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On estimation of limiting variance of partial sums of functions of associated random variables

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Abstract

We discuss three different estimators for estimating the limiting variance of partial sums of functions of associated random variables. The first two estimators are based on a Subsampling method, while the third is obtained using Circular block bootstrap. As an application, we also obtain estimators for the limiting variance for U-statistics based on stationary associated random variables.

Keywords: *Associated random variables; Circular Block Bootstrap; Subsampling; U-statistics.*

1 Introduction

There exist several instances when the underlying random variables of interest are not independent. For example, the components of the moving average process $\{X_n; n \geq 1\}$ defined as $X_n = a_0\epsilon_n + ... + a_q\epsilon_{n-q}$, where $\{\epsilon_n\}_n$ are independent random variables, and $a_0, ..., a_q$ have the same sign are dependent; in reliability studies the lifetimes of components in structures in which components share the load so that failure of one component results in increased load on each of the remaining components are dependent. In both these examples the random variables are not independent but associated. Associated random variables are defined as follows.

Definition 1.1. *(Esary et al. (1967)) A finite collection of random variables $\{X_j, 1 \leq j \leq n\}$ is said to be associated, if for any choice of component-wise nondecreasing functions k_1, k_2 : $\mathbb{R}^n \to \mathbb{R}$, we have,*

$$Cov(k_1(X_1, \ldots, X_n), k_2(X_1, \ldots, X_n)) \geq 0$$

whenever it exists. An infinite collection of random variables $\{X_j, j \geq 1\}$ is associated if every finite sub-collection is associated.

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Apart from the reliability and survival studies, applications of associated random variables can also be found in statistical mechanics, percolation theory and interacting particle systems. Any set of independent random variables is associated, and nondecreasing functions of associated random variables are associated (cf. Esary et al. (1967)). A detailed presentation of the asymptotic results and examples relating to associated sequences can be found in Bulinski and Shashkin (2007, 2009), Oliveira (2012) and Prakasa Rao (2012).

Let $\mathbf{X} = \{X_n, n \geq 1\}$ be a sequence of associated random variables (not necessarily stationary) satisfying $E(X_1^2) < \infty$. For each $j \geq 1$, let A_j be a finite subset of $\{k, k \geq 1\}$ with cardinality denoted as $\#A_j$, and

$$Y_j = g_j(\mathbf{X}_{A_j}), j \geq 1, \tag{1.1}$$

where $\mathbf{X}_{A_j} = \{X_i, i \in A_j\}$, $j \geq 1$. We assume that there exists a $\tilde{g}_j$ such that $g_j \ll \tilde{g}_j$, where g_j, $\tilde{g}_j : \mathbb{R}^{\#A_j} \to \mathbb{R}$, $j \geq 1$. The relation $'' \ll''$ is defined as follows:

Definition 1.2. *(Newman (1984)) If g and $\tilde{g}$ are two real-valued functions on $\mathbb{R}^m$, $m \in \mathbb{N}$, then $g \ll \tilde{g}$ iff $\tilde{g} + g$ and $\tilde{g} - g$ are both coordinate-wise nondecreasing. If $g \ll \tilde{g}$, then $\tilde{g}$ will be coordinate-wise nondecreasing.*

Further let,

$$\tilde{Y}_j = \tilde{g}_j(\mathbf{X}_{A_j}), j \geq 1. \tag{1.2}$$

g_j, $\tilde{g}_j$, and A_j, $j \geq 1$ are such that $\{Y_j, j \geq 1\}$ and $\{\tilde{Y}_j, j \geq 1\}$ are stationary sequences. Under the condition $\sum_{j=1}^{\infty} |Cov(Y_1, Y_j)| < \infty$, the limiting variance of partial sums of $\{Y_j, j \geq 1\}$ is,

$$\sigma_g^2 = \lim_{n \to \infty} Var(\sum_{j=1}^{n} g_j(\mathbf{X}_{A_j})/\sqrt{n}) = Var(Y_1) + 2 \sum_{j=2}^{\infty} Cov(Y_1, Y_j). \tag{1.3}$$

Assume $\sigma_g^2 > 0$.

In this paper, we look at following three estimators of σ_g^2. Under suitable conditions, these estimators are shown to be consistent.

(PS) Peligrad and Suresh (1995) had obtained a consistent estimator for $\lim_{n \to \infty} (Var(\frac{\sum_{j=1}^{n} X_j}{\sqrt{n}}))$, where $\{X_n, n \geq 1\}$ is a sequence of stationary associated random variables. Their estimator was based on overlapping subseries of the underlying sample $\{X_i, 1 \leq i \leq n\}$. Using the same technique we obtain a consistent estimator of σ_g^2.

(PR) The second estimator is based on the estimator of the limiting variance of mean discussed in Politis and Romano (1993). This estimator is also based on overlapping subseries of the underlying sample. Politis and Romano (1993) had proved the consistency of the estimator when the underlying sample is from a mixing sequence. We extend the results of Politis and Romano (1993) to show consistency of the estimator when the sample is from $\{Y_j, j \geq 1\}$.

(CBB) The third estimator is based on observations obtained using Circular Block Bootstrap (introduced in Politis and Romano (1992)).

We also obtain estimators for the limiting variance for U-statistics based on stationary associated random variables. Assume $\{X_n, n \geq 1\}$ is a sequence of stationary associated random variables, with F as the common univariate distribution function. Define the U-statistic, $U_n(\rho)$, based on $\{X_j, 1 \leq j \leq n\}$, where $\rho : \mathbb{R}^2 \to \mathbb{R}$ is a symmetric function of degree two, by,

$$U_n(\rho) = \frac{2}{n(n-1)} \sum_{1 \leq i < j \leq n} \rho(X_i, X_j).$$

Define $\theta = \int_{\mathbb{R}^2} \rho(x,y) dF(x) dF(y)$, and $\rho_1(x_1) = \int_{\mathbb{R}} \rho(x_1, x_2)\ dF(x_2)$. Let,

$$h^{(1)}(x_1) = \rho_1(x_1) - \theta, \text{ and,} \tag{1.4}$$

$$h^{(2)}(x_1, x_2) = \rho(x_1, x_2) - h^{(1)}(x_1) - h^{(1)}(x_2) - \theta. \tag{1.5}$$

Then, using Hoeffding's decomposition,

$$U_n(\rho) = \theta + \frac{2}{n} \sum_{i=1}^{n} h^{(1)}(X_i) + \frac{2}{n(n-1)} \sum_{1 \leq i < j \leq n} h^{(2)}(X_i, X_j). \tag{1.6}$$

Dewan and Prakasa Rao (2001) gave a central limit theorem for degenerate and non-degenerate U-statistics based on $\{X_n, n \geq 1\}$ using an orthogonal expansion of the underlying kernel. Dewan and Prakasa (2002) and its corrigendum (2015) obtained a CLT for U-statistics with component-wise monotonic, differentiable, and non-degenerate kernels of degree 2, using Hoeffding's decomposition. The limiting distribution of U-statistics based on $\{X_n, n \geq 1\}$ can also be obtained using the results of Beutner and Zähle (2012, 2014). The approach of Beutner and Zähle (2012) is based on a modified delta method and quasi-Hadamard differentiability, while Beutner and Zähle (2014) propose a continuous mapping approach. Garg and Dewan (2015) obtained the limiting distribution of U-statistics based on kernels which are functions of Hardy-Krause variation, when the underlying sample is from $\{X_n, n \geq 1\}$.

The asymptotic normality of U-statistics is often used to obtain critical points, level of significance and power for tests based on U-statistics. For applying the Central limit theorem for U-statistics we need a consistent estimator for $\underset{n\to\infty}{lim} Var(\sqrt{n} U_n(\rho))$, which is generally based on unknown population parameters. Under suitable conditions, it can be shown that,

$$\underset{n\to\infty}{lim} Var(\sqrt{n} U_n(\rho)) = 4\sigma_U^2, \text{ where} \tag{1.7}$$

$$\sigma_U^2 = Var(h^{(1)}(X_1)) + 2\sum_{j=2}^{\infty} Cov(h^{(1)}(X_1), h^{(1)}(X_j)) = Var(\rho_1(X_1)) + 2\sum_{j=2}^{\infty} Cov(\rho_1(X_1), \rho_1(X_j)). \tag{1.8}$$

We modify the estimators discussed in (PS), (PR), and (CBB) to obtain consistent estimators for $4\sigma_U^2$.

The paper is organized as following. The three estimators for σ_g^2 and $4\sigma_U^2$ based on (PS), (PR), and (CBB) have been discussed in Sections 2-4, respectively. In Section 5, we compare the three estimators using simulations. A brief discussion on the results obtained in this paper and our intended future work is in Section 6. The proofs of the main results discussed are given in Section 7.

2 Estimation of σ_g^2 and $4\sigma_U^2$ based on (PS)

2.1 Estimation of σ_g^2

The results of this section extend the results of Peligrad and Suresh (1995) to functions of associated random variables. Under the conditions given by Theorems 2.1 − 2.2, the estimator for σ_g^2 can be shown to be consistent. The information on the limiting behavior of the estimator is given by Theorem 2.3.

Theorem 2.1. *Let $\{\ell_n, n \geq 1\}$ be a sequence of positive integers with $1 \leq \ell_n \leq n$ and $\ell_n = o(n)$ as $n \to \infty$. Set $S_j(k) = \sum_{i=j+1}^{j+k} Y_i$, $\bar{Y_n} = \frac{1}{n}\sum_{j=1}^{n} Y_j$. Let $E(Y_1) = \mu$ and $E(Y_1^2) < \infty$. Define, (write $\ell = \ell_n$),*

$$B_{PS_n} = \frac{1}{n-\ell+1}\left(\sum_{j=0}^{n-\ell} \frac{|S_j(\ell) - \ell\bar{Y_n}|}{\sqrt{\ell}}\right). \tag{2.1}$$

Assume,

$$\sum_{j=1}^{\infty} Cov(\tilde{Y_1}, \tilde{Y_j}) < \infty. \tag{2.2}$$

Then,

$$B_{PS_n} \to \sigma_g\sqrt{\frac{2}{\pi}} \quad in\ L_2\ \ as\ \ n \to \infty. \tag{2.3}$$

Theorem 2.2. *In addition to the conditions of Theorem 2.1 if we assume $\ell_n = O(\frac{n}{(logn)^2})$ the convergence in (2.3) is also in almost sure sense.*

Information on the rate of convergence of B_{PS_n} is provided by the following.

Theorem 2.3. *If the conditions of Theorem 2.1 are true, then as $n \to \infty$,*

$$\sqrt{\frac{n}{\ell}}\Big(\frac{\sqrt{\frac{\pi}{2}}}{n-\ell+1}\sum_{j=0}^{n-\ell}\Big(\Big|\frac{S_j(\ell)-\ell\mu}{\sqrt{\ell}}\Big|\Big) - \sqrt{\frac{\pi}{2}}E\Big|\frac{S_o(\ell)-\ell\mu}{\sqrt{\ell}}\Big|\Big)\Big) \xrightarrow{\mathcal{L}} N\Big(0, \frac{3\pi-8}{4}\sigma_g^2\Big). \tag{2.4}$$

For every $x > 0$,

$$\limsup_{n\to\infty} P\Big(\sqrt{\frac{n}{\ell}}\Big|B_{PS_n} - E\Big|\frac{S_o(\ell)-\ell\mu}{\sqrt{\ell}}\Big|\Big| > Ax\Big) \leq 2P(|N| > x), \tag{2.5}$$

where N is a standard normal variable and $A = \Big(\sqrt{\frac{3\pi-8}{2\pi}} + 1\Big)\sigma_g$.

The proofs of Theorems 2.1 − 2.3 are in Section 7.2.

Remark 2.4. *Using* (2.3)*, as* $n \to \infty$*,*

$$\frac{\pi}{2} B^2_{PS_n} = \frac{\pi}{2} \left(\frac{1}{n-\ell+1} \sum_{j=0}^{n-\ell} \frac{|S_j(\ell) - \ell \bar{Y}_n|}{\sqrt{\ell}} \right)^2 \xrightarrow{p} \sigma_g^2. \tag{2.6}$$

Under the conditions of Theorem 2.3*,* $Var(B_{PS_n}) = O(\frac{\ell}{n})$*, or* $Var(B^2_{PS_n}) = O(\frac{\ell}{n})$*,* $n \to \infty$*.*

2.2 Estimation of $4\sigma_U^2$

In general, $h^{(1)}$ or ρ_1 would not be known as they depend on unknown underlying parameters. We next obtain a consistent estimators for $4\sigma_U^2$, denoted here by $B(U_n)_{PS}$.

We first cite the following result from Garg and Dewan (2015) that is used to obtain a consistent estimator for $4\sigma_U^2$.

Lemma 2.5. *(Garg and Dewan (2015)) Let* $P(|X_n| \leq C_1) = 1$ *for some* $0 < C_1 < \infty$*. Let* $h^{(2)}(x, y)$ *be a degenerate kernel of degree two (i.e.* $\int_{\mathbb{R}} h^{(2)}(x, y) dF(y) = 0$ *for all* $x \in \mathbb{R}$*), and* $|h^{(2)}(x, y)| \leq M(C_1)$*, for some* $0 < M(C_1) < \infty$*, for all* $x, y \in [-C_1, C_1]$*. Assume that the density function of* X_1 *is bounded and* $h^{(2)}$ *is of bounded Hardy-Krause variation and left-continuous. Further, let* $\sum_{j=1}^{\infty} Cov(X_1, X_j)^\gamma < \infty$*, for some* $0 < \gamma < 1/6$*. Then, as* $n \to \infty$*,*

$$\sum_{1 \leq i < j \leq n} \sum_{1 \leq k < j \leq n} |E(h^{(2)}(X_i, X_j) h^{(2)}(X_k, X_l))| = O(n^2). \tag{2.7}$$

Let there exist a function $\tilde{h}^{(1)}(\cdot)$ *such that* $h^{(1)} \ll \tilde{h}^{(1)}$ *and,*

$$\sum_{j=1}^{\infty} Cov(\tilde{h}^{(1)}(X_1), \tilde{h}^{(1)}(X_j)) < \infty.$$

Assume further that $\sigma_U^2 > 0$ *(defined in (1.8)). Then, as* $n \to \infty$*,*

$$\begin{aligned} &Var(\sqrt{n} U_n) = 4\sigma_U^2 + o(1), \\ &\frac{\sqrt{n}(U_n - \theta)}{2\sigma_U} \xrightarrow{\mathcal{L}} N(0, 1) \text{ as } n \to \infty. \end{aligned} \tag{2.8}$$

Remark 2.6. *If* $h_1 \ll \tilde{h}_1$*, then* $|Cov(h_1(X_1), h_1(X_j))| \leq C\, Cov(\tilde{h}_1(X_1), \tilde{h}_1(X_j))$*,* $j \geq 1$*. If* $h^{(1)}$ *is monotonic, then* $\tilde{h}_1 \equiv h^{(1)}$ *and* $\{h^{(1)}(X_n), n \geq 1\}$ *is a sequence of stationary associated random variables.*

Assume $\ell \geq 2$. Define,

$$B(U_n)_{PS} = \frac{\sqrt{\ell}}{n-\ell+1} \left(\sum_{i=1}^{n-\ell+1} \left| \binom{\ell}{2}^{-1} \sum_{i \leq j < k \leq i+\ell-1} \rho(X_j, X_k) - U_n(\rho) \right| \right). \tag{2.9}$$

Theorem 2.7. *Under the conditions of Lemma* 2.5*, as* $n \to \infty$,

$$B(U_n)_{PS} \xrightarrow{p} 2\sigma_U \sqrt{\frac{2}{\pi}}, \ \ or \ \ \frac{\pi}{2} B^2(U_n)_{PS} \xrightarrow{p} 4\sigma_U^2. \tag{2.10}$$

The proof of Theorem 2.7 is discussed in Section 7.2.

Remark 2.8. *The condition that the random variables are uniformly bounded is only required to prove* (2.7)*.* (2.7) *can be extended to random variables which are not uniformly bounded by the usual truncation techniques.*

Remark 2.9. *The results of Lemma* 2.5*, and hence the results of Theorem* 2.7 *can be easily extended to non-degenerate U-statistics based on kernels of finite degrees greater than two.*

Remark 2.10. *We have assumed that the U-statistics should be based on kernels which are of bounded Hardy-Krause variation. Examples include the U-statistic estimators of moments and L-moments.*

3 Estimation of σ_g^2 and $4\sigma_U^2$ based on (PR)

3.1 Estimation of σ_g^2

Politis and Romano (1993) had discussed a nonparametric estimator for the sample variance of linear statistics derived from mixing sequences. The variance estimator for σ_g^2 discussed in this section is the estimator for the sample variance of mean discussed in Politis and Romano (1993). The consistency of this estimator for σ_g^2 is discussed in Theorem 3.1. The order of the variance of the estimator has been discussed in Theorem 3.2.

Let $\{\ell_n, n \geq 1\}$ be a sequence of positive integers with $1 \leq \ell_n \leq n$ and $\ell_n = o(n)$ as $n \to \infty$. Let $\ell \equiv \ell_n$.

Theorem 3.1. *Assume*

$$\sum_{j=1}^{\infty} Cov(\tilde{Y}_1, \tilde{Y}_j)^{1/3} < \infty. \tag{3.1}$$

Further, suppose that $(S_0(\ell) - \ell E(Y_1))/\sqrt{\ell}$ *has a bounded continuous density for all* $\ell \in \mathbb{N}$*. Then,* $n \to \infty$,

$$B_{PR_n}^2 = \frac{1}{n-\ell+1} \sum_{j=0}^{n-\ell} \left(\frac{S_j(\ell) - \ell \bar{Y}_n}{\sqrt{\ell}} \right)^2 \xrightarrow{p} \sigma_g^2, \tag{3.2}$$

where $S_j(\ell)$*,* $j = 0, 1, \cdots, n-\ell+1$ *are defined in Theorem* 2.1

Theorem 3.2. *Suppose* $E\left(\frac{S_0(\ell)-\ell E(Y_1)}{\sqrt{\ell}}\right)^{4+\nu} < \infty$*, for all* $\ell \in \mathbb{N}$*, and* $\sum_{j=1}^{\infty} Cov(\tilde{Y}_1, \tilde{Y}_j)^{\nu/3(4+\nu)} < \infty$*, for some* $\nu > 0$*. Further, suppose that* $(S_0(\ell) - \ell E(Y_1))/\sqrt{\ell}$ *has a bounded continuous density for all* $\ell \in \mathbb{N}$*, and* $\ell = o(\sqrt{n})$ *as* $n \to \infty$*. Then,*

$$Var\left(B_{PR_n}^2 \right) = O(\frac{\ell}{n}), \ \ as\ n \to \infty. \tag{3.3}$$

Proofs of Theorems 3.1 − 3.2 are given in Section 7.2.

Remark 3.3. *Assume* $E(Y_1) = 0$, *and* $E|Y_1|^{4+\delta+\nu}$, $E|\tilde{Y}_1|^{4+\delta+\nu} < \infty$, *for some* $\delta, \nu > 0$. *Let* $\sum_{j=n+1}^{\infty} Cov(\tilde{Y}_1, \tilde{Y}_j) = O(n^{-(2+\nu)(4+\nu+\delta)/2\delta})$. *Then,*

$$E\Big(\frac{S_0(\ell)}{\sqrt{\ell}}\Big)^{4+\nu} < \infty. \tag{3.4}$$

This follows using Lemma 7.3.

3.2 Estimation of $4\sigma_U^2$

We obtain a consistent estimators for $4\sigma_U^2$, denoted here by $B(U_n)_{PR}$. Define,

$$B^2(U_n)_{PR_n} = \frac{\ell}{(n-\ell+1)} \sum_{i=1}^{n-\ell+1} \Big(\binom{\ell}{2}^{-1} \sum_{i \le j < k \le i+\ell-1} \rho(X_j, X_k) - U_n(\rho)\Big)^2. \tag{3.5}$$

Theorem 3.4. *Under the conditions of Lemma* 2.5*, and*

$$\sum_{j=1}^{\infty} Cov(\tilde{h}^{(1)}(X_1), \tilde{h}^{(1)}(X_j))^{1/3} < \infty.$$

Further, suppose that $(\sum_{j=1}^{\ell} h^{(1)}(X_j) - \ell\theta)/\sqrt{\ell}$ *has a bounded continuous density for all* $\ell \in \mathbb{N}$. *Then,*

$$B^2(U_n)_{PR_n} \xrightarrow{p} 4\sigma_U^2, \text{ as } n \to \infty. \tag{3.6}$$

Proof of the above is in Section 7.2.

4 Estimation of σ_g^2 and $4\sigma_U^2$ based on (CBB)

Let $\Omega_n = \{Y_i, 1 \le i \le n\}$ have a common one-dimensional marginal distribution function and $E(Y_1) = \mu$. Define,

$$T_n = \sqrt{n}(\bar{Y}_n - \mu). \tag{4.1}$$

Then, $\underset{n\to\infty}{lim} Var(T_n) = \sigma_g^2$. The Circular Block Bootstrap (CBB) method was proposed by Politis and Romano (1992). This method re-samples overlapping and periodically extended blocks of a given length $\ell \equiv \ell_n$, (ℓ is a positive integer) satisfying $\ell = o(n)$ as $n \to \infty$ from $\{B(1,\ell), \cdots, B(n,\ell)\}$. $B(i,\ell)$, $i = 1, \cdots, n$, are defined as follows.

$$B(i,\ell) = (Y_{n,i}, \cdots, Y_{n,i+\ell-1}), \text{ where}$$

$$\begin{aligned} Y_{n,i} &= Y_i, \text{ if } i = 1, \cdots, n, \\ &= Y_j \quad \text{if } j = i-n, \; i = n+1, \cdots, n+(\ell-1). \end{aligned} \tag{4.2}$$

To obtain the CBB sample, randomly select k blocks from $\{B(1,\ell),\cdots,B(n,\ell)\}$ with replacement. The sample size is $m=k\ell$. Let $\Omega_m^\star=\{Y_i^\star, 1\le i\le m\}$ denote the CBB sample of size m from Ω_n.

Let $\{B^\star(1,\ell),\cdots,B^\star(k,\ell)\}$ denote the selected sample of blocks and the elements in $B^\star(j,\ell)$ be denoted as $(Y^\star_{(j-1)\ell+1},\cdots,Y^\star_{j\ell})$, $j=1,2,\cdots,k$.

$$P_\star((Y_1^\star,\cdots,Y_\ell^\star)'=(Y_{n,i},\cdots,Y_{n,i+\ell-1})')=\frac{1}{n},\ \ i=1,\cdots,n, \tag{4.3}$$

where $P_\star$ denotes the conditional probability given Ω_n. Note that in CBB equal weights are given to each of the observations $Y_1,\cdots,Y_n$.

For our calculations, we used $k=[\frac{n}{\ell}]$ and $m=k\ell$. Let $E_\star$ and $Var_\star$ respectively denote the conditional expectation and conditional variance, given Ω_n. Then, the bootstrap version of T_n is given by,

$$T_n^\star=\sqrt{m}(\bar{Y}_m^\star-E_\star\bar{Y}_m^\star). \tag{4.4}$$

Under CBB, $E_\star\bar{Y}_m^\star=\bar{Y}_n$ (from Lahiri (2003) (Section 2.7.1, (2.18))).

4.1 Estimation of σ_g^2

Let,

$$U_i=\frac{Y_{n,i}+\cdots+Y_{n,i+\ell-1}}{\ell}, \tag{4.5}$$

be the average of $B(i,\ell)$, $i=1,\cdots,n$. As the re-sampled blocks are independent,

$$B^2_{(CBB)_n}=Var_\star(T_n^\star)=\ell\Big[n^{-1}\sum_{i=1}^n U_i^2-\bar{Y}_n^2\Big]=\ell\Big[n^{-1}\sum_{i=1}^n(U_i-\bar{Y}_n)^2\Big]. \tag{4.6}$$

The consistency of $B_{(CBB)_n}$ for σ_g^2 is discussed in Theorem 4.1. The order of the variance of the estimator has been discussed in Theorem 4.2.

Theorem 4.1. *Under the conditions of Theorem* 3.1,

$$B^2_{(CBB)_n}\xrightarrow{p}\sigma_g^2,\ \ as\ n\to\infty. \tag{4.7}$$

Theorem 4.2. *Under the conditions of Theorem* 3.2,

$$Var\left(B^2_{(CBB)_n}\right)=O(\frac{\ell}{n}),\ \ as\ n\to\infty. \tag{4.8}$$

Proofs of Theorems 4.1 – 4.2 are in Section 7.2.

4.2 Estimation of $4\sigma_U^2$

Let $\{X_{n,i}, i = 1, \cdots, n\}$ be the periodically extended series of $\{X_i, 1 \leq i \leq n\}$. We assume that $\ell \geq 2$. The consistent estimator for $4\sigma_U^2$ is,

$$B^2(U_n)_{CBB} = \ell \left[n^{-1} \sum_{i=1}^{n} \left(\sum_{i \leq j < k \leq i+\ell-1} \frac{\rho(X_{n,j}, X_{n,k})}{\binom{\ell}{2}} - U_n(\rho) \right)^2 \right]. \quad (4.9)$$

Theorem 4.3. *Under the conditions of Theorem* 3.4*,*

$$B^2(U_n)_{CBB} \xrightarrow{p} 4\sigma_U^2, \ \text{as } n \to \infty. \quad (4.10)$$

5 Simulations - A comparison of the three estimators

We used the statistical software R (http://www.r-project.org; R Development Core Team (2014)) for our simulations. The results are based on a $r = 10000$ iterations of samples of size n. We took the block length $\ell_n = \lceil n^{1/3} \rceil$, $n = 50, 100, 200, 500, 1000$.

5.1 Estimation of σ_g^2

The three estimators for σ_g^2 based on samples of size n are $\frac{\pi}{2} B^2_{PS_n}$ (given in (2.6)), $B^2_{PR_n}$ (given in (3.2)), and $B^2_{CBB_n}$ (given in (4.6)). For the simulation results given in Table 5.1, we generated the samples $\{Y_i, 1 \leq i \leq n\}$ in the following 3 ways.

(S1) $Y_j = min(Z_j, Z_{j+1})$, $j = 1, 2, \cdots, n$, where $Z_j's$ are $i.i.d$ $Exp(1/2)$.

(S2) $Y_j = -0.5Y_{j-1} + \frac{\sqrt{3}}{2} Z_j$, $j = 1, 2, \cdots, n$, where Y_0 is a N(0,1) random variable and $Z_j's, j = 1, 2, \cdots, n$ are $i.i.d$ $N(0, 1)$.

(S3) $Y_j = 2min(Y_{j-1}, Z_j)$, $j = 1, 2, \cdots, n$, where Y_0 is a U(0,1) random variable, and $Z_j's, j = 1, 2, \cdots, n$ are $i.i.d$ with probability density function $f_{Z_1}(x) = \frac{1}{(1-x)^2}$, $x \in (0, \frac{1}{2})$, and 0, for $x \notin (0, \frac{1}{2})$.

$(S1)$ and $(S3)$ generates a sequence of associated random variables. In $(S2)$ we generate a sequence of non-monotonic functions of associated random variables. The true values for σ_g^2 for $(S1)$, $(S2)$, and $(S3)$, are $1.6667, 1/3$, and $1/4$, respectively.

Observations from Table 5.1

(1) All three estimators seem to be performing similarly.

(2) In $(S1)$ we generated random variables which are "almost independent". As expected, the convergences are faster in this case in comparison to the random variables generated using $(S2)$ and $(S3)$. Larger sample sizes are needed to obtain viable consistent estimates for $(S2)$ and $(S3)$.

Table 5.1 *Estimation of* σ_g^2

$(S1)$ $(\sigma_g^2 = 1.6667)$	n=50	n=100	n= 200	n= 500	n= 1000
$\frac{\pi}{2}\bar{B^2}_{PS_n}$	1.3648	1.4260	1.4771	1.5233	1.5485
Est. MSE $(\frac{\pi}{2}B^2_{PS_n})$	0.6646	0.3851	0.2212	0.1109	0.0671
$\bar{B^2}_{PR_n}$	1.3655	1.4505	1.5145	1.5632	1.5850
Est. MSE $(B^2_{PR_n})$	0.7262	0.4239	0.2433	0.1175	0.0679
$\bar{B^2}_{CBB_n}$	1.3564	1.4454	1.5126	1.5625	1.5844
Est. MSE $(B^2_{CBB_n})$	0.7013	0.4132	0.2402	0.1166	0.0676
$(S2)$ $(\sigma_g^2 = 1/3)$	n=50	n=100	n= 200	n= 500	n= 1000
$\frac{\pi}{2}\bar{B^2}_{PS_n}$	0.4208	0.4134	0.3990	0.3845	0.3751
Est. MSE $(\frac{\pi}{2}B^2_{PS_n})$	0.0236	0.0156	0.0094	0.0052	0.0033
$\bar{B^2}_{PR_n}$	0.4108	0.4082	0.3961	0.3830	0.3741
Est. MSE $(B^2_{PR_n})$	0.0197	0.0139	0.0086	0.0048	0.0031
$\bar{B^2}_{CBB_n}$	0.4179	0.4117	0.3982	0.3839	0.3745
Est. MSE $(B^2_{CBB_n})$	0.0212	0.0144	0.0089	0.0049	0.0032
$(S3)$ $(\sigma_g^2 = 1/4)$	n=50	n=100	n= 200	n= 500	n= 1000
$\frac{\pi}{2}\bar{B^2}_{PS_n}$	0.1681	0.1869	0.1998	0.2136	0.2214
Est. MSE $(\frac{\pi}{2}B^2_{PS_n})$	0.0111	0.0073	0.0047	0.0025	0.0016
$\bar{B^2}_{PR_n}$	0.1526	0.1736	0.1885	0.2051	0.2146
Est. MSE $(B^2_{PR_n})$	0.0123	0.0081	0.0053	0.0030	0.0019
$\bar{B^2}_{CBB_n}$	0.1507	0.1722	0.1877	0.2046	0.2144
Est. MSE $(B^2_{CBB_n})$	0.0125	0.0082	0.0054	0.0030	0.0019

In Table 5.1, (1) $\bar{B^2}_{PS_n} = \sum_{i=1}^{r} B^2{}_{PS_n}(i)/r$, where $B^2{}_{PS_n}(i)$ is the estimate of σ_g^2 for the sample of size n at the i^{th} iteration, $i = 1, \cdots, r$. (2) Est. MSE $(B^2_{PS_n}) = \sum_{i=1}^{r} \frac{(B^2{}_{PS_n}(i) - \sigma_g^2)^2}{2}$. Similarly, the values of $\bar{B^2}_{PR_n}$, Est. MSE $(B^2_{PR_n})$, $\bar{B^2}_{CBB_n}$ and Est. MSE $(B^2_{CBB_n})$ were obtained.

5.2 Estimation of $4\sigma_U^2$

We considered $U_n = \frac{1}{\binom{n}{2}} \sum_{1 \le i < j \le n} |X_i - X_j|$, where $\{X_j, 1 \le j \le n\}$ is the sample of stationary associated random variables. The discussed consistent estimators for $4\sigma_U^2$ are: $\frac{\pi}{2}B^2(U_n)_{PS}$ (given in (2.9)), $B^2(U_n)_{PR}$ (given in (3.5)), and $B^2(U_n)_{CBB}$ (given in (4.9)). We generated $\{X_j, 1 \le j \le n\}$, in the following 2 ways.

(S4) $X_j = min(Z_j, Z_{j+1})$, $j = 1, 2, \cdots, n$, where $Z_j's$ are $i.i.d$ $Exp(1/2)$.

(S5) $X_j = min(Z_j, Z_{j+1}, \cdots, Z_{j+9})$, $j = 1, 2, \cdots, n$, where $Z_j's$ are $i.i.d$ $Exp(1/10)$.

The true values for $4\sigma_U^2$ for $(S4)$, and $(S5)$, are 1.9430, and 8.3961, respectively.

Observations from Table 5.2

(1) For the samples generated, it can be seen that the estimator $\frac{\pi}{2}B^2(U_n)_{PS}$ performs better than the other two estimators.

(2) In $(S4)$ we generated random variables which are "almost independent". As expected, the convergences are faster in this case in comparison to the random variables generated using $(S5)$. Larger sample sizes are needed to obtain viable consistent estimates in $(S5)$.

Table 5.2 ***Estimation of*** $4\sigma_U^2$

$(S4)$ $(4\sigma_U^2 = 1.9430)$	n=50	n=100	n= 200	n= 500	n= 1000
$\frac{\pi}{2}\bar{B}^2(U_n)_{PS}$	1.7966	1.7929	1.8090	1.8210	1.8348
$Est.MSE(\frac{\pi}{2}B^2(U_n)_{PS})$	1.2069	0.6463	0.3598	0.1777	0.1054
$\bar{B}^2(U_n)_{PR}$	1.6440	1.6894	1.7375	1.7818	1.8098
$Est.MSE(B^2(U_n)_{PR})$	1.1686	0.6805	0.3908	0.1947	0.1161
$\bar{B}^2(U_n)_{CBB}$	1.6390	1.6879	1.7362	1.7815	1.8097
$Est.MSE(B^2(U_n)_{CBB})$	1.1364	0.6703	0.3871	0.1935	0.1157
$(S5)$ $(4\sigma_U^2 = 8.3961)$	n=50	n=100	n= 200	n= 500	n= 1000
$\frac{\pi}{2}\bar{B}^2(U_n)_{PS}$	3.7350	4.5140	5.1284	5.8882	6.2785
$Est.MSE(\frac{\pi}{2}B^2(U_n)_{PS})$	32.0246	22.4448	15.5086	9.1360	6.2628
$\bar{B}^2(U_n)_{PR}$	2.7247	3.3440	3.8578	4.5560	4.9678
$Est.MSE(B^2(U_n)_{PR})$	37.4503	29.3843	23.2071	16.3825	12.8232
$\bar{B}^2(U_n)_{CBB}$	2.6678	3.2996	3.8270	4.5385	4.9578
$Est.MSE(B^2(U_n)_{CBB})$	37.9008	29.7312	23.4459	16.5033	12.8879

In Table 5.2, (1) $\frac{\pi}{2}\bar{B}^2(U_n)_{PS} = \sum_{i=1}^{r} \frac{\pi}{2}B^2(U_n)_{PS}(i)/r$, where $\frac{\pi}{2}B^2(U_n)_{PS}(i)$ is the estimate of $4\sigma_U^2$ for the sample of size n at the i^{th} iteration, $i = 1, \cdots, r$. (2) Est. MSE $(\frac{\pi}{2}B^2(U_n)_{PS}) = \sum_{i=1}^{r} \frac{(\frac{\pi}{2}B^2(U_n)_{PS}(i) - 4\sigma_U^2)^2}{r}$. Similarly, the others values were obtained.

6 Discussions and Future Work

In this paper we have discussed three different consistent estimators for σ_g^2 ((1.3)). The first two estimators of σ_g^2, ${B^2}_{PS_n}$ (Section 2.1), and ${B^2}_{PR_n}$ (Section 3.1) are based on a Subsampling method, while the third estimator ${B^2}_{CBB_n}$ (Section 4.1) is based on Circular block bootstrap. It can be shown that under suitable conditions, the order of the variances of these three estimators is $O(\ell_n/n)$, where ℓ_n denotes the block size, and n is the sample size. The estimators of σ_g^2 discussed in Section 2.1 (${B^2}_{PS_n}$), and Section 3.1 (${B^2}_{PR_n}$) are based on the same Subsampling method. It can be seen that the difference between the two estimators is that ${B^2}_{PS_n}$ is based on mean of the absolute values of the differences, while ${B^2}_{PR_n}$ is based on the mean of the squares of the differences. Both ${B^2}_{PR_n}$, and ${B^2}_{CBB_n}$ are based on squares of differences. However, for the latter, as the name suggests is based on the overlapping blocks of the periodically extended series of the sample. The consistency of estimators of variance based on Moving and Non-overlapping block bootstrap can be proved similarly as the results discussed for Circular block bootstrap.

We have also obtained consistent estimators for limiting variance of U-statistics based on kernels of degree two (defined in (1.7)). The estimators for U-statistics based on kernels of degree greater than two can be obtained similarly. The discussed estimators use U-statistic values based on sub-samples from the original sample. Hence, for larger samples and U-statistics based on higher degrees these may be computationally cumbersome. Another way would be to use appropriate estimates of ρ_1 and use the consistent estimators for σ_g^2 to obtain estimates for σ_U^2. For example, Dewan and Prakasa Rao (2003) had obtained an estimate of the limiting variance of the Mann-Whitney test statistic under H_0, by estimating $\rho_1(X_i) = F(X_i)$, $1 \leq i \leq n$ by the empirical distribution function based on the underlying sample $\{X_i, 1 \leq i \leq n\}$, and using these estimates in the variance estimator discussed in Peligrad and Suresh (1995).

Simulation results in Table 5.1 show that all the three estimators of σ_g^2 perform similarly for samples sizes considered ($n = 50, 100, 200, 500, 1000$). A comparison of the estimators of $4\sigma_U^2$ in Table 5.2 show that for all sample sizes considered ($n = 50, 100, 200, 500, 1000$) the estimates based on $B^2(U_n)_{PS}$ seem to be closer to the actual values, than the estimates based on $B^2(U_n)_{PR}$ and $B^2(U_n)_{CBB}$.

We have not discussed the optimal value of ℓ_n for the proposed estimators in this paper. It is under preparation.

7 Preliminary Results and Proofs of the Theorems given in Sections $2-4$

7.1 Preliminaries

In this section we give results and definitions which will be needed to prove our main results given in Sections $2-4$. For the results discussed in this sub-section, assume $\{X_n, n \geq 1\}$ is a sequence of associated random variables, $\{Y_n, n \geq 1\}$, $\{\tilde{Y_n}, n \geq 1\}$, and σ_g^2 (assume $\sigma_g^2 > 0$) are defined in (1.1), (1.2) and (1.3) respectively.

Lemma 7.1. *(Newman (1980)) Suppose X and Y be two random variables with $E(X^2) < \infty$ and $E(Y^2) < \infty$. Let h and t be differentiable functions with $E(h^2(X)) < \infty$, $E(t^2(Y)) < \infty$, and finite derivates $h'(\cdot)$ and $t'(\cdot)$. Then,*

$$|Cov(h(X), t(Y)| = \int_{\mathbb{R}^2} h'(x)t'(y)[P(X \leq x, Y \leq y) - P(X \leq x)P(Y \leq y)]dxdy. \tag{7.1}$$

Lemma 7.2. *(Sadikova (1966)) Let $F(x, y)$ and $G(x, y)$ be two bivariate distribution functions with characteristic functions $f(s, t)$ and $g(s, t)$ respectively. Define,*

$$\hat{f}(s,t) = f(s,t) - f(s,0)f(0,t), \text{ and } \hat{g}(s,t) = g(s,t) - g(s,0)g(0,t).$$

Suppose that the partial derivatives of G with respect to x and y exist. Let,

$$A_1 = \sup_{x,y} \frac{\partial G(x,y)}{\partial x}, \text{ and } A_2 = \sup_{x,y} \frac{\partial G(x,y)}{\partial y}.$$

Suppose A_1 and A_2 be finite. Then, for any $T > 0$,

$$\sup_{x,y} |F(x,y) - G(x,y)| \leq \frac{1}{4\pi^2} \int_{-T}^{T} \int_{-T}^{T} \Big| \frac{\hat{f}(s,t) - \hat{f}(s,t)}{st} \Big| + 2\sup_x |F(x,\infty) - G(x,\infty)|$$
$$+ 2\sup_y |F(\infty,y) - G(\infty,y)| + 2\frac{A_1 + A_2}{T}(3\sqrt{2} + 4\sqrt{3}). \tag{7.2}$$

Lemma 7.3. *(Matula (2001)) Assume* $0 < Var(Y_1) < \infty$ *and* $E(Y_1) = 0$*. Suppose that,* $E|Y_1|^{r+\delta} < 0$*, and* $E|\tilde{Y}_1|^{r+\delta} < 0$ *for some* $r > 2$ *and* $\delta > 0$*. Assume that,*

$$\sum_{j=n+1}^{\infty} Cov(\tilde{Y}_1, \tilde{Y}_j) = O(n^{-(r-2)(r+\delta)/2\delta}).$$

Let $S_k = \sum_{j=1}^{k} Y_j$*,* $k \in \mathbb{N}$*. Then, there is a constant* $B > 0$ *not dependening on* n *such that,*

$$\sup_{m\geq 0} E|S_{n+m} - S_m|^r \leq Bn^{r/2}, \text{ for all } n \in \mathbb{N}. \tag{7.3}$$

Lemma 7.4. *(Newman (1984)) Let* $0 < Var(Y_1) < \infty$*, and* $\sum_{j=1}^{\infty} Cov(\tilde{Y}_1, \tilde{Y}_j) < \infty$*. Then,*

$$\frac{\sum_{j=1}^{n}(Y_j - E(Y_j))}{\sqrt{n}\sigma_g} \xrightarrow{\mathcal{L}} N(0, 1) \text{ as } n \to \infty. \tag{7.4}$$

Lemma 7.5. *(Newman (1984))*

$$|\phi - \prod_{j=1}^{n} \phi_j| \leq 2 \sum_{1\leq k<l\leq n} |r_k||r_l| Cov(\tilde{Y}_k, \tilde{Y}_l) \tag{7.5}$$

where ϕ *and* ϕ_j *are given by* $\phi = E(exp(i\sum_{j=1}^{n} r_j Y_j))$ *and* $\phi_j = E(exp(ir_j Y_j))$*.*

In the following, $h_j \ll_A \tilde{h}_j$ if $h_j \ll \tilde{h}_j$ and both h_j and $\tilde{h}_j$ depend only on $x_m's$ with $m \in A$. A is a finite subset of $\{k, k \geq 1\}$.

Lemma 7.6. *(Newman (1984)) Let* $h_1 \ll_A \tilde{h}_1$ *and* $h_2 \ll_A \tilde{h}_2$*. Then,*

$$|Cov(h_1(X_1, X_2, ...), h_2(X_1, X_2, ...))| \leq Cov(\tilde{h}_1(X_1, X_2, ...), \tilde{h}_2(X_1, X_2, ...)) \tag{7.6}$$

Lemma 7.7. *(Matula (2001)) Let* $E(Y_n) = 0$ *for* $n \in \mathbb{N}$*, then for every* $\epsilon > 0$*,*

$$P[\max_{1\leq k\leq n} |S_k| \geq \epsilon] \leq 160\epsilon^{-2} Var\tilde{S}_n. \tag{7.7}$$

Here, $S_k = \sum_{i=1}^{k} Y_i$*,* $1 \leq k \leq n$ *and* $\tilde{S}_n = \sum_{i=1}^{n} \tilde{Y}_i$*.*

7.2 Proofs

7.2.1 Proofs of Theorems stated in Section 2

The proofs require the following lemmas.

Lemma 7.8. *For* $i, j \geq 0$

$$Cov[S_i(\ell), S_j(\ell)] \leq Cov[\tilde{S}_i(\ell), \tilde{S}_j(\ell)], \tag{7.8}$$

$$Cov[|S_i(\ell)|, |S_j(\ell)|] \leq Cov[\tilde{S}_i(\ell), \tilde{S}_j(\ell)], \tag{7.9}$$

where $\tilde{S}_j(\ell) = \sum_{i=j+1}^{j+\ell} \tilde{Y}_i$*.*

Proof. $S_i(\ell) \ll \tilde{S}_i(\ell)$, and $|S_i(\ell)| \ll \tilde{S}_i(\ell)$, for all $i \geq 0$. Using Lemma 7.6, we get (7.8)−(7.9). ☐

Lemma 7.9. *Under the conditions of Theorem* 2.1*,*

$$\lim_{\ell\to\infty} \frac{1}{\ell}\Big[\sum_{j=0}^{\ell-1} Cov\Big\{\Big|\frac{S_0(\ell)}{\sqrt{\ell}}\Big|, \Big|\frac{S_j(\ell)}{\sqrt{\ell}}\Big|\Big\}\Big] = \int_0^1 Cov[|\sigma_g W(1)|, |\sigma_g(W(1+t) - W(t))|]dt \qquad (7.10)$$

where, $\{W(t); t \geq 0\}$ *is the standard Wiener process.*

Proof. An invariance principle for $\frac{S_{[nt]}}{\sqrt{n}\sigma_g}$ is needed. This follows using Lemma 7.4 and Lemma 7.7. Rest follows as Lemma 2.2 in Peligrad and Suresh (1995). ☐

Lemma 7.10. *Under the conditions of Theorem* 2.1*,*

$$\lim_{n\to\infty} \frac{1}{n\ell}\Big[\sum_{j=0}^{n-\ell} Var|\frac{S_j(\ell)}{\sqrt{\ell}}|\Big] = 2\int_0^1 \sigma_g^2 Cov[|W(1)|, |W(1+t) - W(t)|]dt \qquad (7.11)$$

where, $\{W(t); t \geq 0\}$ *is the standard Wiener process.*

Proof. Using inequalities (7.8) − (7.9), and Lemma 7.9, rest of the proof is as Lemma 2.3 of Peligrad and Suresh (1995). ☐

Lemma 7.11. *Let* $\{W(t); t \geq 0\}$ *be the standard Wiener process. Then,*

$$\int_0^1 Cov[|\sigma_g W(1)|, |\sigma_g(W(1+t) - W(t))|]dt = \frac{\sigma_g^2}{4\pi}(3\pi - 8)$$

Proof. The proof is elementary. ☐

Proof of Theorem 2.1**.**

Proof. Using Lemma 7.4, and Lemma 7.10, the proof of the theorem follows as the proof of Theorem 1.1 in Peligrad and Suresh (1995). ☐

Proof of Theorem 2.2**.**

Proof. Let C be a generic positive constant in the sequel. Let $S_n = \sum_{i=1}^n Y_i$ and $\tilde{S}_n = \sum_{i=1}^n \tilde{Y}_i$. For any $\epsilon > 0$,

$$\sum_{n=1}^{\infty} P\Big[\frac{|(S_{2^n} - 2^n\mu)|}{2^{n/2}log2^n} \geq \epsilon\Big] \leq C\epsilon^{-2}\sum_{n=1}^{\infty} 2^{-n}n^{-2}Var(\tilde{S}_{2^n}) \leq C\epsilon^{-2}\sum_{n=1}^{\infty}\frac{1}{n^2} < \infty. \qquad (7.12)$$

The second inequality follows as $S_n \ll \tilde{S}_n$.

$$\sum_{n=1}^{\infty} P\Big[\frac{1}{2^{n/2}log2^n}\max_{2^n<k\leq 2^{n+1}}|S_k - k\mu - (S_{2^n} - 2^n\mu)| \geq \epsilon\Big] \leq C\epsilon^{-2}\sum_{n=1}^{\infty}\frac{1}{2^n n^2}Var(\tilde{S}_{2^{n+1}} - \tilde{S}_{2^n})$$

$$\leq C\epsilon^{-2}\sum_{n=1}^{\infty}\frac{1}{2^n n^2}Var(\tilde{S}_{2^{n+1}}) \leq C\epsilon^{-2}\sum_{n=1}^{\infty}\frac{1}{n^2} < \infty. \qquad (7.13)$$

The second inequality follows from Lemma 7.7. Using Borel-Cantelli lemma we have,

$$\lim_{n\to\infty} \frac{|S_n - n\mu|}{n^{1/2} logn} = 0 \text{ a.s.}$$

Using the central limit theorem for S_n, (Lemma 7.4) which holds under the conditions of Theorem 2.1, we have,

$$E\Big|\frac{S_j(\ell) - \ell\mu}{\sqrt{\ell}}\Big| \to \sigma_g\sqrt{2/\pi} \text{ as } \ell \to \infty.$$

So, we just need to prove,

$$\frac{1}{n-\ell+1}\sum_{j=0}^{n-\ell}\Big|\frac{S_j(\ell)-\ell\mu}{\sqrt{\ell}}\Big| - E\Big|\frac{S_j(\ell)-\ell\mu}{\sqrt{\ell}}\Big| \to 0 \text{ a.s. as } n \to \infty. \tag{7.14}$$

As $Y_k - E(Y_k) \ll \tilde{Y}_k - E(Y_k)$ for all $k \in \mathbb{N}$, without loss of generality, we can assume $\mu = 0$. Observe that under (2.2) and solving as Lemma 7.10, we have,

$$Var\Big[\sum_{j=0}^{n-\ell}\frac{\tilde{S}_j(\ell)}{\sqrt{\ell}}\Big] = O(\frac{n^2}{(logn)^2}) \text{ as } n \to \infty. \tag{7.15}$$

Now, $|S_j(\ell)|$ is a sequence in j and is dominated by the associated sequence $\tilde{S}_j(\ell)$, as $|S_j(\ell)| \ll \tilde{S}_j(\ell)$. From (7.15) and Lemma 7.7, for any $\epsilon > 0$,

$$P\Big[\max_{\ell<n\leq 2^{j+1}}\frac{1}{n-\ell+1}\Big|\frac{\sum_{j=0}^{n-\ell}|S_j(\ell)|-E|S_j(\ell)|}{\sqrt{\ell}}\Big| > \epsilon\Big] = O(\frac{1}{j^2}) \text{ as } j \to \infty.$$

For a large $N \in \mathbb{N}$,

$$\begin{aligned} &P\Big[\bigcup_{n>N}\frac{1}{n-\ell+1}\Big|\frac{\sum_{j=0}^{n-\ell}|S_j(\ell)|-E|S_j(\ell)|}{\sqrt{\ell}}\Big| > \epsilon\Big] \\ &\leq \sum_{j=[log_2 N]-1} P\Big[\max_{\ell\leq n\leq 2^{j+1}}\frac{1}{n-\ell+1}\Big|\frac{\sum_{j=0}^{n-\ell}|S_j(\ell)|-E|S_j(\ell)|}{\sqrt{\ell}}\Big| > \epsilon\Big]. \end{aligned} \tag{7.16}$$

Using Borel-Cantelli Lemma, (7.14) is proved. □

Proof of Theorem 2.3.

Proof. Without loss of generality, assume $\mu = 0$. $\frac{1}{\sqrt{n\ell}}\sum_{j=0}^{n-\ell}(|S_j(\ell)|-E(|S_j(\ell)|)$ is divided into (k_n+1) blocks where $\left[\frac{n-\ell}{\ell}\right] = k_n$. Define, for all $1 \leq i \leq k_n$,

$$V_i = \sum_{j=0}^{\ell-1}\{|S_{(i-1)\ell+j}(\ell)|-E|S_{(i-1)l+j}(\ell)|\}, \text{ and } \tilde{V}_i = \sum_{j=0}^{\ell-1}\{\tilde{S}_{(i-1)\ell+j}(\ell) - E(\tilde{S}_{(i-1)\ell+j}(\ell))\}.$$

$V_i \ll \tilde{V}_i$, $1 \le i \le k_n$. Using Lemma 7.5, and stationary,

$$|E(e^{(it/\sqrt{n\ell})\sum_{j=1}^{k_n} V_j}) - \prod_{j=1}^{k_n} E(e^{(it/\sqrt{n\ell})V_j})| \le \frac{2t^2}{n\ell^2} \sum_{1\le i<j\le k_n} Cov(\tilde{V}_i, \tilde{V}_j) \tag{7.17}$$

$$\le \frac{t^2}{n\ell^2}\Big[Var(\sum_{i=1}^{k_n}\tilde{V}_i) - \sum_{i=1}^{k_n} Var(\tilde{V}_i)\Big] \to 0 \text{ as } n \to \infty.$$

Under the conditions of Theorem 2.1, the variance of the remainder is,

$$Var\Big[\frac{1}{\sqrt{n\ell}}\sum_{j=k_n\ell}^{n-l}|S_j(\ell)|\Big] \le Var\Big[\frac{1}{\sqrt{n\ell}}\sum_{j=0}^{\ell-1}\tilde{S}_j(\ell)\Big] \to 0 \text{ as } n \to \infty. \tag{7.18}$$

Also by Lemma 7.4, under condition (2.2), a central limit theorem for S_n is implied. Using these, proof of (2.4) follows similarly as proof of Theorem 1.3 in Peligrad and Suresh (1995). Next, observe,

$$\sqrt{\frac{n}{\ell}}\Big|B_{PS_n} - \frac{1}{n-\ell+1}\sum_{j=0}^{n-\ell}\frac{|S_j(\ell)|}{\sqrt{\ell}}\Big| \le \frac{S_n}{\sqrt{n}}.$$

Using (2.4) and Lemma 7.4, (2.5) follows. □

Proof of Theorem 2.7

Proof.

$$\Big|B(U_n)_{PS} - 2\sigma_U\sqrt{\frac{2}{\pi}}\Big|$$
$$= \Big|\frac{1}{(n-\ell+1)}\sum_{i=1}^{n-\ell+1}\Big|\sqrt{\ell}\binom{\ell}{2}^{-1}\sum_{i\le j<k\le i+\ell-1}\rho(X_j,X_k) - \sqrt{\ell}U_n(\rho)\Big| - 2\sigma_U\sqrt{\frac{2}{\pi}}\Big|$$
$$\le J_1 + J_2 \text{ (say)}.$$

where,

$$J_1 = \Big|\frac{1}{(n-\ell+1)}\sum_{i=1}^{n-\ell+1}\Big|\frac{2}{\sqrt{\ell}}\sum_{j=i}^{i+\ell-1}h^{(1)}(X_j) - 2\frac{\sqrt{\ell}}{n}\sum_{j=1}^{n}h^{(1)}(X_j)\Big| - 2\sigma_U\sqrt{\frac{2}{\pi}}\Big|.$$

$$J_2 = \frac{1}{(n-\ell+1)}\sum_{i=1}^{n-\ell+1}\Big|\frac{\sqrt{\ell}}{\binom{\ell}{2}}\sum_{i\le j<k\le i+\ell-1}h^{(2)}(X_j,X_k) - \frac{\sqrt{\ell}}{\binom{n}{2}}\sum_{1\le j<k\le n}h^{(2)}(X_j,X_k)\Big|.$$

Using Theorem 2.1, we get, $J_1 \xrightarrow{p} 0$, as $n \to \infty$. To prove $J_2 \xrightarrow{p} 0$. Under the conditions of Lemma 2.5, as $n \to \infty$,

$$\frac{\ell}{\binom{\ell}{2}^2}E\Big(\sum_{i\le j<k\le i+\ell-1}h^{(2)}(X_j,X_k)\Big)^2 \to 0, \text{ for all, } i = 1,2,\cdots,n-\ell+1.$$

$$\frac{\ell}{\binom{n}{2}^2}E\Big(\sum_{1\le j<k\le n}h^{(2)}(X_j,X_k)\Big)^2 \to 0.$$

Hence, $B(U_n)_{PS} \xrightarrow{p} 2\sigma_U\sqrt{\frac{2}{\pi}}$ as $n \to \infty$, or $\frac{\pi}{2}B^2(U_n)_{PS} \xrightarrow{p} 4\sigma_U^2$. □

7.2.2 Proofs of Theorems stated in Section 3

Proof of Theorem 3.1

Proof. Let $N = n - \ell + 1$, and, define $\mu_N = \frac{\sum_{j=1}^{N} V_j}{N}$, where

$$V_j = \frac{Y_j + Y_{j+1} + \cdots + Y_{j+\ell-1}}{\ell}, \text{ for all } j = 1, 2, \cdots, N.$$

Note that,

$$B^2_{PR_n} = \ell\Big(\frac{1}{N}\sum_{i=1}^{N} V_i^2 - \mu_N^2\Big) + \frac{1}{N}\sum_{i=1}^{N}\Big(\ell\mu_N^2 + \ell\bar{Y}_n^2 - 2\sum_{j=i}^{i+\ell-1} Y_j\bar{Y}_n\Big) = I_1 + I_2. \quad (7.19)$$

Note that, I_1 is the value of $Var_\star(T_n^\star)$, when $T_n^\star$ is based on observations from $\{X_i, 1 \leq i \leq n\}$ using Moving Block Bootstrap. Under the given conditions,

$$I_1 \xrightarrow{p} \sigma_g^2, \text{ as } n \to \infty. \quad (7.20)$$

The proof of (7.20) follows similarly as the proof of Theorem 3.1 in Garg and Dewan (2016). Assume without loss of generality that $E(Y_1) = 0$. Under the given conditions, $\ell\bar{Y}_n^2 \xrightarrow{p} 0$ and $\ell\mu_N^2 \xrightarrow{p} 0$, as $n \to \infty$. Also,

$$\frac{1}{N}\sum_{i=1}^{N}\Big(2\sum_{j=i}^{i+\ell-1} Y_j\bar{Y}_n) = 2\ell\bar{Y}_n\mu_N \xrightarrow{p} 0. \quad (7.21)$$

Using (7.19)-(7.21), we get (3.2). □

To discuss the order of variance we need the following lemma.

Lemma 7.12. *Define,* $S_j(\ell) = \sum_{k=j+1}^{j+\ell} Y_k$, $j = 0, \cdots, n-\ell$. *Assume* $E(Y_1) = 0$. *Suppose* $E\Big(\frac{S_0(\ell)}{\sqrt{\ell}}\Big)^4 < \infty$, *for all* $\ell \in \mathbb{N}$, *and* $\sum_{j=1}^{\infty} Cov(\tilde{Y}_1, \tilde{Y}_j) < \infty$. *Then,*

$$\lim_{\ell\to\infty}\frac{1}{\ell}\Big[\sum_{j=0}^{\ell-1} Cov\Big\{\Big(\frac{S_0(\ell)}{\sqrt{\ell}}\Big)^2, \Big(\frac{S_j(\ell)}{\sqrt{\ell}}\Big)^2\Big\}\Big] = \int_0^1 Cov[(\sigma_g W(1))^2, (\sigma_g(W(1+t) - W(t)))^2]dt \quad (7.22)$$

where, $\{W(t); t \geq 0\}$ *is the standard Wiener process.*

Proof. The proof follows similarly as that of Lemma 7.9. □

Proof of Theorem 3.2

Proof. Let C be a generic positive constant in the sequel. Without loss of generality assume that $E(Y_1) = 0$. Observe that, using Lemmas 7.1, 7.2 and 7.5, along with the usual truncation technique, for $j = 1, \cdots, n-\ell$, we get,

$$Cov\Big(\Big(\frac{S_0(\ell)}{\sqrt{\ell}}\Big)^2, \Big(\frac{S_j(\ell)}{\sqrt{\ell}}\Big)^2\Big) \leq C Cov\Big(\Big(\frac{\tilde{S}_0(\ell)}{\sqrt{\ell}}\Big), \Big(\frac{\tilde{S}_j(\ell)}{\sqrt{\ell}}\Big)\Big)^{\nu/3(4+\nu)}, \quad (7.23)$$

where $\tilde{S}_j(\ell), j = 0, 1, \cdots, n-\ell$ are defined in Lemma 7.8. Hence,

$$\begin{aligned}
&\lim_{n\to\infty} Var\Bigg(B^2_{PR_n}\Bigg) \le \lim_{n\to\infty} CVar\Bigg(\frac{1}{(n-\ell+1)}\sum_{j=0}^{n-\ell}\Big(\frac{S_j(\ell)}{\sqrt{\ell}}\Big)^2\Bigg) \\
&\le \lim_{n\to\infty}\frac{1}{(n-\ell+1)^2} C\Big[(n-\ell+1)E\Big(\frac{S_0(\ell)}{\sqrt{\ell}}\Big)^4 + 2\sum_{0\le i<j\le n-\ell} Cov\Big(\Big(\frac{S_i(\ell)}{\sqrt{\ell}}\Big)^2, \Big(\frac{S_j(\ell)}{\sqrt{\ell}}\Big)^2\Big)\Big] \\
&= \lim_{n\to\infty}[I_1 + 2I_2]. \qquad (7.24)
\end{aligned}$$

Note that, $I_1 = O(\frac{1}{n})$, which tends to 0 as $n \to \infty$. As done in Peligrad and Suresh (1995), we can decompose I_2 as following: ($u = u_n$, such that $u_n \to \infty$ and $u_n = o(\ell_n)$ as $n \to \infty$)

$$\begin{aligned}
I_2 &\le \frac{1}{(n-\ell+1)^2}\Bigg[\sum_{i=0}^{n-2\ell-1}\sum_{j=i+1}^{i+\ell-1} Cov\Big(\Big(\frac{S_i(\ell)}{\sqrt{\ell}}\Big)^2, \Big(\frac{S_j(\ell)}{\sqrt{\ell}}\Big)^2\Big) + \sum_{i=0}^{n-2\ell}\sum_{j=i+\ell}^{i+\ell+u} Cov\Big(\Big(\frac{S_i(\ell)}{\sqrt{\ell}}\Big)^2, \Big(\frac{S_j(\ell)}{\sqrt{\ell}}\Big)^2\Big) \\
&+ \sum_{i=0}^{n-2\ell}\sum_{j=i+u+\ell+1}^{n-\ell} Cov\Big(\Big(\frac{S_i(\ell)}{\sqrt{\ell}}\Big)^2, \Big(\frac{S_j(\ell)}{\sqrt{\ell}}\Big)^2\Big) + \sum_{i=n-2\ell+1}^{n-\ell-1}\sum_{j=i+1}^{n-\ell} Cov\Big(\Big(\frac{S_i(\ell)}{\sqrt{\ell}}\Big)^2, \Big(\frac{S_j(\ell)}{\sqrt{\ell}}\Big)^2\Big)\Bigg] \\
&= \frac{1}{(n-\ell+1)^2}\Bigg[J_1 + J_2 + J_3 + J_4\Bigg]. \qquad (7.25)
\end{aligned}$$

From Lemma 7.12, we have $\frac{J_1}{(n-\ell+1)^2} = O(\frac{\ell}{n})$ as $n \to \infty$. Using (7.23),

$$\begin{aligned}
J_2 &\le n\sum_{j=\ell}^{\ell+u} Cov\Big(\Big(\frac{S_0(\ell)}{\sqrt{\ell}}\Big)^2, \Big(\frac{S_j(\ell)}{\sqrt{\ell}}\Big)^2\Big) \le n\sum_{j=\ell}^{\ell+u} Cov\Big(\Big(\frac{\tilde{S}_0(\ell)}{\sqrt{\ell}}\Big), \Big(\frac{\tilde{S}_j(\ell)}{\sqrt{\ell}}\Big)\Big)^{\nu/3(4+\nu)} \\
&\le nu\frac{\ell}{\ell^{\nu/3(4+\nu)}}\sum_{j=2}^{\infty} Cov(\tilde{Y}_1, \tilde{Y}_j)^{\nu/3(4+\nu)}, \qquad (7.26)
\end{aligned}$$

i.e. $\frac{J_2}{(n-\ell+1)^2} \to 0$, as $n \to \infty$. Similarly, $\frac{J_3}{(n-\ell+1)^2} \to 0$, and $\frac{J_4}{(n-\ell+1)^2} \to 0$, as $n \to \infty$. Hence, $I_2 = O(\frac{\ell}{n})$ as $n \to \infty$ and (3.5) is proved. □

Proof of Theorem 3.4

Proof. The result follows similarly as the proof of Theorem 2.7. □

7.2.3 Proofs of Theorems stated in Section 4

Proof of Theorem 4.1

Proof. The result follows similarly as the proof of Theorem 3.1 in Garg and Dewan (2016). □

Proof of Theorem 4.2

Proof. The result follows similarly as the proof of Lemma 3.2. □

Proof of Theorem 4.3

Proof. The result follows similarly as the proof of Theorem 2.7. □

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