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On the reliability of a two component system

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Abstract

In this paper, we propose a non-parametric estimator of the reliability of a system with two independent components. The estimator is motivated from an approximation of the survival function. We derive the exact distribution of the proposed estimator. The new estimator is shown to be consistent and has asymptotically standard normal distribution. Simulation studies are also carried out to assess the performance of the proposed estimator.

Keywords: Distribution function, reliability function, Nnonparametric estimator, Independent random variables

1 Introduction

The sum of random variables plays a vital role in a wide range of areas such as wireless communication and insurance etc. The sum of random variables arise naturally in reliability theory as the lifetime of a system with several subsystems. In many modern systems, in aerospace, electronic and other industries, consist of several different subsystems. For example, a modern aircraft flight typically involves subsystems like automated take-off, ascent, level flight, altered flight due to interferences, descent and landing etc. Each subsystem works independently and has different configurations. The total operational lifetime of the system is the sum of the life times of the subsystems. For various application of sum of random variables in reliability one could refer to Trivedi (2008), Kordecki (1997), Bolch et al. (2006) and Zhang (2005). In insurance, assume that insurer has available various type of insurance claims from a particular line of business. From the standpoint of the insurer, the distribution of sum of claims is of interest. A detailed application of sum of random variables, in the field of health insurance can be found in Panjer and Willmot (1992) and Willmot and Woo (2007).

A detailed review of some known results on the sum of identical random variables can be seen in Nadarajah (2008).

In general, exact distributions of sum random variables, is not available in the explicit form and it involves complex computations. For example, distribution function of sum of Weibull random variables has no closed form. Hence, in order to estimate distribution function or reliability function, of sums of random variables, non-parametric methods are often used (Frees (1994) and Saavedra and Cao (2000)).

In this paper we first propose a new approximation for the distribution function of sum of two independent random variables. This approximation is general in the sense that where the component random variables need not be identical. Using this approximation we suggest a non-parametric estimator for the distribution function of sum of two independent random variables. We derive the exact and the asymptotic distribution of the proposed estimator.

2 Sum of two independent random variables

Suppose that X and Y are two independent, continuous non-negative random variables with quantile functions $Q_X(.)$ and $Q_Y(.)$ respectively. Suppose $F_X(.)$ and $F_Y(.)$ be the distribution functions and $f_X(.)$ and $f_Y(.)$ be the density functions of X and Y respectively. Our aim is to find the distribution function of $Z = X + Y$. Note that $Q_X(u) + Q_Y(u),\ 0 \le u \le 1$, is a quantile function of the random variable $X + Q_Y(F_X(X))$ or $Y + Q_X(F_Y(Y))$ (Sankaran et al. (2014)).

But we know that if $0 \le u_1, u_2 \le 1$ are two independent uniform random numbers then $Q_X(u_1) + Q_Y(u_2)$ gives the random number from the distribution of $X + Y$. Consider the contour of $Q_X(u_1) + Q_Y(u_2)$ at level z, in $0 \le u_1, u_2 \le 1$ plane which is a unit square. For specific value of z the equation corresponding to contour line is

$$Q_X(u_1) + Q_Y(u_2) = z(u_1, u_2). \tag{1}$$

In Figure 1, contour plot of sum of two quantile functions, corresponding to exponential distributions, with parameters values 2 and 3 is given. In the next theorem we prove that the area under the contour line in the unit square plane $0 \le u_1, u_2 \le 1$ gives the distribution function of Z.

Theorem 1. *Area under the curve* $Q_X(u_1)+Q_Y(u_2) = z$ *in the unit square plane* $0 \le u_1, u_2 \le$

1, *gives the distribution function of* Z*, denoted by* $F_Z(z) = P(Z \leq z)$*.*

Proof. Solving (1) for u_2 gives

$$u_2 = F_Y(z - Q_X(u_1)). \tag{2}$$

Now area under the curve (2) can be divided into four cases depending upon the value of z.

1. When $z \leq \min(Q_X(1), Q_Y(1))$, the area is

$$A = \int_0^{F_X(z-Q_Y(0))} F_Y(z - Q_X(u_1))du_1 \tag{3}$$

2. When $z > Q_X(1)$ and $z \leq Q_Y(1)$, the area is

$$A = \int_0^1 F_Y(z - Q_X(u_1))du_1 \tag{4}$$

3. When $z \leq Q_X(1)$ and $z > Q_Y(1)$, the area is

$$A = F_X(z - Q_Y(1)) + \int_{F_X(z-Q_Y(1))}^{F_X(z-Q_Y(0))} F_Y(z - Q_X(u_1))du_1 \tag{5}$$

4. When $z > Q_X(1)$ and $z > Q_Y(1)$, the area is

$$A = F_X(z - Q_Y(1)) + \int_{F_X(z-Q_Y(1))}^{1} F_Y(z - Q_X(u_1))du_1 \tag{6}$$

Now put $u_1 = F_X(v)$ in (3),(4),(5) and (6). We get

$$A = \begin{cases} 0 & z \leq Q_X(0) + Q_Y(0) \\ \int_{Q_X(0)}^{z-Q_Y(0)} F_Y(z-v)f_X(v)dv & z \leq Q_X(1), z \leq Q_Y(1) \\ \int_{Q_X(0)}^{Q_X(1)} F_Y(z-v)f_X(v)dv & z > Q_X(1), z \leq Q_Y(1) \\ F_X(z-Q_Y(1)) + \int_{z-Q_Y(1)}^{z-Q_Y(0)} F_Y(z-v)f_X(v)dv & z \leq Q_X(1), z > Q_Y(1) \\ F_X(z-Q_Y(1)) + \int_{z-Q_Y(1)}^{Q_X(1)} F_Y(z-v)f_X(v)dv & z > Q_X(1), z > Q_Y(1) \\ 1 & z \geq Q_X(1) + Q_Y(1) \end{cases} \tag{7}$$

Here A gives the $E(F_Y(z-X))$ which is the distribution function of Z and hence the theorem is proved. □

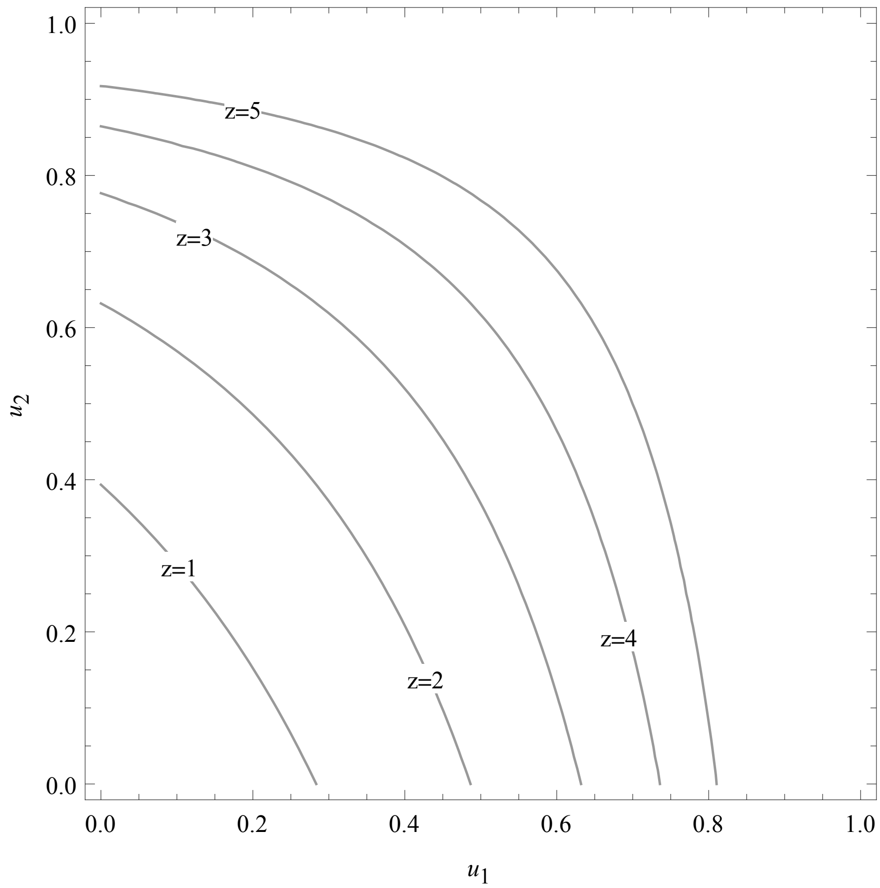


Figure 1: Contour plot

As mentioned in Section 1, there is no explicit form available for the distribution function of sum of two independent random variables for most families of the distributions. Thus, in the next section we derive a simple approximation to the distribution functions of these sums of random variables. Since the reliability function is $1 - F(z)$, the above result will give us an approximation for the reliability function.

2.1 Approximation

We use the trapezoidal rule to approximate the integration part in (3)-(6). Suppose $[F_X(z_0), F_X(z_1)]$ is the domain of integration. We partition the interval $[F_X(z_0), F_X(z_1)]$ to m subintervals by

$$\{F_X(z_0), F_X\left(z_0 + \frac{(z_1 - z_0)}{m}\right), F_X\left(z_0 + \frac{2(z_1 - z_0)}{m}\right) \ldots, F_X\left(z_1\right)\}.$$

Corresponding u_2 values are

$$\{F_Y(z_1 - z_0), F_Y\left(\frac{(m-1)(z_1 - z_0)}{m}\right), F_Y\left(\frac{(m-2)(z_1 - z_0)}{m}\right) \ldots, F_Y\left(0\right)\}.$$

Then by trapezoidal rule the area A can be approximated as

$$\begin{aligned} A \approxeq F_X(z_0) + \frac{1}{2}\sum_{i\,=0}^{m-1}\Bigg(F_X\left(\frac{(i+1)\,(z_1-z_0)}{m}+z_0\right) \\ - F_X\left(\frac{i\,(z_1-z_0)}{m}+z_0\right)\Bigg)\left(F_Y\left(\frac{(z_1-z_0)\,(m-i-1)}{m}\right)+F_Y\left(\frac{(z_1-z_0)\,(m-i)}{m}\right)\right) \end{aligned} \tag{8}$$

When X and Y is in the range $[0,\infty)$ we can write the approximation for $F_Z(z)$ as

$$\begin{aligned} F_Z(z) \approxeq F_{Z_m}(z) = \frac{1}{2}\sum_{i=0}^{m-1}\left(F_X\left(\frac{(i+1)z}{m}\right)-F_X\left(\frac{iz}{m}\right)\right)\Bigg(F_Y\left(\frac{z(m-i-1)}{m}\right) \\ + F_Y\left(\frac{z(m-i)}{m}\right)\Bigg). \end{aligned} \tag{9}$$

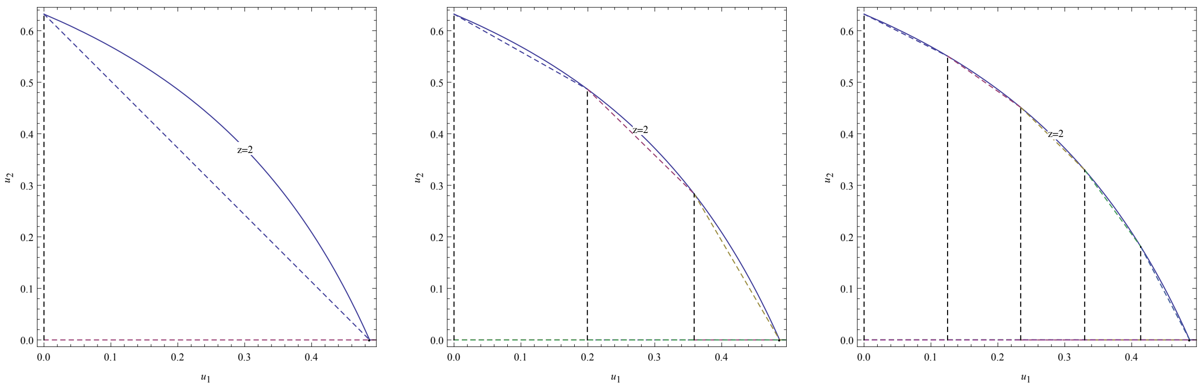
Figure 2 represents approximation to $F_Z(z)$ for various values of m, when X is distributed as $\exp(2)$ and Y is distributed as $\exp(3)$.

Assume that $F_Y(.)$ is twice continuously differentiable. Then, the error of approximation can be obtained as

$$\begin{aligned} E_m &= F_Z(z) - F_{Z_m}(z) \\ &= \int_0^{F_X(z)} F_Y(z-Q_X(u_1))du_1 \\ &\quad - \frac{1}{2}\sum_{i=0}^{n-1}\left(F_X\left(\frac{(i+1)z}{m}\right)-F_X\left(\frac{iz}{m}\right)\right)\left(F_Y\left(\frac{z(n-i-1)}{m}\right)+F_Y\left(\frac{z(m-i)}{m}\right)\right) \\ &= \sum_{i=0}^{m-1}\int_{F_X\left(\frac{iz}{m}\right)}^{F_X\left(\frac{(i+1)z}{m}\right)} F_Y(z-Q_X(u_1))du_1 \\ &\quad - \frac{1}{2}\sum_{i=0}^{m-1}\left(F_X\left(\frac{(i+1)z}{m}\right)-F_X\left(\frac{iz}{m}\right)\right)\left(F_Y\left(\frac{z(m-i-1)}{m}\right)+F_Y\left(\frac{z(m-i)}{m}\right)\right) \\ &= \sum_{i=0}^{m-1}\Bigg[\int_{F_X\left(\frac{iz}{m}\right)}^{F_X\left(\frac{(i+1)z}{m}\right)} F_Y(z-Q_X(u_1))du_1 \\ &\quad - \frac{1}{2}\left(F_X\left(\frac{(i+1)z}{m}\right)-F_X\left(\frac{iz}{m}\right)\right)\left(F_Y\left(\frac{z(m-i-1)}{m}\right)+F_Y\left(\frac{z(m-i)}{m}\right)\right)\Bigg]. \end{aligned}$$

Using (Atkinson, 2008, Equation 5.1.4), there exist a number ξ_i between $\frac{z(m-i-1)}{m}$ and $\frac{z(m-i)}{m}$ such that

$$E_m = \sum_{i=0}^{m-1} -\frac{\left(F_X\left(\frac{(i+1)z}{m}\right)-F_X\left(\frac{iz}{m}\right)\right)^3}{12}F''_Y(\xi_i) \tag{10}$$



$m = 1$ $\qquad$ $m = 3$ $\qquad$ $m = 5$

Figure 2: Approximation

where $\xi_i \in \left[\frac{z(m-i-1)}{m}, \frac{z(m-i)}{m}\right]$. Using mean value theorem we can write (10) as

$$E_m = \sum_{i=0}^{m-1} -\frac{\left(F_X'\left(\tau_i\right) z\right)^3}{12m^3} F_Y''(\xi_i)$$

where $\tau_i \in \left[\frac{z(i)}{m}, \frac{z(i+1)}{m}\right]$. Since $F_Y''(\xi_i)$ and $F_X'\left(\tau_i\right)$ are continous in $[0, z]$, we can write

$$|F_Y''(\xi_i)| \leq \max_{\frac{z(m-i-1)}{m} \leq \xi_i \leq \frac{z(m-i)}{m}} F_Y''(\xi_i) \leq M_1 = \max_{0 \leq \xi \leq z} F_Y''(\xi)$$

and

$$|F_X'(\tau_i)| \leq \max_{\frac{z(i)}{m} \leq \tau_i \leq \frac{z(i+1)}{m}} F_X'(\tau_i) \leq M_2 = \max_{0 \leq \tau \leq z} F_X'(\tau).$$

Thus from (2.1)

$$E_m \leq \frac{-M_1 M_2^3 z^3}{12m^2} \tag{11}$$

which is converging to zero as $m \to \infty$.

3 Non-parametric estimator of distribution function of Z

Suppose X and Y are independent random variables. Let $\{X_1, X_2, \ldots X_{n_1}\}$ and $\{Y_1, Y_2, \ldots Y_{n_2}\}$ be independent copies of X and Y. Let $\hat{F}_X(x)$ and $\hat{F}_Y(y)$ be the empirical estimators of $F_X(x)$ and $F_Y(y)$ respectively. Now using (9), we suggest a non-parametric estimator for $F_Z(z)$ given by

$$\hat{F}_Z(z) = \frac{1}{2} \sum_{i=0}^{m-1} \left(\hat{F}_X\left(\frac{(i+1)z}{m}\right) - \hat{F}_X\left(\frac{iz}{m}\right) \right) \left(\hat{F}_Y\left(\frac{z(m-i-1)}{m}\right) + \hat{F}_Y\left(\frac{z(m-i)}{m}\right) \right) \tag{12}$$

3.1 Asymptotic Properties

In the next theorem we prove that $\hat{F}_Z(z)$ is consistent estimator of $F_Z(z)$.

Theorem 2. *As $n_1, n_2, m \to \infty$*

$$\sup_z |\hat{F}_Z(z) - F_Z(z)| \longrightarrow 0 \ \ a.s.$$

Proof. We have

$$\begin{aligned}
\hat{F}_Z(z) - F_{Z_m}(z) = &\left\{ \frac{1}{2} \sum_{i=0}^{m-1} \left(\hat{F}_X\left(\frac{(i+1)z}{m}\right) - \hat{F}_X\left(\frac{iz}{m}\right)\right) \left(\hat{F}_Y\left(\frac{z(m-i-1)}{m}\right)\right.\right. \\
&\left.\left. + \hat{F}_Y\left(\frac{z(m-i)}{m}\right)\right)\right\} - \left\{ \frac{1}{2} \sum_{i=0}^{m-1} \left(F_X\left(\frac{(i+1)z}{m}\right)\right.\right. \\
&\left.\left. - F_X\left(\frac{iz}{m}\right)\right) \left(F_Y\left(\frac{z(m-i-1)}{m}\right) + F_Y\left(\frac{z(m-i)}{m}\right)\right)\right\}
\end{aligned}$$

$$\begin{aligned}
= \frac{1}{2} \sum_{i=0}^{m-1} &\left\{ \left(\hat{F}_X\left(\frac{(i+1)z}{m}\right) - \hat{F}_X\left(\frac{iz}{m}\right)\right) \left(\hat{F}_Y\left(\frac{z(m-i-1)}{m}\right) + \hat{F}_Y\left(\frac{z(m-i)}{m}\right)\right)\right. \\
&\left. - \left(F_X\left(\frac{(i+1)z}{m}\right) - F_X\left(\frac{iz}{m}\right)\right) \left(F_Y\left(\frac{z(m-i-1)}{m}\right) + F_Y\left(\frac{z(m-i)}{m}\right)\right)\right\}
\end{aligned}$$

$$\begin{aligned}
= \frac{1}{2} \sum_{i=0}^{m-1} &\left\{ \hat{F}_X\left(\frac{(i+1)z}{m}\right) \hat{F}_Y\left(\frac{z(m-i-1)}{m}\right) - \hat{F}_X\left(\frac{iz}{m}\right) \hat{F}_Y\left(\frac{z(m-i-1)}{m}\right)\right. \\
&+ \hat{F}_X\left(\frac{(i+1)z}{m}\right) \hat{F}_Y\left(\frac{z(m-i)}{m}\right) - \hat{F}_X\left(\frac{iz}{m}\right) \hat{F}_Y\left(\frac{z(m-i)}{m}\right) \\
&- F_X\left(\frac{(i+1)z}{m}\right) F_Y\left(\frac{z(m-i-1)}{m}\right) + F_X\left(\frac{iz}{m}\right) F_Y\left(\frac{z(m-i-1)}{m}\right) \\
&\left. - F_X\left(\frac{(i+1)z}{m}\right) F_Y\left(\frac{z(m-i)}{m}\right) + F_X\left(\frac{iz}{m}\right) F_Y\left(\frac{z(m-i)}{m}\right)\right\}
\end{aligned}$$

$$\begin{aligned}
= \frac{1}{2} \sum_{i=0}^{m-1} &\left\{ \left[\hat{F}_X\left(\frac{(i+1)z}{m}\right) \hat{F}_Y\left(\frac{z(m-i-1)}{m}\right) - F_X\left(\frac{(i+1)z}{m}\right) F_Y\left(\frac{z(m-i-1)}{m}\right)\right]\right. \\
&+ \left[F_X\left(\frac{iz}{m}\right) F_Y\left(\frac{z(m-i-1)}{m}\right) - \hat{F}_X\left(\frac{iz}{m}\right) \hat{F}_Y\left(\frac{z(m-i-1)}{m}\right)\right] \\
&+ \left[\hat{F}_X\left(\frac{(i+1)z}{m}\right) \hat{F}_Y\left(\frac{z(m-i)}{m}\right) - F_X\left(\frac{(i+1)z}{m}\right) F_Y\left(\frac{z(m-i)}{m}\right)\right] \\
&\left. + \left[F_X\left(\frac{iz}{m}\right) F_Y\left(\frac{z(m-i)}{m}\right) - \hat{F}_X\left(\frac{iz}{m}\right) \hat{F}_Y\left(\frac{z(m-i)}{m}\right)\right]\right\}
\end{aligned}$$

Therefore

$$\hat{F}_Z(z) - F_{Z_m}(z) = \frac{1}{2} \sum_{i=0}^{m-1} \{A_i(z) + B_i(z) + C_i(z) + D_i(z)\} \tag{13}$$

where

$$A_i(z) = \hat{F}_X\left(\frac{(i+1)z}{m}\right)\hat{F}_Y\left(\frac{z(m-i-1)}{m}\right) - F_X\left(\frac{(i+1)z}{m}\right)F_Y\left(\frac{z(m-i-1)}{m}\right),$$

$$B_i(z) = F_X\left(\frac{iz}{m}\right)F_Y\left(\frac{z(m-i-1)}{m}\right) - \hat{F}_X\left(\frac{iz}{m}\right)\hat{F}_Y\left(\frac{z(m-i-1)}{m}\right),$$

$$C_i(z) = \hat{F}_X\left(\frac{(i+1)z}{m}\right)\hat{F}_Y\left(\frac{z(m-i)}{m}\right) - F_X\left(\frac{(i+1)z}{m}\right)F_Y\left(\frac{z(m-i)}{m}\right)$$

and

$$D_i(z) = F_X\left(\frac{iz}{m}\right)F_Y\left(\frac{z(m-i)}{m}\right) - \hat{F}_X\left(\frac{iz}{m}\right)\hat{F}_Y\left(\frac{z(m-i)}{m}\right)$$

Now

$$\sup_z|A_i(z)| = \sup_z\left|\hat{F}_X\left(\frac{(i+1)z}{m}\right)\hat{F}_Y\left(\frac{z(m-i-1)}{m}\right) - F_X\left(\frac{(i+1)z}{m}\right)F_Y\left(\frac{z(m-i-1)}{m}\right)\right|$$

$$= \sup_z\left|\hat{F}_X\left(\frac{(i+1)z}{m}\right)\hat{F}_Y\left(\frac{z(m-i-1)}{m}\right) - F_X\left(\frac{(i+1)z}{m}\right)\hat{F}_Y\left(\frac{z(m-i-1)}{m}\right)\right.$$
$$\left.+ F_X\left(\frac{(i+1)z}{m}\right)\hat{F}_Y\left(\frac{z(m-i-1)}{m}\right) - F_X\left(\frac{(i+1)z}{m}\right)F_Y\left(\frac{z(m-i-1)}{m}\right)\right|$$

$$= \sup_z\left|\hat{F}_Y\left(\frac{z(m-i-1)}{m}\right)\left(\hat{F}_X\left(\frac{(i+1)z}{m}\right) - F_X\left(\frac{(i+1)z}{m}\right)\right)\right.$$
$$\left.+ F_X\left(\frac{(i+1)z}{m}\right)\left(\hat{F}_Y\left(\frac{z(m-i-1)}{m}\right) - F_Y\left(\frac{z(m-i-1)}{m}\right)\right)\right|$$

$$\leq \sup_z\left|\hat{F}_Y\left(\frac{z(m-i-1)}{m}\right)\left(\hat{F}_X\left(\frac{(i+1)z}{m}\right) - F_X\left(\frac{(i+1)z}{m}\right)\right)\right|$$
$$+ \sup_z\left|F_X\left(\frac{(i+1)z}{m}\right)\left(\hat{F}_Y\left(\frac{z(m-i-1)}{m}\right) - F_Y\left(\frac{z(m-i-1)}{m}\right)\right)\right|$$

Since as $n_1, n_2 \to \infty$, $\sup_z|\hat{F}_X(z) - F_X(z)| \to 0$ and $\sup_z|\hat{F}_Y(z) - F_Y(z)| \to 0$, we have $\sup_z|A_i(z)| \to 0 \;\; a.s.$

On similar lines we can prove that $\sup_z|B_i(z)| \to 0$, $\sup_z|C_i(z)| \to 0$ and $\sup_z|D_i(z)| \to 0$. Now using (13), we can write

$$\sup_z|\hat{F}_Z(z) - F_{Z_m}(z)| = \sup_z\left|\frac{1}{2}\sum_{i=0}^{m-1}\{A_i(z) + B_i(z) + C_i(z) + D_i(z)\}\right|$$

$$\leq \frac{1}{2}\sum_{i=0}^{m-1}\sup_z|A_i(z)| + \frac{1}{2}\sum_{i=0}^{m-1}\sup_z|B_i(z)| + \frac{1}{2}\sum_{i=0}^{m-1}\sup_z|C_i(z)| + \frac{1}{2}\sum_{i=0}^{m-1}\sup_z|D_i(z)| \to 0 \;\; a.s.$$

We can write

$$\sup_z |\hat{F}_Z(z) - F_Z(z)| \leq \sup_z |\hat{F}_Z(z) - F_{Z_m}(z)| + \sup_z |F_{Z_m}(z) - F_Z(z)|$$

As $n_1, n_2, m \to \infty$ the righ hand side of the above expression converges to zero a.s. $\square$

4 Exact distribution of $\hat{F}_Z(z)$

In this section we find the exact distribution of $\hat{F}_Z(z)$, the non-parametric estimator of the unknown distribution function $F_Z(z)$.

Theorem 3. *$2n_1n_2\hat{F}_Z(z)$ is distributed with the probability mass function(p.m.f.) given by*

$$P\left(2n_1n_2\hat{F}_Z(z) = k\right) = \begin{cases} \sum_{j=0}^{\frac{1}{4}\left(2k+(-1)^k-1\right)} \frac{(n_1n_2)!}{j!(k-2j)!(n_1n_2-k+j)!} q_1^j q_2^{k-2j} q_3^{n_1n_2-k+j} & \text{if } k \leq n_1n_2 \\ \sum_{j=k-n_1n_2}^{\frac{1}{4}\left(2k+(-1)^k-1\right)} \frac{(n_1n_2)!}{j!(k-2j)!(n_1n_2-k+j)!} q_1^j q_2^{k-2j} q_3^{n_1n_2-k+j} & \text{if } k > n_1n_2 \end{cases} \tag{14}$$

where $k = 0, 1 \ldots 2n_1n_2$,

$$q_1 = \sum_{i=0}^{m-1} \left(F_X\left(\frac{(i+1)z}{m}\right) - F_X\left(\frac{iz}{m}\right)\right) F_Y\left(\frac{z(m-i-1)}{m}\right),$$

$$q_2 = \sum_{i=0}^{m-1} \left(F_X\left(\frac{(i+1)z}{m}\right) - F_X\left(\frac{iz}{m}\right)\right)\left(F_Y\left(\frac{z(m-i)}{m}\right) - F_Y\left(\frac{z(m-i-1)}{m}\right)\right)$$

and

$$q_3 = 1 - q_1 - q_2.$$

Before proceeding to the proof of the main theorem, we prove the following lemma.

Lemma 1. *If the random vector $\{X_1, X_2, X_3\}$ is distributed as multinomial distribution with parameters n and $\{q_1, q_2, q_3\}$, where $q_1 + q_2 + q_3 = 1$, then $2X_1 + X_2$ is distributed with the p.m.f given by*

$$P(2X_1 + X_2 = k) = \begin{cases} \sum_{j=0}^{\frac{1}{4}\left(2k+(-1)^k-1\right)} \frac{n!}{j!(k-2j)!(n-k+j)!} q_1^j q_2^{k-2j} q_3^{n-k+j} & \text{if } k \leq n \\ \sum_{j=k-n}^{\frac{1}{4}\left(2k+(-1)^k-1\right)} \frac{n!}{j!(k-2j)!(n-k+j)!} q_1^j q_2^{k-2j} q_3^{n-k+j} & \text{if } k > n \end{cases} \tag{15}$$

where $k = 0, 1 \ldots, 2n$.

Proof. The p.m.f. of $\{X_1, X_2, X_3\}$ is given by

$$P(X_1 = x_1, X_2 = x_2, X_3 = n-x_1-x_2) = \frac{n!}{x_1!\, x_2!\, (n-x_1-x_2)!} q_1^{x_1} q_2^{x_2} q_3^{n-x_1-x_2},\ \ x_1, x_2 \geq 0, x_1+x_2 \leq n.$$

To find the $P(2X_1 + X_2 = k)$, we consider four cases

1. If $k \leq n$ and k is even,

$$\begin{aligned}
&P(2X_1 + X_2 = k)\\
&= P(X_1 = 0, X_2 = k, X_3 = n-k)\\
&\quad + P(X_1 = 1, X_2 = k-2, X_3 = n-k+1) + \cdots + P(X_1 = \frac{k}{2}, X_2 = 0, X_3 = n - \frac{k}{2})\\
&= \sum_{j=0}^{\frac{k}{2}} P(X_1 = j, X_2 = k-2j, X_3 = n-k+j)\\
&= \sum_{j=0}^{\frac{k}{2}} \frac{n!}{j!\,(k-2j)!\,(n-k+j)!} q_1^j q_2^{k-2j} q_3^{n-k+j}.
\end{aligned}$$

2. If $k \leq n$ and k is odd,

$$\begin{aligned}
&P(2X_1 + X_2 = k)\\
&= P(X_1 = 0, X_2 = k, X_3 = n-k)\\
&\quad + P(X_1 = 1, X_2 = k-2, X_3 = n-k+1) + \cdots + P(X_1 = \frac{k-1}{2}, X_2 = 1, X_3 = n - \frac{k+1}{2})\\
&= \sum_{j=0}^{\frac{k-1}{2}} P(X_1 = j, X_2 = k-2j, X_3 = n-k+j)\\
&= \sum_{j=0}^{\frac{k-1}{2}} \frac{n!}{j!\,(k-2j)!\,(n-k+j)!} q_1^j q_2^{k-2j} q_3^{n-k+j}.
\end{aligned}$$

3. $k > n$ and k is even,

$$\begin{aligned}
&P(2X_1 + X_2 = k) \\
&= P(X_1 = k-n, X_2 = 2n-k, X_3 = 0) \\
&\quad + P(X_1 = k-n+1, X_2 = 2n-k-2, X_3 = 1) + \cdots + P(X_1 = \frac{k}{2}, X_2 = 0, X_3 = n - \frac{k}{2}) \\
&= \sum_{j=k-n}^{\frac{k}{2}} P(X_1 = j, X_2 = k-2j, X_3 = n-k+j) \\
&= \sum_{j=k-n}^{\frac{k}{2}} \frac{n!}{j!\,(k-2j)!\,(n-k+j)!} q_1^j q_2^{k-2j} q_3^{n-k+j}.
\end{aligned}$$

4. $k > n$ and k is odd,

$$\begin{aligned}
&P(2X_1 + X_2 = k) \\
&= P(X_1 = k-n, X_2 = 2n-k, X_3 = 0) \\
&\quad + P(X_1 = k-n+1, X_2 = 2n-k-2, X_3 = 1) + \cdots + P(X_1 = \frac{k-1}{2}, X_2 = 1, X_3 = n - \frac{k+1}{2}) \\
&= \sum_{j=k-n}^{\frac{k-1}{2}} P(X_1 = j, X_2 = k-2j, X_3 = n-k+j) \\
&= \sum_{j=k-n}^{\frac{k-1}{2}} \frac{n!}{j!\,(k-2j)!\,(n-k+j)!} q_1^j q_2^{k-2j} q_3^{n-k+j}.
\end{aligned}$$

Combining all four cases, we can write

$$P(2X_1 + X_2 = k) = \begin{cases} \sum_{j=0}^{\frac{1}{4}\left(2k+(-1)^k-1\right)} \frac{n!}{j!(k-2j)!(n-k+j)!} q_1^j q_2^{k-2j} q_3^{n-k+j} & \text{if } k \leq n \\ \sum_{j=k-n}^{\frac{1}{4}\left(2k+(-1)^k-1\right)} \frac{n!}{j!(k-2j)!(n-k+j)!} q_1^j q_2^{k-2j} q_3^{n-k+j} & \text{if } k > n \end{cases}$$

$\square$

Now we prove the main theorem.

Proof. Consider $\hat{F}_Z(z)$

$$\begin{aligned}
\hat{F}_Z(z) &= \sum_{i=0}^{m-1}\left[\left(\hat{F}_X\left(\frac{(i+1)z}{m}\right) - \hat{F}_X\left(\frac{iz}{m}\right)\right)\frac{\left(\hat{F}_Y\left(\frac{z(m-i-1)}{m}\right) + \hat{F}_Y\left(\frac{z(m-i)}{m}\right)\right)}{2}\right] \\
&= \frac{1}{2}\sum_{i=0}^{m-1}\left[\left(\frac{\#x_v's \le \frac{(i+1)z}{m}}{n_1} - \frac{\#x_v's \le \frac{(i)z}{m}}{n_1}\right)\left(\frac{\#y_w's \le \frac{(m-i)z}{m}}{n_2} + \frac{\#y_w's \le \frac{(m-i-1)z}{m}}{n_2}\right)\right], \\
&\quad \text{where } v = 1, 2 \dots n_1, w = 1, 2 \dots n_2 \\
&= \frac{1}{2}\sum_{i=0}^{m-1}\left[\left(\frac{\#x_v's \text{ between } \frac{(i)z}{m} \text{ and } \frac{(i+1)z}{m}}{n_1}\right)\right. \\
&\quad \left.\left(\frac{\#y_w's \text{ between } \frac{(m-i-1)z}{m} \text{ and } \frac{(m-i)z}{m}}{n_2} + 2\frac{\#y_w's \le \frac{(m-i-1)z}{m}}{n_2}\right)\right] \\
&= \frac{1}{2n_1n_2}\sum_{i=0}^{m-1}\left[\left(\#x_v's \text{ between } \frac{(i)z}{m} \text{ and } \frac{(i+1)z}{m}\right)\right. \\
&\quad \left.\left(\left(\#y_w's \text{ between } \frac{(m-i-1)z}{m} \text{ and } \frac{(m-i)z}{m}\right) + 2\ \left(\#y_w's \le \frac{(m-i-1)z}{m}\right)\right)\right] \\
&= \frac{1}{2n_1n_2}\left\{\sum_{i=0}^{m-1}\left[\left(\#x_v's \text{ between } \frac{(i)z}{m} \text{ and } \frac{(i+1)z}{m}\right)\right.\right. \\
&\quad \left.\times\left(\#y_w's \text{ between } \frac{(m-i-1)z}{m} \text{ and } \frac{(m-i)z}{m}\right)\right] \\
&\quad \left.+ 2\ \sum_{i=0}^{m-1}\left[\left(\#x_v's \text{ between } \frac{(i)z}{m} \text{ and } \frac{(i+1)z}{m}\right)\times\left(\#y_w's \le \frac{(m-i-1)z}{m}\right)\right]\right\} \\
&= \frac{1}{2n_1n_2}(C_2 + 2C_1),
\end{aligned}$$

where

$$C_1 = \sum_{i=0}^{m-1}\left[\left(\#x_v's \text{ between } \frac{(i)z}{m} \text{ and } \frac{(i+1)z}{m}\right)\times\left(\#y_w's \le \frac{(m-i-1)z}{m}\right)\right]$$

and

$$C_2 = \sum_{i=0}^{m-1}\left[\left(\#x_v's \text{ between } \frac{(i)z}{m} \text{ and } \frac{(i+1)z}{m}\right)\times\left(\#y_w's \text{ between } \frac{(m-i-1)z}{m} \text{ and } \frac{(m-i)z}{m}\right)\right].$$

Let us denote

$$A_i = \left\{(x_v, y_w)\middle| x_v \in \left(\frac{(i)z}{m}, \frac{(i+1)z}{m}\right], y_w \in \left(\frac{(m-i-1)z}{m}, \frac{(m-i)z}{m}\right]\right\}_{v=1,2\dots n_1, w=1,2\dots n_2}, i = 0, 1 \dots m-1$$

$$B_i = \left\{ (x_v, y_w) \middle| x_v \in \left(\frac{(i)z}{m}, \frac{(i+1)z}{m}\right], y_w \in \left(0, \frac{(m-i-1)z}{m}\right] \right\}_{v=1,2\ldots n_1, w=1,2\ldots n_2}, i = 0, 1\ldots m-1$$

Here $A_i \cap A_j = B_i \cap B_j = \emptyset, i \neq j = 0, 1...m-1$ and $A_i \cap B_j = \emptyset, i, j = 0, ..m-1$, where $\emptyset$ denotes empty set. Also it can be seen that $\cup_{i=0}^{m-1} A_i$ and $\cup_{i=0}^{m-1} B_i$ are also disjoint. $\{\cup_{i=0}^{m-1} A_i, \cup_{i=0}^{m-1} B_i, \left(\cup_{i=0}^{m-1}(A_i \cup B_i)\right)^c\}$ is a partition of $(0, \infty) \times (0, \infty)$. Now

$$C_1 = \text{Number of elements in } \cup_{i=0}^{m-1} B_i,$$

$$C_2 = \text{Number of elements in } \cup_{i=0}^{m-1} A_i,$$

and let

$$C_3 = \text{Number of elements in } \left(\cup_{i=0}^{m-1}(A_i \cup B_i)\right)^c = n_1 n_2 - C_1 - C_2.$$

Then it can be observed that (C_1, C_2, C_3) is distributed with multinomial distribution with parameter $(n_1 n_2, q_1, q_2, q_3)$, where

$$q_1 = \sum_{i=0}^{m-1} \left(F_X\left(\frac{(i+1)z}{m}\right) - F_X\left(\frac{iz}{m}\right) \right) F_Y\left(\frac{z(m-i-1)}{m}\right),$$

$$q_2 = \sum_{i=0}^{m-1} \left(F_X\left(\frac{(i+1)z}{m}\right) - F_X\left(\frac{iz}{m}\right) \right) \left(F_Y\left(\frac{z(m-i)}{m}\right) - F_Y\left(\frac{z(m-i-1)}{m}\right) \right)$$

and

$$q_3 = 1 - q_1 - q_2.$$

Thus using Lemma 1, $2n_1 n_2 \hat{F}_Z(z) = C_2 + 2C_1$ is distributed with p.m.f. given in (14). □

Lemma 2. *When n is large,*

$$\frac{2X_1 + X_2 - \mu}{\sigma} \sim N(0, 1)$$

where mean $\mu = n(2q_1 + q_2)$ and variance is given by

$$\sigma^2 = n(q_1 q_2 + q_3 q_2 + 4q_1 q_3)$$

Proof. Since $\mathbf{X} = \{X_1, X_2, X_3\}$ follows multinomial distribution with parameters n and $\{q_1, q_2, q_3\}$, we have moment generating function (M.G.F.) of $\mathbf{X}$ is given by

$$E\left(e^{\mathbf{i}(t_1 X_1 + t_2 X_2 + t_3 X_3)}\right) = (q_1 e^{t_1} + q_2 e^{t_2} + q_3 e^{t_3})^n$$

M.G.F. of $2X_1 + X_2$ is given by

$$\begin{aligned}\phi_{2X_1+X_2}(t) &= E\left[e^{t(2X_1+X_2)}\right] \\ &= E\left[e^{(2tX_1+tX_2)}\right] \\ &= (q_1e^{2t} + q_2e^t + q_3)^n.\end{aligned}$$

M.G.F. of $\frac{2X_1+X_2-\mu}{\sigma}$ is given by

$$\begin{aligned}E\left[e^{t\left(\frac{2X_1+X_2-\mu}{\sigma}\right)}\right] &= e^{\frac{-\mu t}{\sigma}} E\left[e^{\frac{t}{\sigma}(2X_1+X_2)}\right] \\ &= e^{\frac{-\mu t}{\sigma}}(q_1e^{2\frac{t}{\sigma}} + q_2e^{\frac{t}{\sigma}} + q_3)^n \\ &= e^{\ln\left((q_1e^{2\frac{t}{\sigma}}+q_2e^{\frac{t}{\sigma}}+q_3)^n\right)-\frac{\mu t}{\sigma}}\end{aligned}$$

Taylor's series expansion of $\ln\left((q_1e^{2\frac{t}{\sigma}} + q_2e^{\frac{t}{\sigma}} + q_3)^n\right)$ is

$$\ln\left((q_1e^{2\frac{t}{\sigma}} + q_2e^{\frac{t}{\sigma}} + q_3)^n\right) = \ln\left(q_1+q_2+q_3\right)^n + \frac{t\left(2nq_1+nq_2\right)}{\sigma(q_1+q_2+q_3)} + \frac{t^2\left(nq_1q_2+nq_3q_2+4nq_1q_3\right)}{2\sigma^2\left(q_1+q_2+q_3\right)^2} + O\left(\frac{1}{n^{1/2}}\right)$$

$$= \frac{t\mu}{\sigma} + \frac{t^2}{2} + O\left(\frac{1}{n^{1/2}}\right).$$

Thus as $n \to \infty$,

$$E\left[e^{t\left(\frac{2X_1+X_2-\mu}{\sigma}\right)}\right] = e^{\frac{t^2}{2}},$$

which is the M.G.F. of standard normal distribution. □

Theorem 4. *As $n_1, n_2 \to \infty$,*

$$\sqrt{n_1n_2}\left(\hat{F}_Z(z) - F_{Z_m}(z)\right) \sim N\left(0, \frac{1}{2}\sqrt{(q_1q_2+q_3q_2+4q_1q_3)}\right),$$

where

$$q_1 = \sum_{i=0}^{m-1}\left(F_X\left(\frac{(i+1)z}{m}\right) - F_X\left(\frac{iz}{m}\right)\right)F_Y\left(\frac{z(m-i-1)}{m}\right),$$

$$q_2 = \sum_{i=0}^{m-1}\left(F_X\left(\frac{(i+1)z}{m}\right) - F_X\left(\frac{iz}{m}\right)\right)\left(F_Y\left(\frac{z(m-i)}{m}\right) - F_Y\left(\frac{z(m-i-1)}{m}\right)\right)$$

and

$$q_3 = 1 - q_1 - q_2.$$

Proof. From Theorem 3 and Lemma 2 we have, as $n_1, n_2 \to \infty$,

$$2n_1n_2\hat{F}_Z(z) \sim N\left(n_1n_2(2q_1 + q_2), \sqrt{n_1n_2(q_1q_2 + q_3q_2 + 4q_1q_3)}\right).$$

But

$$\begin{aligned}
2q_1 + q2 = {} & 2\sum_{i=0}^{m-1}\left(F_X\left(\frac{(i+1)z}{m}\right) - F_X\left(\frac{iz}{m}\right)\right)F_Y\left(\frac{z(m-i-1)}{m}\right) \\
& + \sum_{i=0}^{m-1}\left(F_X\left(\frac{(i+1)z}{m}\right) - F_X\left(\frac{iz}{m}\right)\right)\left(F_Y\left(\frac{z(m-i)}{m}\right) - F_Y\left(\frac{z(m-i-1)}{m}\right)\right) \\
= {} & \sum_{i=0}^{m-1}\left\{\left(F_X\left(\frac{(i+1)z}{m}\right) - F_X\left(\frac{iz}{m}\right)\right)\right. \\
& \left.\left(2F_Y\left(\frac{z(m-i-1)}{m}\right) + F_Y\left(\frac{z(m-i)}{m}\right) - F_Y\left(\frac{z(m-i-1)}{m}\right)\right)\right\} \\
= {} & \sum_{i=0}^{m-1}\left(F_X\left(\frac{(i+1)z}{m}\right) - F_X\left(\frac{iz}{m}\right)\right)\left(F_Y\left(\frac{z(m-i-1)}{m}\right) + F_Y\left(\frac{z(m-i)}{m}\right)\right) \\
= {} & 2F_{Z_m}(z)
\end{aligned}$$

Hence the theorem is proved. □

5 Simulation study

In this section, we carry out a simulation study to assess the performance of the proposed estimator. For the simulation study we use various parameter combinations of Weibull, lognormal and gamma distributions. For each combination we generate 1000 samples of size 25 from each distribution and we then calculate bias and MSE of the estimator at four probabilities 0.2,0.4,0.6 and 0.8. The chosen values of m are 5,10 and 15. Simulation results are summarized in Tables 1-4. Tables give the approximation error and the bias and the MSE of the nonparametric estimator. From Tables 1-4, it can be seen that approximation error decreases as m increases and approximation error increases as u increases. The maximum reported error is 0.262 for the sum of Weibull(1.0.5) and lognormal(1,0.5), when $u = 0.8$ and $m = 5$. Our estimator gives very small bias and MSE in all cases when $m = 15$. The maximum reported bias is for the sum of Weibull(1.0.5) and lognormal(1,0.5), when $u = 0.8$ and $m = 5$. The reason for the high bias is due to the high approximation error. But bias reduces significantly when $m = 15$. In Figure 3, we plot $F_Z(z)$, the approximation $F_{Z_{15}}(z)$ and the non-parametric estimator $\hat{F}_Z(z)$ for $m = 15$. The name of the distributions considered are labelled on top of each figure.

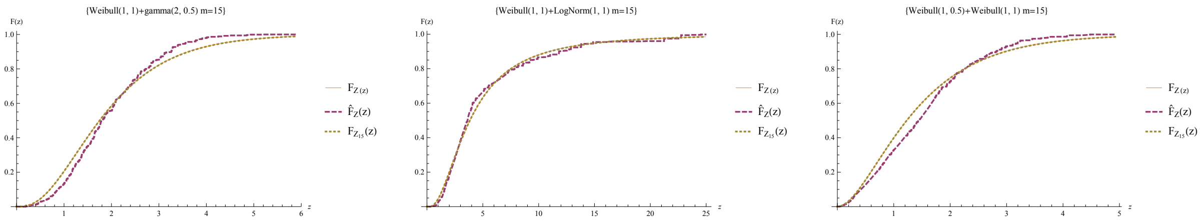


Figure 3: Plot of $F_Z(z), F_{Z_{15}}(z)$ and $\hat{F}_Z(z)$ for $m = 15$

6 Conclusion

The present study provided a non-parametric estimator based on an approximation of the distribution function of sum of two independent random variables. We derived the exact distribution of the estimator. We established the consistency and asymptotic normality of the estimator. Simulation studies showed that the estimator has small bias and small MSE. The result will be useful in estimating the reliability of the system with two independent components, where the system-life length is the sum of the component life lengths. The general case with more than two components is still under investigation.

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Table 1: Simulation Results

Distribution 1	Distribution 2	u	$m = 5$			$m = 10$			$m = 15$		
			error	bias	MSE	error	bias	MSE	error	bias	MSE
Weibull(1,0.5)	Weibull(1,0.5)	0.2	0.0016	0.0009	0.0038	0.0004	-0.0004	0.0039	0.0002	-0.0004	0.0039
		0.4	0.0044	0.0061	0.0068	0.0011	0.0029	0.0069	0.0005	0.0022	0.0069
		0.6	0.0074	0.0056	0.0074	0.0018	0.0005	0.0075	0.0008	-0.0005	0.0075
		0.8	0.0091	0.0082	0.0048	0.0023	0.0013	0.0047	0.0010	0.0000	0.0048
	Weibull(1,1)	0.2	0.0017	0.0016	0.0037	0.0004	0.0003	0.0038	0.0002	0.0000	0.0038
		0.4	0.0047	0.0062	0.0068	0.0012	0.0027	0.0070	0.0005	0.0022	0.0070
		0.6	0.0078	0.0089	0.0074	0.0019	0.0035	0.0076	0.0009	0.0024	0.0075
		0.8	0.0097	0.0100	0.0051	0.0024	0.0023	0.0051	0.0011	0.0010	0.0051
	LogNorm(1,0.5)	0.2	0.0068	0.0071	0.0040	0.0018	0.0017	0.0045	0.0008	0.0007	0.0045
		0.4	0.0171	0.0169	0.0073	0.0044	0.0036	0.0078	0.0019	0.0009	0.0077
		0.6	0.0250	0.0283	0.0077	0.0063	0.0085	0.0078	0.0028	0.0045	0.0079
		0.8	0.0262	0.0268	0.0054	0.0064	0.0074	0.0054	0.0029	0.0041	0.0055
	LogNorm(1,1)	0.2	0.0037	0.0020	0.0047	0.0009	-0.0012	0.0048	0.0004	-0.0016	0.0049
		0.4	0.0094	0.0057	0.0077	0.0024	-0.0013	0.0078	0.0011	-0.0028	0.0078
		0.6	0.0141	0.0176	0.0087	0.0035	0.0068	0.0088	0.0016	0.0047	0.0089
		0.8	0.0155	0.0150	0.0060	0.0040	0.0037	0.0060	0.0018	0.0014	0.0059
	gamma(2,0.5)	0.2	0.0014	0.0000	0.0039	0.0003	-0.0011	0.0040	0.0002	-0.0014	0.0040
		0.4	0.0052	0.0045	0.0066	0.0013	0.0004	0.0068	0.0006	0.0000	0.0068
		0.6	0.0094	0.0086	0.0067	0.0024	0.0024	0.0069	0.0010	0.0009	0.0069
		0.8	0.0120	0.0116	0.0047	0.0030	0.0029	0.0047	0.0013	0.0012	0.0047
	gamma(2,1)	0.2	0.0026	0.0026	0.0042	0.0007	0.0012	0.0044	0.0003	0.0011	0.0044
		0.4	0.0079	0.0123	0.0074	0.0020	0.0064	0.0075	0.0009	0.0049	0.0076
		0.6	0.0135	0.0080	0.0076	0.0034	-0.0021	0.0076	0.0015	-0.0037	0.0076
		0.8	0.0165	0.0168	0.0049	0.0041	0.0050	0.0049	0.0018	0.0028	0.0049

Table 2: Simulation Results

Distribution 1	Distribution 2	u	$m = 5$			$m = 10$			$m = 15$		
			error	bias	MSE	error	bias	MSE	error	bias	MSE
Weibull(1,1)	Weibull(1,1)	0.2	0.0016	0.0014	0.0036	0.0004	0.0002	0.0037	0.0002	-0.0003	0.0037
		0.4	0.0044	0.0077	0.0067	0.0011	0.0048	0.0068	0.0005	0.0039	0.0068
		0.6	0.0074	0.0088	0.0070	0.0018	0.0032	0.0070	0.0008	0.0018	0.0071
		0.8	0.0091	0.0095	0.0048	0.0023	0.0025	0.0048	0.0010	0.0012	0.0048
	LogNorm(1,0.5)	0.2	0.0032	0.0039	0.0038	0.0008	0.0015	0.0040	0.0004	0.0010	0.0040
		0.4	0.0101	0.0024	0.0065	0.0025	-0.0060	0.0068	0.0011	-0.0074	0.0069
		0.6	0.0160	0.0169	0.0068	0.0040	0.0052	0.0070	0.0018	0.0024	0.0071
		0.8	0.0178	0.0133	0.0050	0.0044	0.0000	0.0048	0.0019	-0.0022	0.0048
	LogNorm(1,1)	0.2	0.0022	0.0017	0.0040	0.0005	0.0006	0.0040	0.0002	0.0001	0.0041
		0.4	0.0063	0.0114	0.0075	0.0016	0.0070	0.0077	0.0007	0.0063	0.0078
		0.6	0.0100	0.0074	0.0073	0.0025	0.0004	0.0074	0.0011	-0.0013	0.0075
		0.8	0.0110	0.0135	0.0055	0.0027	0.0051	0.0055	0.0012	0.0040	0.0055
	gamma(2,0.5)	0.2	0.0007	0.0020	0.0033	0.0002	0.0013	0.0033	0.0001	0.0011	0.0033
		0.4	0.0036	0.0020	0.0068	0.0009	-0.0009	0.0069	0.0004	-0.0014	0.0070
		0.6	0.0069	0.0059	0.0066	0.0017	0.0007	0.0068	0.0008	-0.0001	0.0068
		0.8	0.0089	0.0104	0.0050	0.0022	0.0036	0.0051	0.0010	0.0025	0.0051
	gamma(2,1)	0.2	0.0014	0.0004	0.0039	0.0003	-0.0009	0.0039	0.0002	-0.0011	0.0040
		0.4	0.0052	-0.0004	0.0064	0.0013	-0.0042	0.0065	0.0006	-0.0053	0.0066
		0.6	0.0094	0.0061	0.0068	0.0024	-0.0010	0.0070	0.0010	-0.0022	0.0070
		0.8	0.0120	0.0102	0.0049	0.0030	0.0010	0.0049	0.0013	-0.0007	0.0049

Table 3: Simulation Results

Distribution 1	Distribution 2	u	$m=5$			$m=10$			$m=15$		
			error	bias	MSE	error	bias	MSE	error	bias	MSE
LogNorm(1,0.5)	LogNorm(1,0.5)	0.2	-0.0067	-0.0078	0.0035	-0.0017	-0.0023	0.0036	-0.0008	-0.0013	0.0036
		0.4	0.0021	0.0037	0.0062	0.0005	0.0023	0.0064	0.0002	0.0018	0.0065
		0.6	0.0120	0.0140	0.0060	0.0030	0.0042	0.0063	0.0014	0.0030	0.0064
		0.8	0.0172	0.0168	0.0043	0.0043	0.0046	0.0044	0.0019	0.0021	0.0044
	LogNorm(1,1)	0.2	-0.0023	-0.0015	0.0038	-0.0005	0.0006	0.0039	-0.0002	0.0005	0.0039
		0.4	0.0031	0.0017	0.0064	0.0008	-0.0003	0.0066	0.0004	-0.0007	0.0067
		0.6	0.0076	0.0110	0.0073	0.0019	0.0053	0.0074	0.0008	0.0040	0.0075
		0.8	0.0075	0.0086	0.0047	0.0019	0.0029	0.0049	0.0009	0.0018	0.0049
	gamma(2,0.5)	0.2	-0.0042	-0.0071	0.0038	-0.0011	-0.0037	0.0039	-0.0005	-0.0029	0.0039
		0.4	0.0024	0.0050	0.0065	0.0005	0.0043	0.0070	0.0002	0.0033	0.0071
		0.6	0.0102	0.0088	0.0069	0.0025	0.0014	0.0071	0.0011	-0.0005	0.0071
		0.8	0.0146	0.0112	0.0050	0.0035	0.0005	0.0050	0.0015	-0.0018	0.0050
	gamma(2,1)	0.2	-0.0042	-0.0051	0.0036	-0.0011	-0.0013	0.0037	-0.0005	-0.0008	0.0038
		0.4	0.0014	0.0016	0.0056	0.0004	0.0003	0.0058	0.0002	-0.0005	0.0058
		0.6	0.0087	0.0091	0.0061	0.0022	0.0027	0.0063	0.0010	0.0016	0.0064
		0.8	0.0137	0.0131	0.0043	0.0034	0.0034	0.0045	0.0015	0.0016	0.0044

Table 4: Simulation Results

Distribution 1	Distribution 2	u	$m = 5$			$m = 10$			$m = 15$		
			error	bias	MSE	error	bias	MSE	error	bias	MSE
LogNorm(1,1)	LogNorm(1,1)	0.2	-0.0007	-0.0006	0.0037	-0.0001	0.0002	0.0037	0.0000	0.0002	0.0037
		0.4	0.0019	0.0055	0.0064	0.0005	0.0038	0.0065	0.0002	0.0038	0.0065
		0.6	0.0046	0.0022	0.0070	0.0011	-0.0011	0.0072	0.0005	-0.0016	0.0072
		0.8	0.0059	0.0074	0.0051	0.0012	0.0025	0.0052	0.0005	0.0017	0.0052
	gamma(2,0.5)	0.2	-0.0005	0.0004	0.0044	-0.0001	0.0010	0.0045	-0.0001	0.0010	0.0045
		0.4	0.0031	0.0045	0.0072	0.0007	0.0025	0.0074	0.0003	0.0023	0.0074
		0.6	0.0060	0.0039	0.0074	0.0014	-0.0009	0.0076	0.0006	-0.0015	0.0076
		0.8	0.0070	0.0051	0.0056	0.0014	-0.0004	0.0056	0.0006	-0.0012	0.0055
	gamma(2,1)	0.2	-0.0012	-0.0030	0.0039	-0.0003	-0.0018	0.0040	-0.0001	-0.0017	0.0040
		0.4	0.0022	0.0014	0.0068	0.0006	-0.0003	0.0070	0.0003	-0.0008	0.0070
		0.6	0.0058	0.0078	0.0074	0.0015	0.0036	0.0076	0.0007	0.0030	0.0078
		0.8	0.0068	0.0069	0.0051	0.0016	0.0019	0.0051	0.0007	0.0009	0.0051
gamma(2,0.5)	gamma(2,0.5)	0.2	-0.0022	-0.0038	0.0035	-0.0005	-0.0021	0.0035	-0.0002	-0.0016	0.0035
		0.4	0.0009	0.0042	0.0063	0.0002	0.0033	0.0063	0.0001	0.0034	0.0064
		0.6	0.0059	0.0053	0.0070	0.0015	0.0007	0.0072	0.0007	-0.0003	0.0073
		0.8	0.0103	0.0133	0.0047	0.0026	0.0059	0.0048	0.0012	0.0042	0.0048
	gamma(2,1)	0.2	-0.0019	-0.0029	0.0037	-0.0005	-0.0014	0.0038	-0.0002	-0.0009	0.0038
		0.4	0.0013	0.0003	0.0065	0.0003	-0.0002	0.0067	0.0001	-0.0004	0.0068
		0.6	0.0060	0.0064	0.0074	0.0015	0.0011	0.0077	0.0007	0.0004	0.0078
		0.8	0.0099	0.0105	0.0047	0.0024	0.0032	0.0048	0.0011	0.0015	0.0049
gamma(2,1)	gamma(2,1)	0.2	-0.0022	-0.0027	0.0035	-0.0005	-0.0011	0.0035	-0.0002	-0.0007	0.0036
		0.4	0.0009	0.0015	0.0064	0.0002	0.0010	0.0067	0.0001	0.0012	0.0066
		0.6	0.0059	0.0108	0.0067	0.0015	0.0061	0.0068	0.0007	0.0051	0.0069
		0.8	0.0103	0.0126	0.0042	0.0026	0.0049	0.0043	0.0012	0.0036	0.0043

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