X}$ we will use in the rest of the paper. Among others, these assumptions will guarantee that our process is very smooth. Our main interest lies in stationary Gaussian processes, and for these processes the assumptions are easy to state. However, our main results in the subsequent sections do not depend on the stationarity of the process. Rather, they depend on certain properties of the process which follow, in the stationary case, from a small number of basic assumptions. These properties are discussed in the remainder of this section.

We use the notation

$$R(s,t) := \mathrm{E}(X_s X_t)\,, s,t \in \mathbb{R}$$

for the covariance function of the process $\mathbf{X}$. If the process is stationary, then its covariance function is related to the spectral measure of the process by writing (with the usual abuse of notation related to the dual use of R to denote both a function of one variable and a function of two variables)

$$R(s,t) = R(t-s) = \int_{-\infty}^{\infty} e^{\iota(t-s)x} F_X(dx)\,, s,t \in \mathbb{R}\,,$$

where $\iota := \sqrt{-1}$. Recall that the spectral measure F_X of the process $\mathbf{X}$ is a finite symmetric Borel measure on $\mathbb{R}$.

When the process $\mathbf{X}$ is stationary, we will impose the following conditions on the spectral measure F_X.

S1. For all $t \in \mathbb{R}$,

$$\int_{-\infty}^{\infty} e^{tx} F_X(dx) < \infty\,. \tag{2.1}$$

S2. The support of F_X has at least one accumulation point.

The canonical example of such a Gaussian process is the process with the Gaussian spectral density

$$F(dx) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx,\ x \in \mathbb{R}\,. \tag{2.2}$$

This process was considered in detail in Adler et al. (2014), and we will use it in this paper to illustrate our results.

The following proposition establishes certain consequences of the conditions S1 and S2 in the case of a stationary process. It is these consequences, rather than stationarity itself, that will be used in much of the paper.

Proposition 2.1. *Let* $\mathbf{X}$ *be a stationary Gaussian process whose spectral measure satisfies S1 and S2. Then the process* $\mathbf{X}$ *has the following properties.*

Property 1. *The function* $R(\cdot,\cdot)$ *has a power expansion*

$$R(s,t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} r_{mn} s^m t^n \,, s,t \in \mathbb{R}\,,$$

for some $(r_{mn} : m,n \geq 0) \subset \mathbb{R}$.

Property 2. *For any compact interval* $[a,b]$*, the family* $(X_t : t \in [a,b])$ *is non-negatively non-degenerate. That is, for any probability measure* ν *on* $[a,b]$*,*

$$\mathrm{Var}\left(\int_a^b X_t \nu(dt)\right) = \int_a^b \int_a^b R(s,t)\nu(ds)\nu(dt) > 0\,.$$

Property 3. *The sample paths of* $\mathbf{X}$ *are infinitely differentiable, and the covariance matrix of any finite sub-collection of the family* $(X_t^{(n)} : t \in \mathbb{R},\, n = 0,1,\ldots)$ *is non-singular. Here and elsewhere, for any function* f *and* $n \geq 0$*,* $f^{(n)}$ *denotes its* n*-th derivative whenever it exists, with* $f^{(0)} = f$.

Proof. The integral

$$\tilde{R}(z) := \int_{\mathbb{R}} e^{\iota z x}\, F_X(dx)\,, z \in \mathbb{C}\,,$$

defines, clearly, an analytic function, and then $R(s,t) = \tilde{R}(t-s)$ for $s,t \in \mathbb{R}$ has Property 1.

To check Property 2, suppose for the sake of contradiction, that there exists a probability measure ν on $[a,b]$ such that

$$\int_a^b \int_a^b R(s,t)\nu(ds)\nu(dt) = 0\,.$$

Clearly, the left hand side is the same as

$$\int_{-\infty}^{\infty} \left|\int_a^b e^{\iota t x}\nu(dt)\right|^2 F_X(dx)\,.$$

Therefore, it follows that

$$\int_a^b e^{\iota t x}\nu(dt) = 0\,,$$

for every x in the support of F_X. The left hand side is an analytic function of the complex variable x. Since the equality holds on the support of F_X, which has an accumulation point, by the assumption S2, the equality holds for all $x \in \mathbb{C}$ including $x = 0$. This contradicts the fact that ν is a probability measure and thus verifies Property 2.

For Property 3 notice that the integral

$$Y(z) := \int_{\mathbb{R}} e^{\iota z x} Z(dx)\,, z \in \mathbb{C}\,, \tag{2.3}$$

where Z is a complex-valued Gaussian random measure with control measure F_X, defines a random analytic function whose restriction to $z \in \mathbb{R}$ coincides distributionally with $\mathbf{X}$. Therefore, the sample paths of $\mathbf{X}$ are infinitely differentiable.

Suppose that for some array $(a_{mj} : m = 0, 1, \ldots, n,\ j = 1, \ldots, k)$ of real numbers and some $-\infty < s_1 < \ldots < s_k < \infty$,

$$\sum_{m=0}^{n} \sum_{j=1}^{k} a_{mj} X_{s_j}^{(m)} = 0\,.$$

Letting Y be as in (2.3), and using the fact that the restriction of Y to $\mathbb{R}$ has the same distribution as $\mathbf{X}$, it follows that

$$\int_{-\infty}^{\infty} \left(\sum_{m=0}^{n} \sum_{j=1}^{k} a_{mj} (\iota x)^m e^{\iota s_j x} \right) Z(dx) = 0 \text{ a.s. } .$$

Then

$$\sum_{m=0}^{n} \sum_{j=1}^{k} a_{mj} (\iota x)^m e^{\iota x s_j} = 0 \text{ for all } x \text{ in the support of } F_X. \tag{2.4}$$

Note that the left hand side of (2.4), viewed as a function of $x \in \mathbb{C}$, is an analytic function. This function vanishes on the support of F_X, which was assumed to have an accumulation point. Therefore, the analytic function vanishes everywhere, in particular for $x \in \mathbb{R}$.

We claim that this is possible only if $a_{mj} = 0$ for all $m = 0, \ldots, n$, $j = 1, \ldots, k$. If $n = 0$, then the claim follows from the uniqueness of the Fourier transform of finite signed measures, so assume that $n \geq 1$. We start with proving that $a_{nj} = 0$ for all $j = 1, \ldots, k$. Consider the function

$$h(x) = \sum_{j=1}^{k} a_{nj} (\iota x) e^{\iota x s_j},\ x \in \mathbb{R}\,.$$

Observe that the the sum in the left hand side of (2.4) can be written as

$$(\iota x)^{n-1} h(x) + g(x),\ x \in \mathbb{R}\,,$$

where $|g(x)| \leq C|x|^{n-1}$ for some $0 < C < \infty$. Therefore, the function h must be bounded on $\mathbb{R}$, and this implies that

$$\left| \sum_{j=1}^{k} a_{nj} e^{\iota x s_j} \right| =: |\psi(x)| = O(1/|x|) \text{ as } |x| \to \infty.$$

Unless $a_{nj} = 0$ for all $j = 1, \ldots, k$ this is impossible since the implication would be that the function $|\psi|^2$ was integrable over $\mathbb{R}$, which contradicts the fact that this function is the Fourier transform of a discrete signed measure.

Since we have shown that $a_{nj} = 0$ for all $j = 1, \ldots, k$, we have succeeded in reducing n by 1, so we can proceed in an inductive manner.

This completes the proof of the proposition. □

3. Preliminary results

We fix an interval $[a, b]$, once and for all, where $-\infty < a < b < \infty$, and proceed with a number of preliminary results that are important for the main results in the subsequent sections. Most of these results address several issues related to the optimization problem

$$\min_{\nu \in M_1[a,b]} \int_a^b \int_a^b R(s,t)\nu(ds)\nu(dt)\,, \tag{3.1}$$

where the minimum is taken over all Borel probability measures on $[a, b]$. The importance of this problem to the questions studied in this paper was shows in Adler et al. (2014).

Proposition 3.1. *(a) Let* $\mathbf{X}$ *be a Gaussian process with Property 1, Property 2 and Property 3. Assume, additionally, that for every fixed* $s \in [a, b]$,

$$\limsup_{t\to\infty} R(s,t) \le 0\,. \tag{3.2}$$

Then the minimization problem (3.1) *has a unique minimizer* ν_**. Furthermore, the support of* ν_* *has a finite cardinality and the minimum value in* (3.1) *is strictly positive.*

(b) If $\mathbf{X}$ *is a stationary Gaussian process whose spectral measure satisfies S1 and S2, then the conclusion in part (a) holds.*

Proof. We start with part (a). The fact that the minimum is achieved follows from continuity of the functional being optimized in the topology of weak convergence on $M_1[a, b]$ and compactness of that space. The claim that the minimum value is positive follows from Property 2. Let ν_* be a minimizer in the optimization problem (3.1), and define

$$Y := \int_a^b X_s\,\nu_*(ds)\,. \tag{3.3}$$

Let

$$\hat{\mu}(t) := \mathrm{E}\,(X_t | Y = 1)\,,\ t \in \mathbb{R}\,,$$

and observe that

$$\hat{\mu}(t) = \frac{\mathrm{Cov}(X_t, Y)}{\mathrm{Var}(Y)} = \frac{\int_a^b R(s,t)\,\nu_*(ds)}{\mathrm{Var}(Y)},\ t \in \mathbb{R}\,. \tag{3.4}$$

By Theorem 5.1 in Adler et al. (2014), $\hat{\mu} \geq 1$ on $[a, b]$, and $\hat{\mu} = 1$ on the support of ν_*. Since $\hat{\mu}$ is real analytic on $\mathbb{R}$ by Property 1, there are two possibilities. Either

$$\hat{\mu}(t) = 1 \text{ for all } t \in \mathbb{R}\,, \tag{3.5}$$

or the set

$$\hat{S} := \{t \in [a, b] : \hat{\mu}(t) = 1\} \tag{3.6}$$

has no accumulation points and, hence, is a set of finite cardinality. In the latter case, the support of ν_* is also a finite set. We will show that (3.5) is impossible and, hence, the latter option is the only possible one.

Indeed, suppose that (3.5) holds. Then the function

$$g(t) := \int_a^b R(s, t)\, \nu_*(ds)\,, t \in \mathbb{R}$$

is a positive constant. However, by (3.2) and Fatou's lemma, it follows that

$$\limsup_{t\to\infty} g(t) \leq 0\,,$$

leading to a contradiction. We conclude that $\hat{S}$ is a finite set and so is the support of ν_*.

In order to prove the uniqueness of an optimal measure, suppose that ν_1 and ν_2 are two different optimal measures. By Property 3, the finitely many random variables X_t for t in the union of the supports of the two measures are linearly independent and, hence, the function

$$\alpha \mapsto \mathrm{Var}\left(\int_a^b X_s\big(\alpha\nu_1(ds) + (1-\alpha)\nu_2(ds)\big)\right)$$

is strictly convex on $[0, 1]$. Such a function cannot take the same minimal value at the two endpoints 0 and 1, and the uniqueness follows.

We now prove part (b) of the proposition. By Proposition 2.1 the assumptions S1 and S2 on the spectral measure of a stationary Gaussian process $\mathbf{X}$ imply Property 1, Property 2 and Property 3, so the only ingredient missing in an attempt to apply the statement of part (a) to part (b) is that, in part (b), we have not assumed (3.2). Since the only place in the proof of part (a) where (3.2) is used, is in ruling out (3.5), we only need to show that (3.5) can be ruled out under the assumptions of part (b) as well.

Indeed, suppose that (3.5) holds. The assumption S1 implies that R can be extended to an analytic function on $\mathbb{C} \times \mathbb{C}$ by

$$R(s, t) = \int_{-\infty}^{\infty} e^{\iota(t-s)x} F_X(dx),\ s, t \in \mathbb{C}\,.$$

Then the analytic function

$$g(t) = \int_a^b R(s, t)\, \nu(ds),\ t \in \mathbb{C}$$

must be a real constant. Note that

$$g(t) = \int_{-\infty}^{\infty} e^{\iota t x} h(x)\, F_X(dx)\,,$$

where

$$h(x) = \int_a^b e^{-\iota s x}\, \nu(ds), \quad x \in \mathbb{R}\,.$$

Write

$$h(x) = h_1(x) + \iota h_2(x), \ x \in \mathbb{R}\,,$$

where h_1 is a real even function and h_2 is a real odd function. Then, since g is a real constant, we have

$$g(t) = \int_{-\infty}^{\infty} \cos tx\, h_1(x)\, F_X(dx) - \int_{-\infty}^{\infty} \sin tx\, h_2(x)\, F_X(dx), \ t \in \mathbb{R}\,.$$

Since g is an even function (a constant one), the second term in the right hand side vanishes, so that

$$g(t) = \int_{-\infty}^{\infty} \cos tx\, h_1(x)\, F_X(dx), \ t \in \mathbb{R}\,.$$

By the uniqueness of the Fourier transform, the only finite signed measures to have constant transforms are point masses at the origin, so we must have $h_1 = 0$ F_X-a.e. on $\{x \neq 0\}$. Since the function h_1 is real analytic, and the support of F_X has an accumulation point, we conclude that $h_1 = 0$ everywhere. This is not possible since h_1 is the characteristic function of the probability measure obtained by making ν a symmetric probability measure on $[a, b] \cup [-b, -a]$. This rules out (3.5). □

Remark 1. The property (3.2) will not be used again in the sequel. It is clear that, in the stationary case, (3.2) is not implied by the assumptions S1 and S2 (but (3.2) is not needed in the stationary case for the statement of Proposition 3.1 to be true). Interestingly, in the non-stationary case the statement of Proposition 3.1 might be false if (3.2) is not assumed. To see this, consider the following example. Let Y be a standard normal random variable and $(Z_t : t \in \mathbb{R})$ be a stationary mean zero Gaussian process, independent of Y, with covariance

$$\mathrm{E}(Z_s Z_t) = e^{-(s-t)^2/2}\,, s, t \in \mathbb{R}\,.$$

Define

$$X_t = Y + Z_t - \int_0^1 Z_s ds\,, t \in \mathbb{R}\,.$$

Clearly, the process $(X_t : t \in \mathbb{R})$ has Property 1, Property 2 and Property 3. However, if $[a, b] = [0, 1]$, then the minimizer in (3.1) is the Lebesgue measure on $[0, 1]$, and it does not have a support of a finite cardinality.

We denote by S the support of the unique minimizer ν_* in the minimization problem (3.1). Two objects related to this set will be of crucial importance in the sequel. First of all, we let

$$\mu(t) := \mathrm{E}\left(X_t | X_s = 1 \text{ for all } s \in S\right), t \in \mathbb{R}. \tag{3.7}$$

The importance of the function μ stems from the following claim: conditionally on the event B_u in (1.1), as $u \to \infty$,

$$\big(u^{-1}X(t),\, a \le t \le b\big) \to \big(\mu(t),\, a \le t \le b\big) \tag{3.8}$$

in probability, in $C[a,b]$. Indeed, it was shown in Adler et al. (2014) that (3.8) holds with the function μ replaced by the function $\hat{\mu}$ defined in the proof of Proposition 3.1. Therefore, we only need to show that $\mu = \hat{\mu}$. Enumerate the elements of S as $\{t_1, \dots, t_k\}$ and write

$$\mathrm{E}\big(X_t | X_{t_1}, \dots, X_{t_k}\big) = \sum_{j=1}^{k} a_j(t) X_{t_j},\ t \in \mathbb{R}.$$

Then, with Y defined in (3.3),

$$\begin{aligned}
\hat{\mu}(t) Y = \mathrm{E}\big(X_t | Y\big) &= \mathrm{E}\Big(\mathrm{E}\big(X_t | X_{t_1}, \dots, X_{t_k}\big) \big| Y\Big) \\
&= \mathrm{E}\left(\sum_{j=1}^{k} a_j(t) X_{t_j} \Big| Y\right) = \sum_{j=1}^{k} a_j(t) \hat{\mu}(t_j) Y \\
&= \sum_{j=1}^{k} a_j(t) Y
\end{aligned}$$

since $\hat{\mu}$ is equal to one at all points of the support of the measure ν. That is,

$$\hat{\mu}(t) = \sum_{j=1}^{k} a_j(t) = \mu(t),$$

as required.

We record for future use several useful facts about the function μ.

Lemma 3.1. *The function μ is a restriction to $\mathbb{R}$ of an analytic function on $\mathbb{C}$ and for each $j = 0, 1, 2, \dots$ and $t \in \mathbb{R}$,*

$$\mathrm{E}\left(X_t^{(j)} | X_s = 1 \text{ for all } s \in S\right) = \mu^{(j)}(t),\ t \in \mathbb{R}. \tag{3.9}$$

Further, for each $s \in S \cap (a,b)$ there exists an even positive integer n such that

$$\mu^{(1)}(s) = \ldots = \mu^{(n-1)}(s) = 0 < \mu^{(n)}(s).$$

In particular, there is $\varepsilon > 0$ such that $\mu^{(2)}(t) > 0$ for each $t \in (s-\varepsilon, s) \cup (s, s+\varepsilon)$.

Similarly, if $a \in S$, then $\mu^{(1)}(a) \geq 0$. If equality holds, then there is $\varepsilon > 0$ such that

$$\mu^{(2)}(t) > 0, \text{ for each } t \in (a, a+\varepsilon)\,.$$

If $b \in S$, then $\mu^{(1)}(b) \leq 0$. If equality holds, then there is $\varepsilon > 0$ such that

$$\mu^{(2)}(t) > 0, \text{ for each } t \in (b-\varepsilon, b)\,.$$

Proof. We already know that μ is a restriction to $\mathbb{R}$ of an analytic function, and (3.9) is obvious. The final property of μ follows from the fact that it is analytic. □

If we define the "essential" set by

$$E := \{t \in [a,b] : \mu(t) = 1\}\,, \tag{3.10}$$

then we have proved above that $E = \hat{S}$, defined in (3.6). In particular, we showed in the proof of Proposition 3.1 that under the assumptions of the proposition, the essential set $E \supseteq S$ is a finite set as well.

It turns out that in many cases the support S of the optimal measure for the optimization problem (3.1) contains the endpoint of the interval. This holds, in particular, under certain monotonicity assumption in the covariance function of the process. Specific sufficient conditions are given in the following proposition.

Proposition 3.2. *(a) Suppose that the following two conditions hold: for all $a \leq s_1 \leq s_2 \leq s_3 \leq b$,*

$$R(s_1, s_3) \leq \min\{R(s_1, s_2), R(s_2, s_3)\}\,, \tag{3.11}$$

and for all $s, t \in [a,b]$,

$$R(s,t) < \min\{R(s,s), R(t,t)\} \text{ whenever } s \neq t\,. \tag{3.12}$$

Then any finite support optimal measure for the optimization problem (3.1) *puts positive masses at the endpoints a and b of the interval.*

(b) Suppose that $\mathbf{X}$ is a stationary Gaussian process whose covariance function R is nonincreasing on $[0, b-a]$. If the spectral measure F_X is not a point mass at the origin, then the conclusion of part (a) holds.

Proof. We start with part (a). Assume, to the contrary, that ν puts no mass at the point a. Enumerate the elements of the support of ν (which we still denote by S) as $\{t_1, \ldots, t_k\}$, with the smallest of these elements, $t_1 > a$. If $\nu(\{t_i\}) = \alpha_i > 0$, $i = 1, \ldots, k$, then

$$\mathrm{Var}\left(\int_a^b X_s\, \nu(ds)\right) = \sum_{i=1}^k \sum_{j=1}^k \alpha_i \alpha_j R(t_i, t_j)\,.$$

Define a probability measure

$$\nu_\varepsilon := \varepsilon \delta_a + (\alpha_1 - \varepsilon)\delta_{t_1} + \sum_{i=2}^k \alpha_i \delta_{t_i}\,, 0 \leq \varepsilon \leq \alpha_1\,.$$

Notice that $\nu_0 \equiv \nu$, and

$$\begin{aligned} \frac{d}{d\varepsilon}\mathrm{Var}\left(\int_a^b X_s\,\nu_\varepsilon(ds)\right)\Big|_{\varepsilon=0} &= 2\sum_{i=1}^{k}\alpha_i\,[R(a,t_i)-R(t_1,t_i)] \\ &< 0\,. \end{aligned}$$

The inequality follows from the observation that $a < t_1 \le t_i$ for all i and hence by (3.11) the summands are non-positive, and the term with $i = 1$ is strictly negative by (3.12). This contradicts the fact that ν is an optimal measure. Thus, $a \in S$. A similar argument shows that $b \in S$.

For part (b), we only need to check that the assumptions of part (a) hold. The assumption (3.11) follows from monotonicity of the covariance function of the stationary process. The only additional argument needed for (3.12) is the observation that, unless the spectral measure is concentrated at the origin, the covariance function cannot be constant in an neighborhood of the origin. □

We will use in the sequel several facts about the finite-dimensional centered Gaussian vector $(X_t, t \in S)$. These facts are collected in the proposition below. We will use the common notation $f(u) \sim c\,g(u)$ as $u \to \infty$ (where g is a nonvanishing function) to mean that

$$\lim_{u\to\infty}\frac{f(u)}{g(u)} = c\,. \tag{3.13}$$

Note that the possibility $c = 0$ is allowed.

Proposition 3.3. *Under the assumptions of either part (a) or part (b) of Proposition 3.1, let $S := \{t_1, \dots, t_k\}$ be the finite cardinality support of the unique minimizer in* (3.1)*, and let Σ be the covariance matrix of the vector $\big(X_{t_1}, \dots, X_{t_k}\big)$.*

(i) Denote

$$\theta := \Sigma^{-1}\mathbf{1}\,, \tag{3.14}$$

where $\mathbf{1}$ is the column vector of length k with entries equal to 1 (in the sequel, vectors are column vectors unless mentioned otherwise). Then,

$$\theta_j > 0 \text{ for all } 1 \le j \le k\,. \tag{3.15}$$

(ii) Conditionally on the event $\{\min_{t\in S} X_t > u\}$, we have

$$\big(u(X_{t_1}-u), \dots, u(X_{t_k}-u)\big) \Rightarrow \big(E_1, \dots, E_k\big)$$

as $u \to \infty$ (here and in similar statements in the sequel, the law of the random vector in the left hand side is computed, for every $u > 0$, as the conditional law given $\min_{t\in S} X_t > u$). Here $(E_1, \dots, E_k)$ are independent exponential random variables with parameters $\theta_1, \dots, \theta_k$ respectively.

(iii) The distributional tail of the minimal component of the Gaussian vector $\big(X_{t_1}, \ldots, X_{t_k}\big)$ *satisfies*

$$P\big(\min_{t\in S} X_t > u\big) \sim \frac{c}{\theta_1 \ldots \theta_k} \exp\left\{-\frac{1}{2}u^2(\theta_1 + \ldots + \theta_k)\right\} \tag{3.16}$$

as $u \to \infty$*, where*

$$c = (2\pi)^{-k/2}(\det \Sigma)^{-1/2}\,. \tag{3.17}$$

Proof. We start with part (i). Recall that by Property 3, the inverse matrix $\Sigma^{-1} = (\sigma_{ij}^{-1})$ is well defined and, hence, so is the vector θ. Suppose, for instance, that, to the contrary,

$$\sum_{j=1}^{k} \sigma_{1j}^{-1} \le 0\,. \tag{3.18}$$

Define a $k \times 1$ vector $\lambda^{(0)}$ by

$$\lambda_j^{(0)} := \nu(\{t_j\})\,, 1 \le j \le k\,,$$

where ν is the unique minimizer in (3.1). In other words, $\lambda^{(0)}$ is the minimizer in the problem

$$\min_{\lambda\in\mathbb{R}_+^k:\, \sum_{i=1}^k \lambda_i = 1} \lambda^T \Sigma \lambda\,. \tag{3.19}$$

Clearly, $\lambda_i^{(0)} > 0$ for all $i = 1, \ldots, k$. For small $\varepsilon > 0$ consider vectors of the form

$$\lambda^{(\varepsilon)} = \lambda^{(0)} - \varepsilon \Sigma^{-1}\mathbf{e}^{(1)}\,,$$

where $\mathbf{e}^{(1)} = (1, 0, \ldots, 0)$. Then it follows from (3.18) that

$$\sum_{i=1}^{n} \lambda_i^{(\varepsilon)} \ge 1. \tag{3.20}$$

Note that

$$(\lambda^{(\varepsilon)})^T \Sigma \lambda^{(\varepsilon)} = (\lambda^{(0)})^T \Sigma \lambda^{(0)} - 2\varepsilon\lambda_1^{(0)} + \varepsilon^2 (\mathbf{e}^{(1)})^T \Sigma^{-1}\mathbf{e}^{(1)}\,.$$

Since $\lambda_1^{(0)} > 0$, for small $\varepsilon > 0$, this expression is strictly smaller than

$$(\lambda^{(0)})^T \Sigma \lambda^0\,.$$

Additionally, for small $\varepsilon > 0$ the vector $\lambda^{(\varepsilon)}$ has positive components. Recalling (3.20), this contradicts the optimality of $\lambda^{(0)}$ in (3.19). This contradiction shows that $\theta_1 > 0$. Similarly, $\theta_j > 0$ for all $1 \le j \le k$. Hence, (3.15) holds.

For parts (ii) and (iii) we start by noticing that we can write, with $c > 0$ given by (3.17), for any $h_i \geq 0$, $i = 1, \ldots, k$, for $u > 0$,

$$
\begin{aligned}
&P\big(X_{t_i} > u + h_i/u,\ i = 1, \ldots, k\big) \\
=&c \int_{u+h_1/u}^{\infty} \ldots \int_{u+h_k/u}^{\infty} \exp\left\{ -\frac{1}{2} \mathbf{y}^T \Sigma^{-1} \mathbf{y} \right\} dy_1 \ldots dy_k \\
=&c \int_{h_1/u}^{\infty} \ldots \int_{h_k/u}^{\infty} \exp\left\{ -\frac{1}{2} \sum_{i=1}^{k} \sum_{j=1}^{k} (y_i + u)(y_j + u) \sigma_{ij}^{-1} \right\} dy_1 \ldots dy_k \\
=&c \exp\left\{ -\frac{1}{2} u^2 \sum_{i=1}^{k} \sum_{j=1}^{k} \sigma_{ij}^{-1} \right\} \int_{h_1/u}^{\infty} \ldots \\
&\ldots \int_{h_k/u}^{\infty} \exp\left\{ -\frac{1}{2} \sum_{i=1}^{k} \sum_{j=1}^{k} y_i y_j \sigma_{ij}^{-1} \right\} \exp\left\{ -u \sum_{i=1}^{k} y_i \sum_{j=1}^{k} \sigma_{ij}^{-1} \right\} dy_1 \ldots dy_k \\
=&c \exp\left\{ -\frac{1}{2} u^2 (\theta_1 + \ldots + \theta_k) \right\} \\
&\int_{h_1/u}^{\infty} \ldots \int_{h_k/u}^{\infty} \exp\left\{ -\frac{1}{2} \sum_{i=1}^{k} \sum_{j=1}^{k} y_i y_j \sigma_{ij}^{-1} \right\} \exp\left\{ -u \sum_{i=1}^{k} \theta_i y_i \right\} dy_1 \ldots dy_k \\
:=&c \exp\left\{ -\frac{1}{2} u^2 (\theta_1 + \ldots + \theta_k) \right\} I(u) \,.
\end{aligned}
$$

For $\varepsilon > 0$ we write

$$
\begin{aligned}
I(u) = &\int_{h_1/u}^{\infty} \ldots \int_{h_k/u}^{\infty} \mathbf{1}\big(\max(y_1, \ldots, y_k) > \varepsilon\big) \cdots \\
&+ \int_{h_1/u}^{\varepsilon} \ldots \int_{h_k/u}^{\varepsilon} \cdots := I_{\varepsilon,1}(u) + I_{\varepsilon,2}(u) \,.
\end{aligned}
$$

Note that

$$
c I_{\varepsilon,1}(u) \leq \exp\big\{ -\varepsilon u \min_{i=1,\ldots,k} \theta_i \big\} P(X_{t_i} > u,\ i = 1, \ldots, k) \,.
$$

On the other hand,

$$
\begin{aligned}
&I_{\varepsilon,2}(u) \left(\int_{h_1/u}^{\infty} \ldots \int_{h_k/u}^{\infty} \exp\left\{ -u \sum_{i=1}^{k} \theta_i y_i \right\} dy_1 \ldots dy_k \right)^{-1} \\
&\in \left(\exp\left\{ -\frac{1}{2} \varepsilon^2 \sum_{i=1}^{k} \sum_{j=1}^{k} \left| \sigma_{ij}^{-1} \right| \right\}, 1 \right) \,.
\end{aligned}
$$

Since

$$\int_{h_1/u}^{\infty} \dots \int_{h_k/u}^{\infty} \exp\left\{-u\sum_{i=1}^{k}\theta_i y_i\right\} dy_1 \dots dy_k$$
$$= \big(\theta_1 \dots \theta_k\big)^{-1} \exp\big\{-\sum_{i=1}^{k}\theta_i h_i\big\},$$

we conclude that

$$\lim_{u\to\infty} I(u) = \big(\theta_1 \dots \theta_k\big)^{-1} \exp\big\{-\sum_{i=1}^{k}\theta_i h_i\big\}$$

and, hence,

$$P\big(X_{t_i} > u + h_i/u,\ i = 1, \dots, k\big) \tag{3.21}$$
$$\sim \frac{c}{\theta_1 \dots \theta_k} \exp\left\{-\sum_{i=1}^{k}\theta_i h_i\right\} \exp\left\{-\frac{1}{2}u^2(\theta_1 + \dots + \theta_k)\right\}$$

as $u \to \infty$. Putting $h_1 = \dots = h_k = 0$ we obtain (3.16), which together with (3.21) proves the claim of part (ii). □

4. The main results

In this section we will answer Questions 1-4 mentioned in the introduction. These questions are all centered around the overall infimum $\min_{t\in[a,b]} X_t$ of the process $\mathbf{X}$ and the behaviour of the process when the overall infimum is large. It turns out that, when the overall infimum is large, its behaviour is similar, but not identical, to the behaviour of a simpler object - the minimal value in a finite-dimensional Gaussian vector, formed by several key observations of the process. Understanding what happens when the latter minimum is large is an important ingredient in our analysis in this section.

Recall (by Proposition 3.1) that under the assumptions we are imposing in this paper, the optimization problem (3.1) has a unique minimizer, a probability measure with a support of a finite cardinality, which we denote by S. Then the minimal value in a finite-dimensional Gaussian vector mentioned above is simply $\min_{t\in S} X_t$.

We start with Question 1 of the introduction. We will need additional notation, which we introduce now. Let, once again, S be the support of a finite cardinality of the unique minimizer in (3.1). Denote

$$Y_t := \mathrm{E}(X_t | X_s, s \in S), \tag{4.1}$$
$$Z_t := X_t - Y_t, \tag{4.2}$$

$t \in \mathbb{R}$. Then $(Y_t : t \in \mathbb{R})$ and $(Z_t : t \in \mathbb{R})$ are two centered Gaussian processes. Moreover, the process $(Z_t : t \in \mathbb{R})$ is independent of $(X_s, s \in S)$. In particular, the process $(Z_t : t \in \mathbb{R})$ is independent of the random variable

$\min_{t\in S} X_t$. Recall that under the assumptions of Proposition 3.1 (which we will always assume), the sample paths of the processes $\mathbf{Y}$ and $\mathbf{Z}$ are in C^∞.

Recall the definition of the essential set E in (3.10), which is a (not necessarily strict) superset of S. It is a finite set, and we will enumerate its points as $E = \{t_1, \dots, t_l\}$, in such a way that the first k points form the support of the unique minimizer in (3.1), i.e. $S = \{t_1, \dots, t_k\}$, $k \le l$, and $a \le t_1 < \ldots < t_k \le b$.

Let θ be the k-dimensional vector with positive coordinates defined in (3.14), and let μ be the function defined in (3.7). We define several random variables. Let

$$W_{(a,b)} = \exp\left\{ -\frac{1}{2} \sum_{j=1,\dots,k:\, t_j\in(a,b)} \frac{\theta_j}{\mu^{(2)}(t_j)} \big(Z^{(1)}_{t_j}\big)^2 \right\},$$

following the usual convention that a positive number divided by zero is plus infinity, and $e^{-\infty} = 0$. That is, the right hand side of the above is to be interpreted as zero if $\mu^{(2)}(t_j) = 0$ for some j such that $t_j \in (a, b)$. Let, further,

$$W_a = \mathbf{1}\big(Z^{(1)}_{t_1} > 0\big) + \exp\left\{ -\frac{\theta_1}{2\mu^{(2)}(t_1)} \big(Z^{(1)}_{t_1}\big)^2 \right\} \mathbf{1}\big(Z^{(1)}_{t_1} < 0\big)$$

if both $t_1 = a$ and $\mu^{(1)}(t_1) = 0$, and let $W_a = 1$ otherwise. Similarly, we let

$$W_b = \mathbf{1}\big(Z^{(1)}_{t_k} < 0\big) + \exp\left\{ -\frac{\theta_1}{2\mu^{(2)}(t_k)} \big(Z^{(1)}_{t_k}\big)^2 \right\} \mathbf{1}\big(Z^{(1)}_{t_k} > 0\big)$$

if both $t_k = b$ and $\mu^{(1)}(t_k) = 0$, and let $W_b = 1$ otherwise. Finally, we let

$$W = W_{(a,b)} W_a W_b \prod_{j=k+1}^{l} \mathbf{1}\big(Z_{t_j} > 0\big). \tag{4.3}$$

For the rest of this section we will use the notation Σ for the covariance matrix of the Gaussian vector $\big(X_{t_1}, \dots, X_{t_k}\big)$ and V_* for the optimal value in the optimization problem (3.1).

It follows from Adler et al. (2014) that

$$\log P\left(\min_{t\in[a,b]} X_t > u \right) \sim \log P\left(\min_{t\in S} X_t > u \right), \ u \to \infty,$$

in the sense of (3.13). The following theorem both explains how the two tail probabilities are related once the logarithms are removed, and answers Question 1.

Theorem 4.1. *Suppose either that a Gaussian process* $\mathbf{X}$ *has Property 1, Property 2 and Property 3 and* (3.2) *holds, or that* $\mathbf{X}$ *is a stationary Gaussian process whose spectral measure satisfies S1 and S2. Then, as* $u \to \infty$,

$$P\left(\min_{t\in[a,b]} X_t > u \right) \sim \mathrm{E}(W) P\left(\min_{t\in S} X_t > u \right), \tag{4.4}$$

and hence

$$P\left(\min_{t\in[a,b]} X_t > u\right) \sim \frac{\mathrm{E}(W)}{(2\pi)^{k/2}(\theta_1\dots\theta_k)(\det\Sigma)^{1/2}} u^{-k}\exp\left(-\frac{u^2}{2V_*}\right). \tag{4.5}$$

Furthermore, $\mathrm{E}(W) > 0$ *if and only if*

$$\mu^{(2)}(t) > 0 \text{ for all } t \in S\cap(a,b). \tag{4.6}$$

We now proceed to address the rest of the questions in the introduction. From now on we assume (4.6) to hold. The following result answers Question 2.

Theorem 4.2. *Under assumptions of Theorem 4.1, assume also that* (4.6) *holds. Then, as* $u\to\infty$*,*

$$P\left((X_t - u\mu(t) : a\le t\le b)\in\cdot \,\middle|\, \min_{t\in[a,b]} X_t > u\right) \Rightarrow Q_W(\cdot)$$

weakly on $C[a,b]$*, where* Q_W *is a probability measure on* $C[a,b]$ *defined by*

$$Q_W(B) = \frac{1}{\mathrm{E}W}\mathrm{E}\left[\mathbf{1}\left(\left(Z_t : a\le t\le b\right)\in B\right)W\right],\ B\subset C[a,b],\ \textit{Borel},$$

with W *given by* (4.3)*.*

The next result is an answer to Question 3.

Theorem 4.3. *Under assumptions of Theorem 4.1, assume also that* (4.6) *holds. Then, as* $u\to\infty$*, the conditional distribution of* $u\big(\min_{t\in[a,b]} X_t - u\big)$ *given* $\min_{t\in[a,b]} X_t > u$ *converges weakly to an exponential distribution with mean* V_**.*

Finally, we answer Question 4.

Theorem 4.4. *Under assumptions of Theorem 4.1, assume also that* (4.6) *holds. Let*

$$T_* := \arg\min_{s\in[a,b]} X_s,$$

where we choose the leftmost location of the minimum in case there are ties. Then, as $u\to\infty$*,*

$$P\left(T_*\in\cdot\,\middle|\,\min_{s\in[a,b]} X_s > u\right) \Rightarrow \sum_{i=1}^{k}\frac{\theta_i}{\theta_1+\ldots+\theta_k}\delta_{t_i}(\cdot),$$

where δ_t *is the Dirac point mass at* $t\in[a,b]$*.*

5. Proofs

We start with several preliminary results. First, an elementary convergence statement.

Lemma 5.1. *Suppose that three families of random variables* $\mathbb{U} := (U_{iN} : 1 \le i \le m, 1 \le N \le \infty)$, $\mathbb{V} := (V_{iN} : 1 \le i \le n, 1 \le N < \infty)$ *and* $\mathbb{W} := (W_i : 1 \le i \le n)$ *live on the same probability space, and that* $\mathbb{U}$ *and* $\mathbb{W}$ *are independent. Suppose that, as* $N \to \infty$,

$$\begin{aligned} (U_{1N}, \dots, U_{mN}) &\Rightarrow (U_{1\infty}, \dots, U_{m\infty}) \,, \\ V_{iN} &\xrightarrow{P} W_i \,, 1 \le i \le n \,. \end{aligned}$$

Then

$$(U_{1N}, \dots, U_{mN}, V_{1N}, \dots, V_{nN}) \Rightarrow (U_{1\infty}, \dots, U_{m\infty}, W_1, \dots, W_n)$$

as as $N \to \infty$, *where the random vectors* $(U_{1\infty}, \dots, U_{m\infty})$ *and* $(W_1, \dots, W_n)$ *in the right hand side are independent.*

Proof. It is, clearly, enough to check that for any Borel sets $A \subset \mathbb{R}^m$ and $B \subset \mathbb{R}^n$ such that A is a continuity set with respect to the law of $(U_{1\infty}, \dots, U_{m\infty})$ and B s a continuity set with respect to the law of $(W_1, \dots, W_n)$ we have

$$\begin{aligned} &P\Big((U_{1N}, \dots, U_{mN}) \in A, (V_{1N}, \dots, V_{nN}) \in B\Big) \\ &-P\Big((U_{1N}, \dots, U_{mN}) \in A, (W_1, \dots, W_n) \in B\Big) \to 0 \end{aligned} \tag{5.1}$$

as $N \to \infty$. However,

$$\mathbf{1}\Big((V_{1N}, \dots, V_{nN}) \in B\Big) \to \mathbf{1}\Big((W_1, \dots, W_n) \in B\Big)$$

in probability as $N \to \infty$, and, hence, (5.1) follows. □

The next lemma is the first step in the proof of Theorem 4.1.

Lemma 5.2. *Under the assumptions of Theorem 4.1, for all* $\varepsilon > 0$ *and* $n \ge 0$ *we have*

$$\lim_{u\to\infty} P\left(\sup_{a\le t\le b} \left| X_t^{(n)} - u\mu^{(n)}(t) - Z_t^{(n)} \right| > \varepsilon \,\middle|\, \min_{s\in S} X_s > u \right) = 0 \,. \tag{5.2}$$

Proof. Notice that we can write

$$Y_t^{(n)} = \sum_{j=1}^{k} \alpha_j(t) X_{t_j} \,,$$

$t \in \mathbb{R}$, for some continuous functions $\alpha_1, \dots, \alpha_n$ from $\mathbb{R}$ to $\mathbb{R}$. Therefore,

$$\mu^{(n)}(t) = \sum_{j=1}^{k} \alpha_j(t) \,, t \in \mathbb{R} \,,$$

and probability in the left hand side of (5.2) equals

$$\begin{aligned} & P\left(\sup_{a\le t\le b}\left|Y_t^{(n)}-u\mu^{(n)}(t)\right|>\varepsilon\middle|\min_{s\in S}X_s>u\right)\\ = \;& P\left(\sup_{a\le t\le b}\left|\sum_{j=1}^k\alpha_j(t)(X_{t_j}-u)\right|>\varepsilon\middle|\min_{s\in S}X_s>u\right)\\ \le \;& P\left(\sum_{j=1}^k|X_{t_j}-u|>\frac{\varepsilon}{\max_{a\le t\le b,1\le j\le k}|\alpha_j(t)|}\middle|\min_{s\in S}X_s>u\right)\\ \to \;& 0\,, \end{aligned}$$

as $u\to\infty$ because, conditionally on $\{\min_{s\in S}X_s>u\}$, $X_{t_j}-u\stackrel{P}{\longrightarrow}0$ for each $j=1,\ldots,k$ by part (ii) of Proposition 3.3. □

The next theorem is the crucial step towards proving Theorem 4.1. Its statement uses Lemma 3.1.

Theorem 5.1. *Suppose that the assumptions of Theorem 4.1 are satisfied, and let $t\in S$.*

(i) Suppose that $t\in(a,b)$. Let $\varepsilon>0$ be such that $[t-\varepsilon,t+\varepsilon]\subset[a,b]$ and

$$\mu^{(2)}(s)>0 \text{ for all } 0<|s-t|\le\varepsilon\,. \tag{5.3}$$

Then, as $u\to\infty$, conditionally on the event $\{\min_{s\in S}X_s>u\}$,

$$u\left(X_t-\min_{s\in[t-\varepsilon,t+\varepsilon]}X_s\right)\stackrel{P}{\longrightarrow}\frac{1}{2\mu^{(2)}(t)}\left(Z_t^{(1)}\right)^2\,, \tag{5.4}$$

and, as before, we interpret the right hand side as $+\infty$ if $\mu^{(2)}(t)=0$.

(ii) Suppose that $t=a$. Then either $\mu^{(1)}(a)>0$ or $\mu^{(1)}(a)=0$ and $\mu^{(2)}(a)\ge0$. Then for all $\varepsilon>0$ small enough, as $u\to\infty$, conditionally on the event $\{\min_{s\in S}X_s>u\}$,

$$\begin{aligned} & u\left(X_a-\min_{s\in[a,a+\varepsilon]}X_s\right)\\ \stackrel{P}{\longrightarrow}\;& \begin{cases} 0\,, & \mu^{(1)}(a)>0\,,\\ \frac{1}{2\mu^{(2)}(a)}\left(Z_a^{(1)}\right)^2\mathbf{1}\left(Z_a^{(1)}<0\right)\,, & \mu^{(1)}(a)=0,\,\mu^{(2)}(a)>0\,,\\ \infty\mathbf{1}\left(Z_a^{(1)}<0\right)\,, & \mu^{(1)}(a)=0,\,\mu^{(2)}(a)=0\,. \end{cases} \end{aligned}$$

Here and below, we follow the convention that $\infty\cdot0=0\cdot\infty=0$.

(iii) Suppose that $t=b$. Then either $\mu^{(1)}(b)<0$ or $\mu^{(1)}(b)=0$ and $\mu^{(2)}(b)\ge0$. Then for all $\varepsilon>0$ small enough, as $u\to\infty$, conditionally on

the event $\{\min_{s\in S} X_s > u\}$,

$$u\left(X_b - \min_{s\in[b-\varepsilon,b]} X_s\right)$$
$$\xrightarrow{P} \begin{cases} 0\,, & \mu^{(1)}(b) < 0\,, \\ \frac{1}{2\mu^{(2)}(b)}\left(Z_b^{(1)}\right)^2 \mathbf{1}\left(Z_b^{(1)} > 0\right), & \mu^{(1)}(b) = 0,\ \mu^{(2)}(b) > 0\,, \\ \infty \mathbf{1}\left(Z_b^{(1)} > 0\right), & \mu^{(1)}(b) = 0,\ \mu^{(2)}(b) = 0\,. \end{cases}$$

Proof. The claims of parts (ii) and (iii) are similar, so we will only prove the claims of parts (i) and (ii). We start with part (i). Define

$$t_* := \arg \min_{t-\varepsilon\le s\le t+\varepsilon} X_s\,,$$

taken to be the closest to t location of the minimum in case there are ties. We first check that

$$\lim_{u\to\infty} P\left(X_{t_*}^{(1)} = 0 \Big| \min_{s\in S} S_s > u\right) = 1\,. \tag{5.5}$$

To see this, note that by Lemma 5.2,

$$\begin{aligned} P\left(X_{t_*}^{(1)} = 0 \Big| \min_{s\in S} X_s > u\right) &\ge P\left(X_{t+\varepsilon}^{(1)} > 0,\ X_{t-\varepsilon}^{(1)} < 0 \Big| \min_{s\in S} X_s > u\right) \\ &\to 1\,, \end{aligned}$$

as $u\to\infty$. Here we have used the fact that by (5.3),

$$\mu^{(1)}(t-\varepsilon) < 0 < \mu^{(1)}(t+\varepsilon)\,.$$

Keeping the definition of t_* unchanged, but replacing ε by arbitrarily small $0<\varepsilon'<\varepsilon$ in the above argument, shows that, as $u\to\infty$, conditionally on the event $\{\min_{s\in S} X_s > u\}$,

$$t_* \xrightarrow{P} t\,. \tag{5.6}$$

By Lemma 3.1 there exists an even positive integer n such that

$$\mu^{(1)}(t) = \ldots = \mu^{(n-1)}(t) = 0 < \mu^{(n)}(t)\,. \tag{5.7}$$

Consider the series expansion

$$X_{t_*}^{(1)} = X_t^{(1)} + \sum_{j=2}^{n-1} X_t^{(j)} \frac{(t_*-t)^{j-1}}{(j-1)!} + X_{\xi_1}^{(n)} \frac{(t_*-t)^{n-1}}{(n-1)!}\,, \tag{5.8}$$

for some ξ_1 in between t and t_*. By Lemma 5.2 and (5.7) we know that, as $u\to\infty$, conditionally on the event $\{\min_{s\in S} X_s > u\}$,

$$X_t^{(j)} \xrightarrow{P} Z_t^{(j)}\,, 1\le j\le n-1\,. \tag{5.9}$$

The above along with (5.5), (5.6) and (5.8) imply that, as $u\to\infty$, conditionally on the event $\{\min_{s\in S} X_s > u\}$,

$$X_{\xi_1}^{(n)} \frac{(t_*-t)^{n-1}}{(n-1)!} \xrightarrow{P} -Z_t^{(1)}\,.$$

Furthermore, by Lemma 5.2, we also have

$$u^{-1}X^{(n)}_{\xi_1} \xrightarrow{P} \mu^{(n)}(t)\,. \tag{5.10}$$

Therefore,

$$u(t_*-t)^{n-1} \xrightarrow{P} -Z^{(1)}_t \frac{(n-1)!}{\mu^{(n)}(t)}\,. \tag{5.11}$$

Next, we use the series expansion

$$X_{t_*} - X_t = \sum_{j=1}^{n-1} X^{(j)}_t \frac{(t_*-t)^j}{j!} + X^{(n)}_{\xi_2}\frac{(t_*-t)^n}{n!}\,, \tag{5.12}$$

for some ξ_2 in between t and t_*. Since (5.10) also holds with ξ_2 replacing ξ_1, we conclude by (5.9) with $j=1$ and (5.11) that

$$\begin{aligned} & u^{1/(n-1)}\left(X_{t_*}-X_t\right) \\ = \;& u^{1/(n-1)}(t_*-t)X^{(1)}_t + \left(u^{-1}X^{(n)}_{\xi_2}\right)\frac{u^{n/(n-1)}(t_*-t)^n}{n!} \\ & + u^{1/(n-1)}\sum_{j=2}^{n-1} X^{(j)}_t\frac{(t_*-t)^j}{j!} \\ \xrightarrow{P} \;& -|Z^{(1)}_t|^{n/(n-1)}\frac{1}{(\mu^{(n)}(t))^{1/(n-1)}}((n-1)!)^{1/(n-1)}\frac{n-1}{n}\,. \end{aligned} \tag{5.13}$$

When $n=2$, this reduces to (5.4). If $n>2$, i.e. if $\mu^{(2)}(t)=0$, the above limit says that

$$u(X_{t_*}-X_t) \xrightarrow{P} -\infty\,,$$

which is, again, (5.4). This completes the proof of part (i).

We now prove part (ii) of the theorem. The claim will be proved separately for the three cases listed in the statement. We start with the case $\mu^{(1)}(a)>0$. For all $\varepsilon>0$ small enough, such that $a+\varepsilon\le b$ and $\min_{s\in[a,a+\varepsilon]}\mu^{(1)}(s)>0$, we have by Lemma 5.2,

$$\begin{aligned} & P\left(X_a = \min_{s\in[a,a+\varepsilon]} X_s \,\middle|\, \min_{s\in S} X_s > u\right) \\ \ge \;& P\left(\min_{s\in[a,a+\varepsilon]} X^{(1)}_s > 0 \,\middle|\, \min_{s\in S} X_s > u\right) \\ \to \;& 1\,, \end{aligned}$$

as $u\to\infty$, which proves the claim of part (ii) in the case $\mu^{(1)}(a)>0$.

Suppose now that $\mu^{(1)}(a)=0$. By Lemma 3.1, we can choose $\varepsilon>0$ such that

$$\mu^{(2)}(s)>0 \text{ for all } a<s\le a+\varepsilon\,. \tag{5.14}$$

Consider the event

$$B := \left\{X^{(1)}_a > 0 > \min_{s\in[a,a+\varepsilon]} X^{(1)}_s\right\}\,.$$

Our first claim is that

$$\lim_{u\to\infty} P\left(B \,\middle|\, \min_{s\in S} X_s > u\right) = 0\,. \tag{5.15}$$

Indeed, on the event B, the derivative $X_s^{(1)}$ crosses 0 in the interval $[a, a+\varepsilon]$. If we define a random variable

$$s_* := \inf\{s \in [a, a+\varepsilon] : X_s^{(1)} = 0\}\,,$$

then

$$X_{s_*}^{(1)}\, \mathbf{1}_B = 0\,. \tag{5.16}$$

Furthermore, the definition of s_* tells us that

$$(X_{s_*} - X_a)\, \mathbf{1}_B \geq 0\,. \tag{5.17}$$

Note that by (5.14) the second derivative $\mu^{(2)}$ is bounded away from 0 on any interval $[a+\delta, a+\varepsilon]$ for $0 < \delta < \varepsilon$. It follows from Lemma 5.2 that

$$P\left(\min_{a+\delta\leq t\leq a+\varepsilon} X_t^{(2)} < 0 \,\middle|\, \min_{s\in S} X_s > u\right) \to 0$$

as $u \to \infty$. Therefore, conditionally on the event $\{\min_{s\in S} X_s > u\}$, as $u \to \infty$,

$$(s_* - a)\, \mathbf{1}_B \xrightarrow{P} 0\,.$$

Using twice the Taylor expansion and imitating the steps leading to (5.11) and (5.13), in conjunction with (5.16), shows that

$$\lim_{u\to\infty} P\left(X_{s_*} < X_a \,\middle|\, B \cap \{\min_{s\in S}\{X_s > u\}\right) = 1\,.$$

This would contradict (5.17) if (5.15) were false. Thus (5.15) follows.

Write

$$\begin{aligned}
& u\left(X_a - \min_{s\in[a,a+\varepsilon]} X_s\right) \\
= \;& u\left(X_a - \min_{s\in[a,a+\varepsilon]} X_s\right) \mathbf{1}(X_a^{(1)} < 0) \\
+ \;& u\left(X_a - \min_{s\in[a,a+\varepsilon]} X_s\right) \mathbf{1}(\{X_a^{(1)} > 0\} \setminus B) \\
+ \;& u\left(X_a - \min_{s\in[a,a+\varepsilon]} X_s\right) \mathbf{1}(\{X_a^{(1)} > 0\} \cap B)\,.
\end{aligned}$$

By the definition of the event B, the middle term in the right hand side is equal to zero, while by (5.15), the last term in the right hand side goes to zero in probability. It remains, therefore, to consider the first term in the right hand side. By the assumption $\mu^{(1)}(a) = 0$ and Lemma 5.2 we know that $\mathbf{1}(X_a^{(1)} < 0) \to \mathbf{1}(Z_a^{(1)} < 0)$ in probability. For the rest of the that term the same analysis as the one used in the proof of part (i) applies. Specifically, we use the two Taylor expansions (5.8) and (5.12). The only

difference between the two scenarios is that now the integer n does not need to be an even number, but it plays no role in the argument.

This completes the proof of the theorem in all cases. □

We have now all the ingredients needed to prove Theorem 4.1.

Proof of Theorem 4.1. A restatement of (4.4) is

$$\lim_{u\to\infty} P\left(\min_{t\in[a,b]} X_t > u \Big| \min_{t\in S} X_t > u\right) = \mathrm{E}W\,, \tag{5.18}$$

which we proceed to show first. For $u > 0$ let $(\tilde{Y}_t^{(u)} : t \in \mathbb{R})$ be a process with continuous sample paths whose law is the law of the process $\mathbf{Y}$ in (4.1) conditioned on the event $\{\min_{s\in S} X_s > u\}$, and let this process be independent of the process $\mathbf{Z}$ in (4.2). Define

$$\tilde{X}_t^{(u)} := \tilde{Y}_t^{(u)} + Z_t,\, t \in \mathbb{R}\,.$$

Let $\varepsilon > 0$ be small enough such that the convergence in probability in Theorem 5.1 holds. Continuing using the notation $S = \{t_1, \ldots, t_k\}$ and $E \setminus S = \{t_{k+1}, \ldots, t_{k+l}\}$, we define for $1 \le j \le k$

$$V_{ju} := u\left(\tilde{X}_{t_j}^{(u)} - \min_{s\in[t_j-\varepsilon,t_j+\varepsilon]\cap[a,b]} \tilde{X}_s^{(u)}\right)\,,$$

and

$$V_{k+1,\,u} := \inf_{s\in G}\left[\tilde{X}_s^{(u)} - u\mu(s)\right]\,,$$

where

$$G := \{s \in [a,b] : |s - t_j| \le \varepsilon \text{ for some } k+1 \le j \le l\}\,, \tag{5.19}$$

with the convention that infimum over the empty set is defined as $-\infty$.

For $j = 1, \ldots, k$ we denote by W_j (not to be confused with W_a, W_b or $W_{(a,b)}$) the limit in probability of

$$T_{ju} := u\left(X_{t_j} - \min_{s\in[t_j-\varepsilon,t_j+\varepsilon]\cap[a,b]} X_s\right)\,,$$

as $u \to \infty$, conditionally on the event $\{\min_{s\in S} X_s > u\}$, given in Theorem 5.1. Recall that W_j may take the value $+\infty$. We define also

$$U_{ju} := u\big(\tilde{X}_{t_j}^{(u)} - u\big)\,, 1 \le j \le k\,.$$

Clearly, the conditional law of

$$\Big(u(X_{t_1} - u), \ldots, u(X_{t_k} - u), T_{1u}, \ldots, T_{ku}, \min_{s\in G}\left[X_s - u\mu(s)\right]\Big)$$

given $\{\min_{s\in S} X_s > u\}$ coincides with the law of

$$\Big(U_{1u}, \ldots, U_{ku}, V_{1u}, \ldots, V_{k+1,\,u}\Big)\,.$$

By Theorem 5.1 we know that for fixed $1 \le j \le k$,

$$V_{ju} \xrightarrow{P} W_j \tag{5.20}$$

as $u \to \infty$, where we regard W_j as a function of the process $\mathbf{Z}$ in the definition of (V_{ju}). Furthermore, by Proposition 3.3,

$$(U_{1u}, \dots, U_{ku}) \Rightarrow (E_1, \dots, E_k) \tag{5.21}$$

as $u \to \infty$, where $E_1, \dots, E_k$ are independent exponential random variables with parameters given by (3.14). Finally, Lemma 5.2 implies that as $u \to \infty$,

$$V_{k+1,\,u} \xrightarrow{P} \min_{s \in G} Z_s \,. \tag{5.22}$$

We apply now Lemma 5.1 to conclude that, as $u \to \infty$,

$$\left(U_{1u}, \dots, U_{ku}, V_{1u}, \dots, V_{k+1,\,u}\right) \Rightarrow \left(E_1, \dots, E_k, W_1, \dots, W_k, \min_{s \in G} Z_s\right),$$

with $(E_1, \dots, E_k)$ being independent of the rest of the random variables in the right hand side, which implies that, as $u \to \infty$,

$$\left(u(X_{t_1} - u) - T_{1u}, \dots, u(X_{t_k} - u) - T_{ku}, \min_{s \in G} [X_s - u\mu(s)] \Big| \min_{s \in S} X_s > u\right)$$

$$\Rightarrow \left(E_1 - W_1, \dots, E_k - W_k, \min_{s \in G} Z_s\right), \tag{5.23}$$

weakly on $\mathbb{R}^{k+1}$, with the obvious interpretation if some of the W_j take the value $+\infty$. If we denote

$$H := \{s \in [a, b] : |s - t_j| \le \varepsilon \text{ for some } 1 \le j \le k\}, \tag{5.24}$$

then it follows by the continuous mapping theorem that, as $u \to \infty$,

$$\begin{aligned} &\min \left\{ u \left(\min_{s \in H} X_s - u \right), \min_{s \in G} [X_s - u\mu(s)] \right\} \\ \Rightarrow \quad &\min \left\{ E_1 - W_1, \dots, E_k - W_k, \min_{s \in G} Z_s \right\}. \end{aligned} \tag{5.25}$$

Note that (5.25) continues to hold if we use $\varepsilon = 0$ in the definition of G (but not in the definition of H).

Since the function μ is bounded away from 1 on $[a, b] \setminus (G \cup H)$, Lemma 5.2 implies that

$$\lim_{u \to \infty} P\left(\min_{s \in [a,b] \setminus (G \cup H)} X_s > u \Big| \min_{s \in S} X_s > u \right) = 1 \,.$$

Together with the fact that $\mu(s) \geq 1$ for all s, this implies that

$$\begin{aligned}
&\liminf_{u\to\infty} P\left(\min_{t\in[a,b]} X_t > u \Big| \min_{s\in S} X_s > u\right)\\
=&\liminf_{u\to\infty} P\left(\min_{s\in G\cup H} X_s > u \Big| \min_{s\in S} X_s > u\right)\\
\geq&\lim_{u\to\infty} P\left(\min\left\{u\left(\min_{s\in H} X_s - u\right), \min_{s\in G}\left[X_s - u\mu(s)\right]\right\} > 0 \Big| \min_{s\in S} X_s > u\right)\\
\geq&P\left(E_1 - W_1 > 0, \ldots, E_k - W_k > 0, \min_{s\in G} Z_s > 0\right)\\
=&\mathrm{E}\left[\exp\left(-\sum_{j=1}^k \theta_j W_j\right)\mathbf{1}\left(\min_{s\in G} Z_s > 0\right)\right]\\
=&\mathrm{E}\left[W_{(a,b)} W_a W_b\,\mathbf{1}\left(\min_{s\in G} Z_s > 0\right)\right].
\end{aligned}$$

We let now $\varepsilon \downarrow 0$ and use the monotone convergence theorem to conclude that

$$\liminf_{u\to\infty} P\left(\min_{t\in[a,b]} X_t > u \Big| \min_{t\in S} X_t > u\right) \geq \mathrm{E}W\,.$$

On the other hand,

$$\begin{aligned}
&P\left(\min_{t\in[a,b]} X_t > u \Big| \min_{t\in S} X_t > u\right)\\
\leq\;& P\left(\min_{s\in H\cup(E\setminus S)} X_s > u \Big| \min_{s\in S} X_s > u\right)\\
=\;& P\left(\min\left\{u\left(\min_{s\in H} X_s - u\right), \min_{s\in E\setminus S}\left[X_s - u\mu(s)\right]\right\} > 0 \Big| \min_{s\in S} X_s > u\right)\\
\to\;& \mathrm{E}W\,,
\end{aligned}$$

the limit in the last line following from (5.25) with $\varepsilon = 0$ in the definition of G and the fact that by Property 3 the Gaussian random variables Z_s, $s \in E\setminus S$ are nondegenerate. Thus, (5.18) follows.

In view of (4.4) and part (iii) of Proposition 3.3, all that needs to be shown for (4.5) is that

$$\sum_{i=1}^k \theta_i = \frac{1}{V_*}\,, \tag{5.26}$$

where V_* is the optimal value in (3.1) which is strictly positive by Property 2. However, Theorem 5.1 of Adler et al. (2014) implies that

$$\min_{y\in\mathbb{R}^k:\min_{1\leq i\leq k} y_i \geq 1} y^T\Sigma^{-1}y = \frac{1}{V_*}\,,$$

and the unique minimizer is **1**. This in conjunction with (3.14) establishes (5.26). This completes the proof.

For the final claim, recall from the definition that $W_{(a,b)} = 0$ a.s. if (4.6) fails. It immediately follows that $EW > 0$ implies (4.6). For the converse, that is, the 'if' part, suppose that (4.6) holds. Then, $W_{(a,b)} > 0$ a.s. . Property 3 implies that the collection $(Z_t : t \in \mathbb{R} \setminus S) \cup (Z_t^{(1)} : t \in \mathbb{R})$ is linearly independent. The random vector $(Z_a^{(1)}, Z_b^{(1)}, Z_{t_{k+1}}, \ldots, Z_{t_l})$ has a multivariate normal law. The linear independence implies that

$$P\left(Z_a^{(1)} > 0, Z_b^{(1)} < 0, \min_{k+1 \le j \le l} Z_{t_j} > 0\right) > 0\,.$$

It is trivial to check from (4.3) that on this event, $W = W_{(a,b)}$. Thus, the 'if' part follows, which completes the proof. □

Proof of Theorem 4.2. Fix a Borel subset B of $C[a,b]$ such that

$$P\left((Z_t : a \le t \le b) \in \partial B\right) = 0\,,$$

where ∂B denotes the boundary of B in the supremum norm topology, and write

$$\begin{aligned} &P\left((X_t - u\mu(t) : a \le t \le b) \in B \,\middle|\, \min_{t\in[a,b]} X_t > u\right) \\ =&\frac{P\left((X_t - u\mu(t) : a \le t \le b) \in B,\, \min_{t\in[a,b]} X_t > u \,|\min_{t\in S} X_t > u\right)}{P\left(\min_{t\in[a,b]} X_t > u \,|\min_{t\in S} X_t > u\right)}\,. \end{aligned}$$

The denominator converges to EW by Theorem 4.1, as $u \to \infty$, and it is positive since (4.6) is assumed. Furthermore, the same argument as the one used in the proof of Theorem 4.1 gives us

$$\begin{aligned} &\lim_{u\to\infty} P\left((X_t - u\mu(t) : a \le t \le b) \in B,\, \min_{t\in[a,b]} X_t > u \,\middle|\, \min_{t\in S} X_t > u\right) \\ =\;& \mathrm{E}\left[\mathbf{1}\left(\left(Z_t : a \le t \le b\right) \in B\right) W\right]\,, \end{aligned}$$

and the statement of the theorem follows. □

Theorems 4.3 and 4.4 are both based on the following result that we prove first.

Theorem 5.2. *Under assumptions of Theorem 4.1, assume also that* (4.6) *holds. For* $\varepsilon > 0$ *define*

$$M_{j\varepsilon} := \min_{s\in[t_j-\varepsilon, t_j+\varepsilon]\cap[a,b]} X_s,\ 1 \le j \le k\,.$$

Then

$$\lim_{\varepsilon\downarrow 0} \liminf_{u\to\infty} P\left(\min_{1\le j\le k} M_{j\varepsilon} = \min_{t\in[a,b]} X_t \,\middle|\, \min_{t\in[a,b]} X_t > u\right) = 1\,. \tag{5.27}$$

Furthermore, for $\varepsilon > 0$ small enough so that the convergence in all parts of Theorem 5.1 holds, as $u \to \infty$, conditionally on the event $\left\{\min_{t\in[a,b]} X_t > u\right\}$,

$$(u(M_{1\varepsilon} - u), \ldots, u(M_{k\varepsilon} - u)) \Rightarrow (E_1, \ldots, E_k)\,, \tag{5.28}$$

where $E_1, \ldots, E_k$ are independent exponential random variables with respective parameters $\theta_1, \ldots, \theta_k$.

Proof. With the notation $E \setminus S = \{t_{k+1}, \ldots, t_{k+l}\}$ as above and the set G defined in (5.19), we first prove that

$$\lim_{\varepsilon\downarrow 0} \liminf_{u\to\infty} P\left(\min_{1\le j\le k} M_{j\varepsilon} < \min_{s\in G} X_s \,\middle|\, \min_{t\in[a,b]} X_t > u\right) = 1 \tag{5.29}$$

(note that the definition of G depends on $\varepsilon > 0$).

Let $\varepsilon > 0$ be small enough so that the convergence in all parts of Theorem 5.1 holds. As in the proof of Theorem 4.1, we denote by W_j the limit in probability of

$$u\left(X_{t_j} - \min_{s\in[t_j-\varepsilon, t_j+\varepsilon]\cap[a,b]} X_s\right)\,,$$

as $u \to \infty$, conditionally on the event $\left\{\min_{t\in S} X_t > u\right\}$. Arguing, once again, as in the proof of Theorem 4.1, we obtain

$$\begin{aligned} &P\left[\left(u\left(\min_{1\le j\le k} M_{j\varepsilon} - u\right), \min_{s\in G}[X_s - u\mu(s)]\right) \in \cdot \,\middle|\, \min_{t\in[a,b]} X_t > u\right] \\ \Rightarrow\ &\frac{1}{\mathrm{E}W} P\Bigg[\left(\min_{1\le j\le k}(E_j - W_j), \min_{s\in G} Z_s\right) \in \cdot, \\ &\qquad\qquad E_1 - W_1 > 0, \ldots, E_k - W_k > 0, \min_{j=k+1,\ldots,l} Z_{t_j} > 0\Bigg] \end{aligned} \tag{5.30}$$

weakly in $\mathbb{R}^2$. We conclude both that, conditionally given $\left\{\min_{t\in[a,b]} X_t > u\right\}$, as $u \to \infty$,

$$\frac{\min_{1\le j\le k} M_{j\varepsilon} - u}{\min_{s\in G}[X_s - u\mu(s)]} \xrightarrow{P} 0 \tag{5.31}$$

and that

$$P\left(\min_{s\in G}[X_s - u\mu(s)] < 0 \,\middle|\, \min_{t\in[a,b]} X_t > u\right) \to 0\,.$$

Since $\mu(s) \geq 1$ for all $s \in [a,b]$, it follows that

$$\begin{aligned} & P\left(\min_{1\leq j\leq k} M_{j\varepsilon} < \min_{s\in G} X_s \middle| \min_{t\in[a,b]} X_t > u \right) \\ \geq\ & P\left(\min_{1\leq j\leq k} M_{j\varepsilon} - u < \min_{s\in G}[X_s - u\mu(s)] \middle| \min_{t\in[a,b]} X_t > u \right) \\ \geq\ & P\left(\left| \frac{\min_{1\leq j\leq k} M_{j\varepsilon} - u}{\min_{s\in G}[X_s - u\mu(s)]} \right| < 1 \middle| \min_{t\in[a,b]} X_t > u \right) \\ & -P\left(\min_{s\in G}[X_s - u\mu(s)] < 0 \middle| \min_{t\in[a,b]} X_t > u \right) \to 1\,. \end{aligned}$$

Therefore, (5.29) follows. The fact that $\mu(s) > 1$ for all $s \in [a,b] \setminus E$ with an appeal to Theorem 4.2 implies (5.27).

In order to prove (5.28), fix $x_1, \ldots, x_k > 0$. The same argument as in (5.30) gives us

$$\begin{aligned} & \lim_{u\to\infty} P\left(u(M_{1\varepsilon} - u) > x_1, \ldots, u(M_{k\varepsilon} - u) > x_k \middle| \min_{t\in[a,b]} X_t > u \right) \\ =\ & \frac{1}{\mathrm{E}W} P\left(E_j - W_j > x_j\,, 1 \leq j \leq k,\ \min_{j=k+1,\ldots,l} Z_{t_j} > 0 \right) \\ =\ & P\left(E_j > x_j\,, 1 \leq j \leq k \right)\,, \end{aligned}$$

the last equality following by first conditioning on $(Z_t : t \in \mathbb{R})$ and then using the memoryless property of $E_1, \ldots, E_k$. Thus (5.28) follows. □

Proof of Theorem 4.3. By (5.28), (5.31), and the fact that $\mu(s) > 1$ for all $s \in [a,b] \setminus E$ we conclude that for $x > 0$,

$$P\left(u\big(\min_{t\in[a,b]} X_t - u\big) > x \middle| \min_{t\in[a,b]} X_t > u \right) \to e^{-(\theta_1+\ldots+\theta_k)x}\,.$$

By (5.26), the claim of the theorem follows. □

Proof of Theorem 4.4. By (5.27), for each $j = 1, \ldots, k$,

$$\begin{aligned} & \lim_{u\to\infty} P\left(T_* = t_j \middle| \min_{s\in[a,b]} X_s > u \right) \\ & \qquad \to P\big(E_j = \min(E_1, \ldots, E_k)\big) = \frac{\theta_j}{\theta_1 + \ldots + \theta_k}\,, \end{aligned}$$

hence the claim of the theorem. □

6. Examples

In order to illustrate the general results in Section 4 we will, in this section, look at specific examples of Gaussian processes satisfying the assumptions of the general results. We start with a quintessential example of a stationary process.

Example 6.1. Gaussian covariance function

Consider the zero mean stationary Gaussian process $(X_t : t \in \mathbb{R})$ with covariance function

$$R(t) := \mathrm{E}(X_s X_{s+t}) = \exp\left(-t^2/2\right), \, t \geq 0 \, .$$

This process has a spectral density that coincides with the standard Gaussian density, as in (2.2), hence the process has properties S1 and S2 of Section 2. Therefore, the results in Section 4 apply. Recall that some of the results require the assumption (4.6). We will presently see both that this assumption may fail and that this assumptions fails only rarely.

Let

$$0 = a < b < c_1 := \min\left\{y > 0 : 2e^{-y^2/8} - 1 - e^{-y^2/2} = 0\right\} \approx 2.2079 \, . \tag{6.1}$$

It has been shown in Proposition 5.3 and Example 6.1 of Adler et al. (2014) that for such an interval,

$$\begin{aligned} E = S &= \{a, b\} \, , \\ \mu(t) &= \frac{e^{-t^2/2} + e^{-(b-t)^2/2}}{1 + e^{-b^2/2}}, \, t \in [a, b] \, , \\ V_* &= \frac{1 + e^{-b^2/2}}{2} \, . \end{aligned} \tag{6.2}$$

Assumption (4.6) holds automatically, and Theorem 4.1 implies that

$$P\left(\min_{a \leq t \leq b} X_t > u\right) \sim C_1 u^{-2} \exp\left(-\frac{1}{1 + e^{-b^2/2}} u^2\right)$$

for $C_1 > 0$ as $u \to \infty$. In fact, one can check that

$$C_1 := \frac{1}{2\pi} \frac{(1 - e^{-b^2})^{3/2}}{(1 - e^{-b^2/2})^2} \, . \tag{6.3}$$

By Theorem 4.3, conditionally on the event $\left\{\min_{t \in [a,b]} X_t > u\right\}$, the scaled overshoot $u\big(\min_{t \in [a,b]} X_t - u\big)$ converges weakly, as $u \to \infty$, to an exponential random variable with the mean $(1 + e^{-b^2/2})/2$.

Next we consider the situation when $a = 0$ and $b = c_1$, defined by in (6.1). Then μ and V_* are still as in (6.2), but we now have $S = \{a, b\}$ and $E = \{a, b/2, b\}$. Theorem 4.1 still applies, and it gives

$$P\left(\min_{a \leq t \leq b} X_t > u\right) \sim \frac{1}{2} C_1 u^{-2} \exp\left(-\frac{1}{1 + e^{-b^2/2}} u^2\right) \, , \tag{6.4}$$

as $u \to \infty$, where C_1 is as defined in (6.3). The asymptotic conditional distribution of the scaled overshoot $u\big(\min_{t \in [a,b]} X_t - u\big)$ is same as in the case $b < c_1$.

We proceed to the case $a = 0$ and

$$c_1 < b < c_2 := \min\left\{y > c_1 : (1 - \varepsilon(y)) \left(\frac{y^2}{4} - 1\right) e^{-y^2/8} = \varepsilon(y)\right\} \, , \tag{6.5}$$

with

$$\varepsilon(y) := \frac{1 + e^{-y^2/2} - 2e^{-y^2/8}}{3 + e^{-y^2/2} - 4e^{-y^2/8}}\,.$$

The value of $c_2 \approx 3.9283$. Proposition 5.5 of Adler et al. (2014) shows that in this case

$$\begin{aligned} E = S =& \{a, b/2, b\}\,, \\ \mu(t) =& \frac{1}{V_*}\left[\frac{1-\varepsilon(b)}{2}\left(e^{-t^2/2} + e^{-(b-t)^2/2}\right) + \varepsilon(b)e^{-(t-b/2)^2/2}\right],\ t \in \mathbb{R}\,, \\ V_* =& \mathrm{Var}\left(\frac{1-\varepsilon(b)}{2}\left(X_a + X_b\right) + \varepsilon(b)X_{b/2}\right)\,. \end{aligned} \tag{6.6}$$

In this case

$$\mu^{(2)}(b/2) > 0\,, \tag{6.7}$$

so that (4.6) holds. Theorem 4.1 implies that for some $C_2 > 0$,

$$P\left(\min_{a\le t\le b} X_t > u\right) \sim C_2 u^{-3} \exp\left(-\frac{1}{2V_*}u^2\right)\,,$$

as $u \to \infty$. The b-dependent constant C_2 can be explicitly calculated if desired. The asymptotic conditional distribution of the scaled overshoot $u\big(\min_{t\in[a,b]} X_t - u\big)$ is exponential with mean V_*.

When $a = 0$ and $b = c_2$ defined in (6.5), then (6.6) still holds, but (6.7) fails, so the assumption (4.6) no longer holds. In this situation, all Theorem 4.1 implies is that

$$P\left(\min_{a\le t\le b} X_t > u\right) = o\left(u^{-3}\exp\left(-\frac{1}{2V_*}u^2\right)\right)\,.$$

as $u \to \infty$.

The plots on Figure 1 illustrate the different behaviour of the function μ when $c_1 < b < c_2$, and when $b = c_2$.

If a stationary Gaussian process satisfies S1 and S2, then the unique optimizer ν_* of the minimization problem (3.1) must be symmetric around the midpoint of the interval $[a, b]$; indeed, for any probability measure ν supported by $[a, b]$, the measure $\tilde{\nu}$ obtained by reflection around the midpoint leads to the same value in the integral and hence, by convexity, $\nu_* = (\nu_* + \tilde{\nu}_*)/2$ is symmetric. Since the optimal measure cannot be concentrated at the midpoint of the interval, we conclude that the cardinality of the set S in this case is at least 2. However, for a non-stationary process, S may be a singleton. The next example illustrates this fact.

Example 6.2. A non-stationary process

We start with a stationary centered Gaussian process $\mathbf{Z} = (Z_t : t \in \mathbb{R})$ with a spectral measure F_X satisfying S1 and S2. Let Y be a standard normal random variable independent of $\mathbf{Z}$. Define

$$X_t := Y(1 + t^2 - t^4) + Z_t - Z_0,\ t \in \mathbb{R}\,.$$

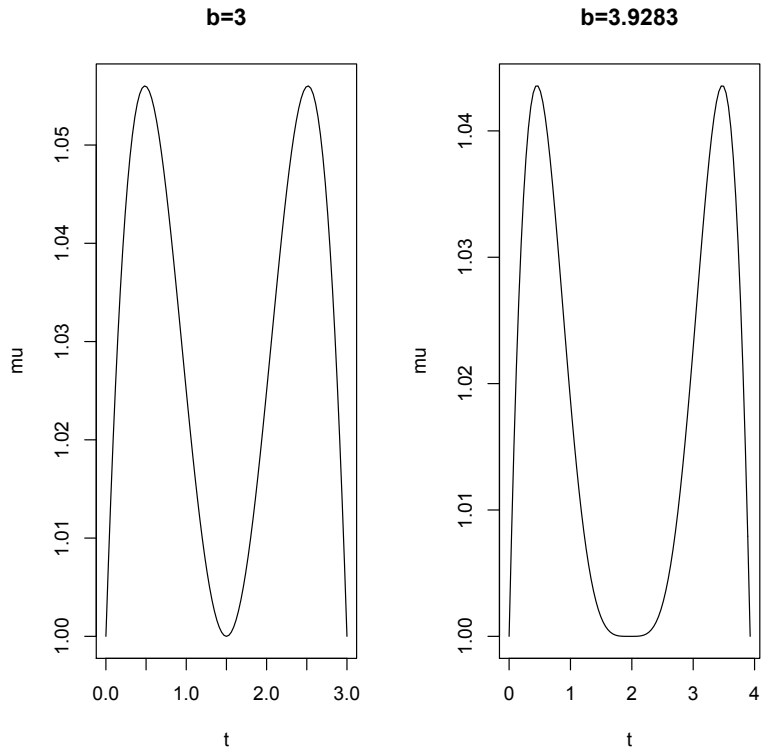

FIGURE 1. Function μ for $b = 3$ and $b = 3.9283$ in Example 6.1

Let $-1 < a < 0 < b < 1$. Note that the process $\mathbf{X} := (X_t : t \in \mathbb{R})$ has Property 1 by construction. It is elementary that the covariance function R of the process $\mathbf{X}$ satisfies (3.2).

Clearly, for any probability measure ν on $[a, b]$,

$$\int_a^b \int_a^b R(s,t)\,\nu(ds)\,\nu(dt) \geq \left(\int_a^b (1 + t^2 - t^4)\,\nu(dt)\right)^2 \geq 1\,. \tag{6.8}$$

Therefore, the process $\mathbf{X}$ has Property 2. In order to check that it also has Property 3 , suppose that for distinct reals numbers $t_0, \ldots, t_k$ and coefficients $\alpha_0, \ldots, \alpha_k, \beta_0, \ldots, \beta_k$, we have

$$\sum_{j=0}^{k} \left(\alpha_j X_{t_j} + \beta_j X^{(1)}_{t_j}\right) = 0 \text{ a.s.}\,.$$

Without loss of generality, we assume $t_0 = 0$. Using the independence of Y and $\mathbf{Z}$ we see that

$$-Z_0 \sum_{j=1}^{k} \alpha_j + \sum_{j=1}^{k} \alpha_j Z_{t_j} + \sum_{j=0}^{k} \beta_j Z^{(1)}_{t_j} = 0 \text{ a.s.}\,.$$

Proposition 2.1 implies that $\alpha_1 = \ldots = \alpha_k = \beta_0 = \ldots = \beta_k = 0$. Thus, also $\alpha_0 = 0$, and so process $\mathbf{X}$ has Property 3.

Note that the choice $\nu = \delta_{\{0\}}$ is, by (6.8), the optimal measure ν_*. Thus, $S = E = \{0\}$. That is, the support S is a singleton which contains none of the endpoints of the interval. That is, the conclusion of Proposition 3.2 indeed fails without appropriate assumptions on the covariance function of the process.

Here

$$\mu(t) = \mathrm{E}(X_t | X_0 = 1) = \mathrm{E}(X_t | Y = 1) = 1 + t^2 - t^4,\ t \in \mathbb{R}\,.$$

Therefore, $\mu^{(2)}(0) = 2 > 0$. By Theorem 4.1,

$$P\left(\min_{a \le t \le b} X_t > u\right) \sim \frac{1}{\sqrt{2\pi(1 + \lambda_2/2)}} u^{-1} e^{-u^2/2}\,,$$

as $u \to \infty$, where λ_2 is the second spectral moment of $\mathbf{Z}$. By Theorem 4.3, conditionally on the event $\{\min_{t\in[a,b]} X_t > u\}$, the scaled overshoot $u\big(\min_{t\in[a,b]} X_t - u\big)$ converges weakly, as $u \to \infty$, to the standard exponential random variable.

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