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# On the equivalence of separability and extendability of quantum states

B. V. Rajarama Bhat, K. R. Parthasarathy
and Ritabrata Sengupta

Indian Statistical Institute, Delhi Centre
7, SJSS Marg, New Delhi–110 016, India

# ON THE EQUIVALENCE OF SEPARABILITY AND EXTENDABILITY OF QUANTUM STATES

B. V. RAJARAMA BHAT, K. R. PARTHASARATHY, AND RITABRATA SENGUPTA

Abstract. Motivated by the notions of $k$-extendability and complete extendability of the state of a finite level quantum system as described by Doherty et al (Phys. Rev. A, 69:022308), we introduce parallel definitions in the context of Gaussian states and using only properties of their covariance matrices derive necessary and sufficient conditions for their complete extendability. It turns out that the complete extendability property is equivalent to the separability property of a bipartite Gaussian state.

Following the proof of quantum de Finetti theorem as outlined in Hudson and Moody (Z. Wahrscheinlichkeitstheorie und Verw. Gebiete, 33(4):343–351), we show that separability is equivalent to complete extendability for a state in a bipartite Hilbert space where at least one of which is of dimension greater than 2. This, in particular, extends the result of Fannes, Lewis, and Verbeure (Lett. Math. Phys. 15(3): 255–260) to the case of an infinite dimensional Hilbert space whose C* algebra of all bounded operators is not separable.

**Keywords.** Gaussian state, exchangeable Gaussian state, extendability, entanglement.

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## 1. Introduction

One of the most important problems in quantum mechanics as well as quantum information theory is to determine whether a given bipartite state is separable or entangled [?]. There are several methods in tackling this problem leading to a long list of important publications. A detailed discussion on this topic is available in the survey articles by Horodecki et al [?], and Gühne and Tóth [?]. One such condition which is both necessary and sufficient for separability in finite dimensional product spaces is complete extendability [?].

**Definition 1.1.** *Let $k \in \mathbb{N}$. A state $\rho \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$ is said to be $k$-extendable with respect to system $B$ if there is a state $\tilde{\rho} \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B^{\otimes k})$ which is invariant under any permutation in $\mathcal{H}_B^{\otimes k}$ and $\rho = \operatorname{Tr}_{\mathcal{H}_B^{\otimes (k-1)}} \tilde{\rho}$, $k \geq 2$.*

*A state $\rho \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$ is said to be completely extendable if it is $k$-extendable for all $k \in \mathbb{N}$.*

The following theorem of Doherty, Parrilo, and Spedalieri [?] emphasizes the importance of the notion of complete extendability.

**Theorem A.** [?] *A bipartite state $\rho \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$ is separable if and only if it is completely extendable with respect to one of its subsystems.*

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It is fairly simple to see that separability implies complete extendability. The proof of the converse depends on an application of the quantum de Finetti theorem [?, ?], according to which any exchangable state is, indeed, separable. The link between separability and extendability has found applications in quantum information theory [?, ?]. Here we study the same in the context of quantum Gaussian states.

The importance of finite mode Gaussian states and their covariance matrices in general quantum theory as well as quantum information has been highlighted extensively in the literature. A comprehensive survey of Gaussian states and their properties can be found in the book of Holevo [?]. For their applications to quantum information theory the reader is referred to the survey article by Weedbrook et al [?] as well as Holevo's book [?]. For our reference we use [?, ?, ?] for Gaussian states and for notations in the following sections we use [?] and [?].

If $\rho$ is a state of a quantum system and $X_i$, $i = 1, 2$ are two real-valued observables, or equivalently, self-adjoint operators with finite second moments in the state $\rho$ then the covariance between $X_1$ and $X_2$ in the state $\rho$ is the scalar quantity

$$\mathrm{Tr}\left(\frac{1}{2}(X_1X_2 + X_2X_1)\rho\right) - (\mathrm{Tr}\, X_1\rho) \cdot (\mathrm{Tr}\, X_2\rho),$$

which is denoted by $\mathrm{Cov}_\rho(X_1, X_2)$. Suppose $q_1, p_1; q_2, p_2; \cdots; q_n, p_n$ are the position - momentum pairs of observables of a quantum system with $n$ degrees of freedom obeying the canonical commutation relations. Then we express

$$(X_1, X_2, \cdots, X_{2n}) = (q_1, -p_1, q_2, -p_2, \cdots, q_n, -p_n).$$

If $\rho$ is a state in which all the $X_j$'s have finite second moments we write

$$S_\rho = [[\mathrm{Cov}_\rho(X_i, X_j)]], \quad i, j \in \{1, 2, \cdots, 2n\}. \tag{1.1}$$

We call $S_\rho$ the covariance matrix of the position momentum observables. If we write

$$J_{2n} = \begin{bmatrix} 0 & 1 & & & & & \\ -1 & 0 & & & & & \\ & & 0 & 1 & & & \\ & & -1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & 1 \\ & & & & & -1 & 0 \end{bmatrix} \tag{1.2}$$

or equivalently $\bigoplus_1^n \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ for the $2n \times 2n$ block diagonal matrix, the complete Heisenberg uncertainty relations for all the position and momentum observables assume the form of the following matrix inequality

$$S_\rho + \frac{\imath}{2} J_{2n} \geq 0. \tag{1.3}$$

Conversely, if $S$ is any real $2n \times 2n$ symmetric matrix obeying the inequality $S + \frac{\imath}{2} J_{2n} \geq 0$, then there exists a state $\rho$ such that $S$ is the covariance matrix $S_\rho$ of the observables $q_1, -p_1; q_2, -p_2; \cdots; q_n, -p_n$. In such a case $\rho$ can be chosen to be a Gaussian state with mean zero. Recall [?], a state $\rho$ in $\Gamma(\mathcal{H})$ with $\mathcal{H} = \mathbb{C}^n$ is an $n$-mode Gaussian state if its Fourier transform $\hat{\rho}$ is given by

$$\hat{\rho}(\boldsymbol{x} + \imath\boldsymbol{y}) = \exp\left[-\imath\sqrt{2}(\boldsymbol{l}^T\boldsymbol{x} - \boldsymbol{m}^T\boldsymbol{y}) - \begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{pmatrix}^T S \begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{pmatrix}\right]. \tag{1.4}$$

for all $\boldsymbol{x},\ \boldsymbol{y} \in \mathbb{R}^n$ where $\boldsymbol{l},\ \boldsymbol{m}$ are the momentum-position mean vectors and $S$ their covariance matrix.

We organise the paper as follows. Motivated by the concept of extendability for finite dimensional systems, we define Gaussian extendability for Gaussian states in §2. We study this extendability problem entirely in terms of covariance matrices. In §3 we look at the same problem in an abstract way. We show that for a bipartite state of arbitrary dimension, complete extendability is equivalent to separability. This proof directly follows from work of Hudson and Moody [?]. The problem of finding necessary and sufficient conditions for $k$-extendability of states in both Gaussian and non-Gaussian cases remains open.

## 2. Gaussian extendability

**Definition 2.1** (Gaussian extendability)**.** *Let $k \in \mathbb{N}$. A Gaussian state $\rho_g$ in $\Gamma(\mathbb{C}^m) \otimes \Gamma(\mathbb{C}^n)$ is said to be Gaussian $k$-extendable with respect to the second system if there is a Gaussian state $\tilde{\rho}_g$ in $\Gamma(\mathbb{C}^m) \otimes \Gamma(\mathbb{C}^n)^{\otimes k}$ which is invariant under any permutation in $\Gamma(\mathbb{C}^n)^{\otimes k}$ and $\rho_g = \mathrm{Tr}_{\Gamma(\mathbb{C}^n)^{\otimes(k-1)}}\tilde{\rho}_g$, $k \geq 2$.*

*A Gaussian state $\rho_g$ in $\Gamma(\mathbb{C}^m) \otimes \Gamma(\mathbb{C}^n)$ is said to be Gaussian completely extendable if it is Gaussian $k$-extendable for every $k \in \mathbb{N}$.*

**Remark 2.1.** *In this section we confine our attention to Gaussian states only and so we use the terms $k$-extendability and complete extendability to mean Gaussian $k$-extendability and Gaussian complete extendability respectively, unless stated otherwise. In the next section §3, we use extendability and complete extendability in its usual sense.*

We shall use the following result.

**Theorem B.** *Let*

$$X = \left[\begin{array}{c|c} A & B \\ \hline B^\dagger & C \end{array}\right]$$

*be a Hermitian block matrix with real or complex entries, $A$ and $C$ being strictly positive matrices of order $m \times m$ and $n \times n$ respectively. Then $X \geq 0$ if and only if*

$$A \geq BC^{-1}B^\dagger.$$

*Proof.* For a proof, see Theorem 1.3.3 in the book of Bhatia [?]. □

Entanglement property of a Gaussian state depends only on its covariance matrix. Hence without loss of generality, we can confine our attention to the Gaussian states with mean zero. Thus an $(m+n)$-mode mean zero Gaussian state in $\Gamma(\mathbb{C}^m) \otimes \Gamma(\mathbb{C}^n)$ is uniquely determined by a $2(m+n) \times 2(m+n)$ covariance matrix

$$S = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}.$$

Here $A$ and $C$ are covariance matrices of the $m$ and $n$-mode marginal states respectively.

If $\rho(\mathbf{0},\mathbf{0};S)$, written in short as $\rho(S)$ in $\Gamma(\mathbb{C}^m)\otimes\Gamma(\mathbb{C}^n)$ is $k$-extendable with respect to the second system, then there exists a real matrix $\theta_k$ of order $2n\times 2n$ such that the extended matrix

$$S_k = \left[\begin{array}{c|cccc} A & B & B & \cdots & B \\ \hline B^T & C & \theta_k & \cdots & \theta_k \\ B^T & \theta_k^T & C & \cdots & \theta_k \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B^T & \theta_k^T & \theta_k^T & \cdots & C \end{array}\right] \tag{2.1}$$

is the covariance matrix of a Gaussian state in $\Gamma(\mathbb{C}^m)\otimes\Gamma(\mathbb{C}^n)^{\otimes k}$. Then it satisfies inequality (1.3) in the form

$$S_k + \frac{\imath}{2} J_{2(m+kn)} \geq 0. \tag{2.2}$$

Now we observe that $\theta_k$ can be chosen independent of $k$. To prove this we need the following theorem which will also be used later in this paper.

**Theorem C.** *Let $A$ and $B$ be positive. Then the matrix $\left[\begin{array}{c|c} A & B \\ \hline B^\dagger & C \end{array}\right]$ is positive if and only if $B = A^{\frac{1}{2}} K C^{\frac{1}{2}}$ for some contraction $K$.*

*Proof.* For a proof, see Proposition 1.3.2 in the book of Bhatia [?]. □

The reason follows form the fact that the marginal covariance matrix $\begin{bmatrix} C & \theta_k \\ \theta_k^T & C \end{bmatrix} \geq 0$. Using the above Theorem C, this is equivalent to the existence of a contraction $K$ with $\|K\|\leq 1$ such that $\theta_k = C^{\frac{1}{2}} K C^{\frac{1}{2}}$. Hence $\|\theta_k\| \leq \|C\|$. Since for a given state, $C$ is fixed, we have $\theta_k$ bounded. Hence, in the set of all $\theta_k$'s there is a convergent subsequence which converges to a $\theta$ and can replace $\theta_k$ in (2.1) by $\theta$. The extension matrix for each $k = 1, 2, 3, \cdots$ will look like

$$S_k = \left[\begin{array}{c|cccc} A & B & B & \cdots & B \\ \hline B^T & C & \theta_k & \cdots & \theta_k \\ B^T & \theta_k^T & C & \cdots & \theta_k \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B^T & \theta_k^T & \theta_k^T & \cdots & C \end{array}\right] \tag{2.3}$$

Hence by Definition 1.1, $\rho(S)$ is completely extendable if the inequality (2.2) holds for every $k = 1,\ 2,\ \cdots$.

Let us denote the marginal covariance matrix corresponding to $\Gamma(\mathbb{C}^n)^{\otimes k}$ by

$$\Sigma_k(C,\theta) = \begin{bmatrix} C & \theta & \cdots & \theta \\ \theta^T & C & \cdots & \theta \\ \vdots & \vdots & \ddots & \vdots \\ \theta^T & \theta^T & \cdots & C \end{bmatrix}.$$

If $\rho$ is completely extendable, $S_k$ is a covariance matrix for each $k$, and hence $\Sigma_k(C,\theta)$ is a covariance matrix for each $k$ as well. Using Theorem 1 of [?] (see also [?]), such a pair $(C,\theta)$ defines a covariance matrix $\Sigma_k(C,\theta)$ for each $k = 1, 2, 3, \cdots$ if and only if

(i) $\theta$ is a real symmetric positive semidefinite matrix, and
(ii) $C - \theta + \frac{\imath}{2} J_{2n} \geq 0$.

In particular, $S_k$ is of the form

$$S_k = \left[\begin{array}{c|ccccc} A & B & B & \cdots & B \\ \hline B^T & C & \theta & \cdots & \theta \\ B^T & \theta & C & \cdots & \theta \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B^T & \theta & \theta & \cdots & C \end{array}\right], \tag{2.4}$$

where $\theta$ is a real positive semidefinite matrix.

Our first theorem gives a necessary and sufficient condition for complete extendability of Gaussian states.

**Lemma 2.1.** *Let $\rho$ be a bipartite Gaussian state in $\Gamma(\mathbb{C}^m) \otimes \Gamma(\mathbb{C}^n)$ with no pure marginal state in $\Gamma(\mathbb{C}^m)$ as well as $\Gamma(\mathbb{C}^n)$. Let $S = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$ be the covariance matrix of $\rho$, where $A$ and $C$ are marginal covariance matrices of the first and second system respectively. Then $\rho$ is completely extendable with respect to the second system if and only if there exists a real positive matrix $\theta$ such that*

$$C + \frac{\imath}{2} J_{2n} \geq \theta \geq B^T \left( A + \frac{\imath}{2} J_{2m} \right)^{-1} B. \tag{2.5}$$

*Proof.* Without loss of generality, we may assume that $A$ and $C$ are written in their Williamson normal forms. Since no pure state is a marginal of $\rho$, $\frac{1}{2} I_2$ is not a sub-matrix of ether $A$ or $C$. This implies $\left(A + \frac{\imath}{2} J_{2m}\right)$ and $\left(C + \frac{\imath}{2} J_{2n}\right)$ are invertible, and hence we can apply Theorem B, when $A$ and $C$ are replaced respectively by $\left(A + \frac{\imath}{2} J_{2m}\right)$ and $\left(C + \frac{\imath}{2} J_{2n}\right)$. Thus,

$$C + \frac{\imath}{2} J_{2n} \geq B^T \left( A + \frac{\imath}{2} J_{2m} \right)^{-1} B.$$

The necessity of the left part of inequality 2.5 is already contained in the discussion above (2.4). Hence, all we need to prove is the right part the same inequality starting from (2.2).

Setting $|\psi_k\rangle = \frac{1}{\sqrt{k}}[1, 1, \cdots, 1]^T \in \mathbb{C}^k$ and $\sqrt{k}\mathcal{B}_k = B \otimes \langle \psi_k|$, the left hand side of (2.2) can be expressed as

$$\begin{bmatrix} A + \frac{\imath}{2} J_{2m} & \sqrt{k}\mathcal{B}_k \\ \sqrt{k}\mathcal{B}_k^T & \Sigma_k + \frac{\imath}{2} J_{2nk} \end{bmatrix}.$$

By Theorem B this matrix is positive if and only if

$$\Sigma_k + \frac{\imath}{2} J_{2nk} \geq k\mathcal{B}_k^T \left( A + \frac{\imath}{2} J_{2m} \right)^{-1} \mathcal{B}_k.$$

By elementary algebra, this is equivalent to

$$\left( C - \theta + \frac{\imath}{2} J_{2n} \right) \otimes (I_k - |\psi_k\rangle\langle\psi_k|) + \left( C + \overline{k-1}\theta + \frac{\imath}{2} J_{2n} \right) \otimes |\psi_k\rangle\langle\psi_k| \geq kB^T \left( A + \frac{\imath}{2} J_{2m} \right)^{-1} B \otimes |\psi_k\rangle\langle\psi_k|.$$

Since $|\psi_k\rangle\langle\psi_k|$ and $I_k - |\psi_k\rangle\langle\psi_k|$ are mutually orthogonal projections, it follows that the inequality above is equivalent to

$$\left(C + \overline{k-1}\theta + \frac{\imath}{2}J_{2n}\right) \geq kB^T \left(A + \frac{\imath}{2}J_{2m}\right)^{-1} B,$$

which can be rewritten as

$$\frac{1}{k}\left(C - \theta + \frac{\imath}{2}J_{2n}\right) + \theta \geq B^T \left(A + \frac{\imath}{2}J_{2m}\right)^{-1} B, \quad \text{for every } k \in \mathbb{N}. \tag{2.6}$$

Since $\left(C - \theta + \frac{\imath}{2}J_{2n}\right)$ is positive and the left hand side decreases monotonically to $\theta$ as $k \to \infty$, it follows that (2.6) is equivalent to

$$\theta \geq B^T \left(A + \frac{\imath}{2}J_{2m}\right)^{-1} B.$$

□

We now consider the case when the Gaussian state $\rho_g$ admits a pure marginal state.

**Proposition 2.1.** *If $X = \begin{bmatrix} A & B \\ B^T & \frac{1}{2}I_{2s} \end{bmatrix}$ is a Gaussian covariance matrix, then $B = 0$.*

*Proof.* Let $C = \frac{1}{2}(I_{2s} + \imath J_{2s})$. Then $C$ is a projection with $C^{\frac{1}{2}} = C$. It follows that $C(I_{2s} - \imath J_{2s}) = 0$. Since $X$ is a Gaussian matrix,

$$X + \frac{\imath}{2}J_{2(n+s)} = \begin{bmatrix} A + \frac{\imath}{2}J_{2n} & B \\ B^T & \frac{1}{2}(I_{2s} + \imath J_{2s}) \end{bmatrix} \geq 0,$$

there is a contraction $D$ such that $B = \left(A + \frac{\imath}{2}J_{2n}\right)^{\frac{1}{2}} D \frac{1}{2}(I_{2s} + \imath J_{2s})^{\frac{1}{2}} = \left(A + \frac{\imath}{2}J_{2n}\right)^{\frac{1}{2}} DC$, where $\|D\| \leq 1$. Then $B(I_{2s} - \imath J_{2s}) = 0$, and hence $B = \imath \mathrm{Re}\,(B - BJ_{2s}) = 0$. □

**Theorem 2.1.** *Let $\rho$ be a bipartite Gaussian state in $\Gamma(\mathbb{C}^m) \otimes \Gamma(\mathbb{C}^n)$ with covariance matrix $S = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$, where $A$ and $C$ are marginal covariance matrices of the first and second system respectively. Then $\rho$ is completely extendable with respect to the second system if and only if there exists a real positive matrix $\theta$ such that*

$$C + \frac{\imath}{2}J_{2n} \geq \theta \geq B^T \left(A + \frac{\imath}{2}J_{2m}\right)^{-} B, \tag{2.7}$$

*where $\left(A + \frac{\imath}{2}J_{2m}\right)^{-}$ is the Moore-Penrose inverse of $A + \frac{\imath}{2}J_{2m}$.*

*Proof.* Since the case where both $A + \frac{\imath}{2}J_{2m}$ and $C + \frac{\imath}{2}J_{2n}$ are invertible has already been dealt with in Lemma 2.1, we only need to prove in the case when $\rho$ admits pure marginal states.

Without loss of generality let us assume that $A$ and $C$ are written in their Williamson normal forms. Let $A = \left(\oplus_1^k \kappa_j I_2\right) \bigoplus \left(\oplus_{k+1}^m \frac{1}{2}I_2\right) = A' \bigoplus \frac{1}{2}I_{2(m-k)}$ and $C = \left(\oplus_1^s \mu_l I_2\right) \bigoplus \left(\oplus_{s+1}^n \frac{1}{2}I_2\right) = C' \bigoplus \frac{1}{2}I_{2(n-s)}$, where $\kappa_j,\ \mu_l > \frac{1}{2}$ for every $j$, $l$. By Proposition 2.1, $B$ has the form

$$B = \left[\begin{array}{c|c} B' & \\ \hline & \end{array}\right],$$

where $B'$ is a real matrix of order $2k \times 2s$ and rest of the entries are zero matrices of appropriate order.

Consider the marginal Gaussian state, whose covariance matrix is

$$\begin{bmatrix} A' & B' \\ B'^T & C' \end{bmatrix}. \tag{2.8}$$

Since $A'$ and $C'$ do not have any principal sub-matrix of the form $\frac{1}{2}I_2$, by Lemma 2.1, the marginal Gaussian state with covariance matrix given by (2.8) is completely extendable if and only if there is a real $2s \times 2s$ matrix $\theta'$ such that

$$C' + \frac{\imath}{2}J_{2s} \geq \theta' \geq B'^T \left(A' + \frac{\imath}{2}J_{2k}\right)^{-1} B'.$$

Observe that

$$\begin{aligned} B^T\left(A + \frac{\imath}{2}J_{2n}\right)^{-} B &= \left[\begin{array}{c|c} B'^T & \\ \hline & \end{array}\right] \left(\left(A' + \frac{\imath}{2}J_{2k}\right)^{-1} \bigoplus \left(\bigoplus_{(m-k)\text{-copies}} \begin{bmatrix} \frac{1}{2} & \frac{\imath}{2} \\ -\frac{\imath}{2} & \frac{1}{2} \end{bmatrix}\right)\right) \left[\begin{array}{c|c} B' & \\ \hline & \end{array}\right] \\ &= \left[\begin{array}{c|c} B'^T\left(A' + \frac{\imath}{2}J_{2k}\right)^{-1} B' & \mathbf{0}_{2s\times 2(n-s)} \\ \hline \mathbf{0}_{2(n-s)\times 2s} & \mathbf{0}_{2(n-s)\times 2(n-s)} \end{array}\right], \end{aligned}$$

$\mathbf{0}$ with indices denoting zero matrices. Set $\theta = \theta' \bigoplus \mathbf{0}_{2(n-s)\times 2(n-s)}$. It is easy to see that such a real matrix $\theta$ satisfies the conditions of inequality (2.7). Hence the theorem is proved. □

**Theorem 2.2.** *Any separable Gaussian state in a bipartite system is completely extendable.*

*Proof.* Let $\rho$ be an $(m+n)$ mode Gaussian state with covariance matrix $\begin{bmatrix} A & B^T \\ B & C \end{bmatrix}$ with $A$ and $C$ being the $m$ and $n$-mode marginal covariance matrices. By a theorem of Werner and Wolf [**?**], $\rho$ is separable if and only if there exist $m$-mode and $n$-mode Gaussian states with covariance matrices $X$ and $Y$ respectively such that

$$\begin{bmatrix} A & B^T \\ B & C \end{bmatrix} \geq \begin{bmatrix} X & \\ & Y \end{bmatrix}.$$

Set $E = A - X$, $G = C - Y$, and $F = B$. Then the above inequality can be expressed as

$$\begin{bmatrix} E & F^T \\ F & G \end{bmatrix} \geq 0.$$

By the previous discussions and Theorem 2.1, we need to construct a real, symmetric, $n \times n$ matrix $\varphi$ such that for every $k$-extension, the matrix

$$\begin{bmatrix} E & F^T & F^T & F^T & \cdots & F^T \\ F & G & \varphi & \varphi & \cdots & \varphi \\ F & \varphi & G & \varphi & \cdots & \varphi \\ F & \varphi & \varphi & G & \cdots & \varphi \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ F & \varphi & \varphi & \varphi & \cdots & G \end{bmatrix} \geq 0.$$

Calculations similar to those in Theorem 2.1, show that this is possible if and only if

$$G \geq \varphi \geq F E^{-} F^T. \tag{2.9}$$

We choose

$$\varphi = tG + (1-t)FE^{-}F^{T}, \quad t \in [0,1]. \tag{2.10}$$

Notice that for every $k = 1, 2, \cdots$,

$$\begin{bmatrix} X & & & \\ & Y & & \\ & & \ddots & \\ & & & Y \end{bmatrix} + \begin{bmatrix} E & F^T & F^T & F^T & \cdots & F^T \\ F & G & \varphi & \varphi & \cdots & \varphi \\ F & \varphi & G & \varphi & \cdots & \varphi \\ F & \varphi & \varphi & G & \cdots & \varphi \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ F & \varphi & \varphi & \varphi & \cdots & G \end{bmatrix} = \begin{bmatrix} A & B^T & B^T & B^T & \cdots & B^T \\ B & C & \varphi & \varphi & \cdots & \varphi \\ B & \varphi & C & \varphi & \cdots & \varphi \\ B & \varphi & \varphi & C & \cdots & \varphi \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ B & \varphi & \varphi & \varphi & \cdots & C \end{bmatrix},$$

where the first term in the left hand side, $X \oplus (\oplus_k Y)$, is a Gaussian covariance matrix, and the second one is a positive matrix thanks to the construction above. Thus the right hand side is also a Gaussian covariance matrix. Hence the theorem is proved with the extension matrix $\varphi$ satisfying equation (2.10). $\square$

**Theorem 2.3.** *Any completely extendable Gaussian state is separable.*

*Proof.* Let $S = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$ be the covariance matrix of a $(m+n)$ mode Gaussian state $\rho$, which is completely extendable by a real symmetric positive matrix $\theta$ satisfying inequalities (2.7) of Theorem 2.1. By the result of Werner and Wolf [**?**], it is enough to find $m$ mode and $n$ mode Gaussian states with covariance matrices $X$ and $Y$ respectively such that

$$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \geq \begin{bmatrix} X & \\ & Y \end{bmatrix}. \tag{2.11}$$

Since $S$ is extendable, $\theta \geq B^T \left(A + \frac{\imath}{2} J_{2m}\right)^{-} B$, which is equivalent to the matrix condition

$$\begin{bmatrix} A + \frac{\imath}{2} J_{2m} & B \\ B^T & \theta \end{bmatrix} \geq 0.$$

Using Theorem C, this is equivalent to the condition that there is a contraction $K$ with $\|K\| \leq 1$ such that $B = \left(A + \frac{\imath}{2} J_{2m}\right)^{\frac{1}{2}} K \theta^{\frac{1}{2}}$. Here $\theta^{\frac{1}{2}} = \theta'^{\frac{1}{2}} \bigoplus \mathbf{0}$, where $\mathbf{0}$ being the zero matrix of appropriate order, as in the proof of Theorem 2.2. In a similar way, we may define $\theta^{-\frac{1}{2}} = \theta'^{-\frac{1}{2}} \oplus \mathbf{0}$, which is a real positive matrix. Since $B$ and $\theta$ are real matrices, so is $B\theta^{-\frac{1}{2}} = \left(A + \frac{\imath}{2} J_{2m}\right)^{\frac{1}{2}} K$. Hence, we can choose the contraction $K$ to be such that the product in the right hand side is a real matrix. Choose

$$\begin{aligned} X &= A - \left(A + \frac{\imath}{2} J_{2m}\right)^{\frac{1}{2}} K K^{\dagger} \left(A + \frac{\imath}{2} J_{2m}\right)^{\frac{1}{2}}, \\ Y &= C - \theta. \end{aligned}$$

By our choice of contraction $K$, $X$ chosen above is a real matrix. It follows from the left half of inequalities (2.7) that $Y$ is a Gaussian covariance matrix. To see that $X$ is so, observe that

$\|K\| \leq 0$, and note that

$$\begin{aligned} X + \frac{\imath}{2} J_{2m} &= \left(A + \frac{\imath}{2} J_{2m}\right) - \left(A + \frac{\imath}{2} J_{2m}\right)^{\frac{1}{2}} K K^{\dagger} \left(A + \frac{\imath}{2} J_{2m}\right)^{\frac{1}{2}} \\ &= \left(A + \frac{\imath}{2} J_{2m}\right)^{\frac{1}{2}} (I - K K^{\dagger}) \left(A + \frac{\imath}{2} J_{2m}\right)^{\frac{1}{2}} \geq 0. \end{aligned}$$

To check that our choice of $X$ and $Y$ satisfies inequality (2.11), we observe that -

$$\begin{aligned} \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} - \begin{bmatrix} X & \\ & Y \end{bmatrix} &= \begin{bmatrix} \left(A + \frac{\imath}{2} J_{2m}\right)^{\frac{1}{2}} K K^{\dagger} \left(A + \frac{\imath}{2} J_{2m}\right)^{\frac{1}{2}} & \left(A + \frac{\imath}{2} J_{2m}\right)^{\frac{1}{2}} K \theta^{\frac{1}{2}} \\ \theta^{\frac{1}{2}} K^{\dagger} \left(A + \frac{\imath}{2} J_{2m}\right)^{\frac{1}{2}} & \theta \end{bmatrix} \\ &= \begin{bmatrix} \left(A + \frac{\imath}{2} J_{2m}\right)^{\frac{1}{2}} & \\ & \theta^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} K \\ I \end{bmatrix} \begin{bmatrix} K^{\dagger} & I \end{bmatrix} \begin{bmatrix} \left(A + \frac{\imath}{2} J_{2m}\right)^{\frac{1}{2}} & \\ & \theta^{\frac{1}{2}} \end{bmatrix} \geq 0. \end{aligned}$$

Hence the theorem is proved. □

We combine Theorem 2.2 and 2.3 to get a necessary and sufficient condition for separability of Gaussian states in the following theorem.

**Theorem 2.4.** *Any bipartite Gaussian state $\rho$ in $\Gamma(\mathbb{C}^m) \otimes \Gamma(\mathbb{C}^n)$ is separable if and only if it is completely extendable.*

## 3. Complete extendability and separability in general case

Consider a separable Hilbert space $\mathfrak{h}$ and denote $\mathcal{B} = \mathcal{B}(\mathfrak{h})$ the C* algebra of all bounded operators on $\mathfrak{h}$. Let $\mathcal{B}_n = \mathcal{B}(\mathfrak{h}^{\otimes n}) = \mathcal{B}^{\otimes n}$ be the $n$-fold tensor product of copies of $\mathcal{B}$. Then $\mathcal{B}_n$ can be embedded as a C* algebra of $\mathcal{B}_{n+1}$ by the map $X \mapsto X \otimes I$, $X \in \mathcal{B}_n$, $I$ being the identity operator in $\mathcal{B}$. This enables the construction of an inductive limit C* algebra $\mathcal{B}^{\infty}$ such that there exists a C* embedding $i_n : \mathcal{B}_n \hookrightarrow \mathcal{B}^{\infty}$ such that $i_{n-1}(X) = i_n(X \otimes I)$ for all $X \in \mathcal{B}_{n-1}$, $n = 2, 3, \cdots$. Let $\mathfrak{S}$ denote the set of all states in $\mathcal{B}^{\infty}$ equipped with the weak* topology. Then $\mathfrak{S}$ is a compact convex set. For any $\omega \in \mathfrak{S}$, define

$$\omega_n(X) = \omega(i_n(X)), \quad X \in \mathcal{B}_n.$$

Then $\omega_n$ is a state in $\mathcal{B}_n$ for all $n$ and

$$\omega_{n-1}(X) = \omega_n(X \otimes I), \quad \forall X \in \mathcal{B}_{n-1},\ n = 2, 3, \cdots .$$

in other words $\{\omega_n\}$ is a consistent family of states in $\{\mathcal{B}_n\}$, $n = 2, 3, \cdots$ with the projective limit $\omega$.

Conversely, let $\omega_n$ be a state in $\mathcal{B}_n$ for each $n = 1, 2, 3, \cdots$ such that $\omega_n(X \otimes I) = \omega_{n-1}(X \otimes I)$, $\forall X \in \mathcal{B}_{n-1}$, $n = 2, 3, \cdots$. Then there exists a unique state $\omega$ in $\mathcal{B}^{\infty}$ such that

$$\omega(i_n(X)) = \omega_n(X), \quad \forall X \in \mathcal{B}_n,\ n = 1, 2, 3, \cdots .$$

**Definition 3.1.** *A state $\omega$ in $\mathcal{B}^{\infty}$ is said to be* locally normal *if each $\omega_n$ in $\mathcal{B}_n$, $n = 1, 2, \cdots$ is determined by a density operator $\rho_n$, $n = 1, 2, \cdots$, i.e., a positive operator $\rho_n$ of unit trace in $\mathfrak{h}^{\otimes n}$ satisfying*

$$\omega_n(X) = \operatorname{Tr} \rho_n X, \quad X \in \mathcal{B}_n,\ n = 1, 2, \cdots .$$

*Then the relative trace of $\rho_n$ in $\mathfrak{h}^{\otimes n}$ over the last copy of $\mathfrak{h}$ is equal to $\rho_{n-1}$ for each $n = 2, 3, \cdots$ .*

**Definition 3.2.** *A state in $\mathcal{B}^\infty$ is said to be* exchangeable *if for any permutation $\pi$ of $\{1, 2, \cdots, n\}$ and operators $X_j \in \mathcal{B}$, $i = 1, 2, \cdots, n$*

$$\omega_n(X_{\pi(1)} \otimes X_{\pi(2)} \otimes \cdots \otimes X_{\pi(n)}) = \omega_n(X_1 \otimes X_2 \otimes \cdots \otimes X_n) = \omega(i_n(X_1 \otimes X_2 \otimes \cdots \otimes X_n)).$$

We shall now describe a version of quantum de Finetti theorem due to Hudson and Moody [?] (see also Størmer [?] for an abstract C* algebraic version) which we shall make use of in our analysis of complete extendability - separability problem. To this end denote by $\mathcal{R}_\mathfrak{h}$ the set of all density operators on $\mathfrak{h}$. Viewing $\mathcal{R}_\mathfrak{h}$ as a subset of the dual of $\mathcal{B} = \mathcal{B}_\mathfrak{h}$, equip it with the relative topology inherited from the weak* topology. Let $\mathcal{P}_\mathfrak{h}$ denote the set of all probability measures on the Borel $\sigma$-algebra of $\mathcal{R}_\mathfrak{h}$.

**Theorem 3.1.** *[Hudson and Moody] A locally normal state $\omega$ on $\mathcal{B}^\infty$ is exchangeable if and only if there exists a probability measure $P_\omega$ in $\mathcal{P}_\mathfrak{h}$ such that*

$$\omega(i_n(X)) = \int_{\mathcal{R}_\mathfrak{h}} \operatorname{Tr} \rho^{\otimes n} X \, P_\omega(\mathrm{d}\rho), \quad \forall X \in \mathcal{B}_n,\ n = 1, 2, \cdots .$$

*The correspondence $\omega \to P_\omega$ between the set of locally normal and exchangeable states and the set $\mathcal{P}_\mathfrak{h}$ of probability measures on $\mathcal{R}_\mathfrak{h}$ is bijective.*

**Remark 3.1.** *Theorem 3.1 shows that exchanbeability property automatically implies that every finite dimensional projection of $\omega$, namely $\omega_n$, is separable. It is natural to expect that complete extendeability would force separability.*

**Theorem 3.2.** *Let $\mathfrak{h}_0$, $\mathfrak{h}$ be Hilbert spaces with $\dim \mathfrak{h}_0 > 2$ and $\rho$ be a density operator in $\mathfrak{h}_0 \otimes \mathfrak{h}$. Let $\mathcal{B}_{n]} = \mathcal{B}(\mathfrak{h}_0 \otimes \mathfrak{h}^{\otimes n})$, $n = 0, 1, 2, \cdots$. Suppose there exist density operators $\rho_n$ in $\mathfrak{h}_0 \otimes \mathfrak{h}^{\otimes n}$, $n = 1, 2, \cdots$ satisfying the following properties:*

*(1) $\rho_1 = \rho$ and*

$$\operatorname{Tr} \rho_n (X \otimes I) = \operatorname{Tr} \rho_{n-1} X, \quad X \in \mathcal{B}_{n]},$$

*$I$ being the identity in $\mathfrak{h}$, $n = 1, 2, \cdots$.*

*(2) For any $X_0 \in \mathcal{B}(\mathfrak{h}_0)$, $Y_j \in \mathcal{B}(\mathfrak{h})$, $j = 1, 2, \cdots, n$ and any permutation $\pi$ of $\{1, 2, \cdots, n\}$*

$$\operatorname{Tr} \rho_n\, X_0 \otimes Y_1 \otimes \cdots \otimes Y_n = \operatorname{Tr} \rho_n\, X_0 \otimes Y_{\pi(1)} \otimes \cdots \otimes Y_{\pi(n)}.$$

*Then $\rho$ is separable in $\mathfrak{h}_0 \otimes \mathfrak{h}$. Furthermore $\rho_n$ is separable in $\mathfrak{h}_0 \otimes \mathfrak{h}^{\otimes n}$, $n = 1, 2, \cdots$.*

*Proof.* We adopt the convention that for any density operator $\rho$ and any operator $X$ in a Hilbert space $\rho(X) = \operatorname{Tr} \rho X$. Let $\mathcal{B}_n$, $i_n$, $\mathcal{B}^\infty$ be as defined at the beginning of this section. Let $\rho_0$ be the relative trace of $\rho$ over $\mathfrak{h}$ in $\mathfrak{h}_0 \otimes \mathfrak{h}$. Choose and fix an operator $0 \leq A \leq I$ in $\mathfrak{h}_0$ such that $\rho_0(A) = \operatorname{Tr} \rho_0 A > 0$. Then there exists a well-defined state $\omega_A$ in the C* algebra $\mathcal{B}^\infty$ satisfying

$$\omega_A(i_n(Y)) = \frac{\rho_n(A \otimes Y)}{\rho_0(A)}, \quad Y \in \mathcal{B}_n,\ n = 1, 2, \cdots . \tag{3.1}$$

Indeed, this follows from property (1) of $\{\rho_n\}$. Now property (2) of $\rho_n$ implies that $\omega_A$ is exchangeable and locally normal. Thus by Theorem 3.1 there exists a unique probability measure $\mu_A$ on the Borel $\sigma$-algebra of $\mathcal{R}_\mathfrak{h}$ such that

$$\omega_A = \int_{\mathcal{R}_\mathfrak{h}} \sigma^\infty \mu_A(\mathrm{d}\sigma) \tag{3.2}$$

where $\sigma^\infty$ denotes the unique state in $\mathcal{B}^\infty$ satisfying

$$\sigma^\infty(i_n(Y_1 \otimes \cdots \otimes Y_n)) = \sigma(Y_1)\sigma(Y_2)\cdots\sigma(Y_n)$$

for all $Y_j \in \mathcal{B}(\mathfrak{h})$, $n = 1, 2, \cdots$.

In particular, the probability measure $\mu_I$ is well-defined. When $\rho_0(A) = 0$ we define $\mu_A$ to be $\mu_I$.

Equations (3.1) and (3.2) imply

$$\rho_n(A \otimes Y) = \rho_0(A) \int_{\mathcal{R}_\mathfrak{h}} \sigma^{\otimes n}(Y)\mu_A(\mathrm{d}\sigma), \quad Y \in \mathcal{B}_n,\ n = 1, 2, \cdots \tag{3.3}$$

whenever $\rho_0(A) > 0$. If $\rho_0(A) = 0$ it follows from the inequality $-\|Y\|A \otimes I \leq A \otimes Y \leq \|Y\|A \otimes I$ that $\rho_n(A \otimes Y) = 0$ whenever $Y$ is self adjoint. Thus $\rho_n(A \otimes Y) = 0$ for any $Y \in \mathcal{B}_n$ whenever $\rho_0(A) = 0$. Hence (3.3) holds for all $0 \leq A \leq I$.

Let $A \geq 0$, $B \geq 0$, $A + B \leq I$ in $\mathcal{B}(\mathfrak{h}_0)$. Writing down (3.3) for $A$, $B$ and $A + B$, adding the first two and comparing with the third we get the relation

$$\int \sigma^{\otimes n}(Y) \left(\rho_0(A)\mu_A + \rho_0(B)\mu_B\right)(\mathrm{d}\sigma) = \int \sigma^{\otimes n}(Y)\rho_0(A+B)\mu_{A+B}(\mathrm{d}\sigma), \quad Y \in \mathcal{B}_n,\ n = 1, 2, \cdots.$$

If $\rho_0(A+B) > 0$, dividing both sides by $\rho_0(A+B)$ and using the uniqueness of the probability measure in Theorem 3.1 we get

$$\rho_0(A + B)\mu_{A+B} = \rho_0(A)\mu_A + \rho_0(B)\mu_B. \tag{3.4}$$

If $\rho_0(A+B) = 0$ then $\rho_0(A) = \rho_0(B) = 0$ and hence the same relation holds trivially. Choosing $0 \leq A \leq I$, $B = I - A$ we have

$$\mu_I = \rho_0(A)\mu_A + \rho_0(I - A)\mu_{I-A}.$$

This shows that as $A$ increases to $I$, the fact that $\rho_0(I - A) \to 0$ implies that $\rho_0(A)\mu_A(F)$ increases to $\rho_0(I)\mu_I(F)$. Let $\{\boldsymbol{u}_j\}$ be an orthonormal basis for $\mathfrak{h}_0$. For any unit vector $\boldsymbol{u} \in \mathfrak{h}_0$ and any Borel set $F \subset \mathcal{R}_\mathfrak{h}$ define

$$f(\boldsymbol{u}, F) = \rho_0(|\boldsymbol{u}\rangle\langle\boldsymbol{u}|)\mu_{|\boldsymbol{u}\rangle\langle\boldsymbol{u}|}(F). \tag{3.5}$$

Since $\sum_{j=1}^n |\boldsymbol{u}_j\rangle\langle\boldsymbol{u}_j|$ increases to the identity operator as $n \to \infty$ it now follows from (3.4) and the remark above

$$\sum_{j=1}^\infty f(\boldsymbol{u}_j, F) = \mu_I(F)$$

for any Borel set $F \subset \mathcal{R}(\mathfrak{h})$ and any orthonormal basis. In other words, for each fixed $F$, the map $\boldsymbol{u} \mapsto f(\boldsymbol{u}, F)$ is a frame function in the sense of Gleason [?] on the unit sphere of $\mathfrak{h}_0$. Hence by Gleason's theorem [?, ?] there exists a positive trace class operator $T(F)$ such that

$$f(\boldsymbol{u}, F) = \langle\boldsymbol{u}|T(F)|\boldsymbol{u}\rangle\,, \quad \forall\boldsymbol{u} \text{ with } \|\boldsymbol{u}\| = 1$$

and any Borel set $F$ in $\mathcal{R}(\mathfrak{h})$. Thus

$$\langle\boldsymbol{u}|T(F)|\boldsymbol{u}\rangle = \rho_0(|\boldsymbol{u}\rangle\langle\boldsymbol{u}|)\mu_{|\boldsymbol{u}\rangle\langle\boldsymbol{u}|}(F). \tag{3.6}$$

This together with (3.3) implies that $T(\cdot)$ is a positive operator-valued measure satisfying

$$\begin{aligned} T(\mathcal{R}(\mathfrak{h})) &= \rho_0, \\ \operatorname{Tr} T(F)A &= \rho_0(A)\mu_A(F) \leq \mu_I(F) \end{aligned} \tag{3.7}$$

for any $0 \leq A \leq I$ in $\mathfrak{h}_0$ and any Borel set $F$ in $\mathcal{R}(\mathfrak{h})$. Now choose and fix an orthonormal basis $\{\boldsymbol{e}_j\}$ in $\mathfrak{h}_0$. Then complex-valued measures $\langle \boldsymbol{e}_i | T(\cdot) | \boldsymbol{e}_j \rangle$ , $i, j = 1, 2, \cdots$ are all absolutely continuous with respect to the measure $\mu_I$ and hence there exist Radon-Nykodym derivatives $f_{ij}$ in $\mathcal{R}(\mathfrak{h})$ satisfying the relations

$$\langle \boldsymbol{e}_i | T(F) | \boldsymbol{e}_j \rangle = \int_F f_{ij}(\sigma)\mu_I(\mathrm{d}\sigma).$$

The positivity of $T(F)$ for all Borel sets $F$ in $\mathcal{R}(\mathfrak{h})$ implies that for any finite $n$ the matrix $((f_{ij}(\sigma)))$, $i, j \in \{1, 2, \cdots, n\}$ is positive semidefinite a.e. $(\mu_I)$ for every $n$. Furthermore,

$$\begin{aligned} \mu_I(F) &= \operatorname{Tr} T(F) \\ &= \sum_{i=1}^{\infty} \langle \boldsymbol{e}_i | T(F) | \boldsymbol{e}_i \rangle \\ &= \int_F \sum_i f_{ii}(\sigma)\mu_I(\mathrm{d}\sigma) \end{aligned}$$

for all $F$. Thus

$$\sum_i f_{ii}(\sigma) = 1, \qquad \text{a.e. } \mu_I.$$

Thus there exist density operators $\tau(\sigma)$, $\sigma \in \mathcal{R}_(\mathfrak{h})$ in $\mathfrak{h}_0$ such that

$$\langle \boldsymbol{e}_i | \tau(\sigma) | \boldsymbol{e}_j \rangle = f_{ij}(\sigma)$$

a.e $\sigma(\mu_I)$. Then

$$T(F) = \int_F \tau(\sigma)\mu_I(\mathrm{d}\sigma)$$

for every Borel set $F$ in $\mathcal{R}_{\mathfrak{h}}$. Now (3.3) and (3.7) imply that for any $|\boldsymbol{u}\rangle\langle\boldsymbol{u}|$ , $\boldsymbol{u}$ a unit vector in $\mathfrak{h}_0$

$$\rho_n\left(|\boldsymbol{u}\rangle\langle\boldsymbol{u}| \otimes Y\right) = \int_{\mathcal{R}_{\mathfrak{h}}} \sigma^{\otimes n}(Y)\tau(\sigma)(|\boldsymbol{u}\rangle\langle\boldsymbol{u}|)\mu_I(\mathrm{d}\sigma), \quad Y \in \mathcal{B}_n,\ n = 1, 2, \cdots.$$

Thus

$$\rho_n(A \otimes Y) = \int_{\mathcal{R}_{\mathfrak{h}}} \tau(\sigma)(A)\sigma^{\otimes n}(Y)\mu_I(\mathrm{d}\sigma)$$

for all $Y \in \mathcal{B}_n$, $A \in \mathcal{B}_0$, $n = 1, 2, \cdots$. In other words each $\rho_n$ is separable in the bipartite product $\mathfrak{h}_0 \otimes [\mathfrak{h}]^{\otimes n}$. This completes the proof. ☐

## 4. Conclusion

Motivated by the notions of extendability and complete extendability of finite level states as described by Doherty et al [?] we introduce similar definitions for Gaussian states. A necessary and sufficient condition is obtained for the complete extendability of a bipartite Gaussian state in terms of its covariance matrix.Using only the properties of covariance matrices we show the separability of any finite mode bipartite Gaussian state is equivalent to its complete extendability. The question of finding a necessary and sufficient condition for $k$-extendability remains open. By exploiting a version of the quantum de Finetti theorem as in Hudson and Moody [?], and Gleason's theorem [?], we prove the equivalence of separability and complete extendibility of a bipartite state whenever one of the Hilbert spaces is of dimension greater than 2. Since the C* algebra of all bounded operators of an infinite dimensional separable Hilbert space is not separable, our result is also an extension of Fannes, Lewis, and Verbeure [?].

Theoretical Statistics and Mathematics Unit, Indian Statistical Institute, Bengalore Centre, 8th Mile, Mysore Road RVCE Post, Bangalore 560 059
*E-mail address*, B. V. Rajarama Bhat: bhat@isibang.ac.in

Theoretical Statistics and Mathematics Unit, Indian Statistical Institute, Delhi Centre, 7 S J S Sansanwal Marg, New Delhi 110 016, India
*E-mail address*, K R Parthasarathy: krp@isid.ac.in

Theoretical Statistics and Mathematics Unit, Indian Statistical Institute, Delhi Centre, 7 S J S Sansanwal Marg, New Delhi 110 016, India
*E-mail address*, Ritabrata Sengupta: rb@isid.ac.in