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On the argmin of a drifted Brownian Motion

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ON THE ARGMIN OF A DRIFTED BROWNIAN MOTION

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Abstract. Consider a two-sided Brownian Motion to which an even continuous drift is added. A further drift which is odd, continuous and increasing is added, such that the infimum of the resultant process over the real line is finite and achieved. It is shown in this note that the positive part of the argmin is stochastically dominated by the negative part of the same.

1. Introduction

The following question came up during an informal discussion over coffee at the Indian Statistical Institute. Let $(W(t) : t \in \mathbb{R})$ be a two-sided Brownian motion with $W(0) = 0$. Let $\psi : \mathbb{R} \to \mathbb{R}$ be a continuous even function such that $\psi(t)/t \to \infty$ as $t \to \infty$. Let $A : [0, \infty) \to [0, \infty)$ be non-decreasing with $A(0) = 0$, the function not being identically zero, and $A(t)/t \to 0$ as $t \to \infty$. For all $t \in \mathbb{R}$, define $\mathcal{A}(t) = A(t)\mathbf{1}(t \geq 0) - A(-t)\mathbf{1}(t < 0)$ and consider the drifted process $Z(t) := W(t) + \psi(t) + \mathcal{A}(t)$.
Question. Denoting by M^+ and M^- the positive and negative parts of

$$M := \arg\min_{t\in\mathbb{R}} Z(t)\,, \tag{1}$$

respectively, is M^+ stochastically dominated by M^-?

In order to make sense out of this question, we first need to show that the argmin in (1) is well defined. However, this is guaranteed by [1]. It has been shown in this note that the answer to the above question is "yes", and in fact, the stochastic domination is strict. This is recorded in the following result.

Theorem 1. *Under the conditions mentioned above, the argmin in* (1) *is uniquely defined. If F_{M^+} and F_{M^-} denote the cumulative distribution functions of M^+ and M^- respectively, then*

$$F_{M^+}(x) \geq F_{M^-}(x) \text{ for all } x \in \mathbb{R}\,, \tag{2}$$

the inequality being strict for some x.

If the inequality in (2) is strict at some point, then right continuity of the cumulative distribution function will make it strict on an interval.

2. Proof

The proof of Theorem 1 will be executed in two steps. We start with showing the relaxed inequality (2).

Proposition 1. *The argmin in* (1) *is well defined, and* (2) *holds.*

Proof. The assumptions on the deterministic functions A and ψ ensure that

$$\lim_{t\to\pm\infty} \frac{1}{t} \left(\psi(t) + \mathcal{A}(t)\right) = \infty\,,$$

and hence almost surely it holds that

$$\lim_{t\to\pm\infty} Z(t) = \infty\,.$$

This shows that

$$\inf_{t\in\mathbb{R}} Z_t > -\infty\,,$$

and the infimum is achieved. Lemma 2.6 of [1] ensures that the infimum cannot be achieved at more than one point. Hence, the definition of M in (1) is unambiguous.

Define for every $t \in \mathbb{R}$,

$$\begin{aligned} \alpha(t) &:= W(t) + \psi(t) - \mathcal{A}(t)\,, && (3)\\ \beta(t) &:= 2\mathcal{A}(t)\,. && (4) \end{aligned}$$

Clearly,

$$M = \arg\min_{t\in\mathbb{R}} \left(\alpha(t) + \beta(t)\right)\,.$$

Let

$$M^* := \arg\min_{t\in\mathbb{R}} \alpha(t)\,,$$

which is also unambiguously defined because of similar arguments as above. Lemma 2.1 of [2] implies that

$$M^* \ge M \text{ almost surely}\,. \qquad (5)$$

The facts that ψ and $\mathcal{A}$ are even and odd functions respectively, and that $W(-t) \stackrel{d}{=} W(t)$, imply that

$$\left(Z(-t) : t \in \mathbb{R}\right) \stackrel{d}{=} \left(\alpha(t) : t \in \mathbb{R}\right)\,.$$

Therefore,

$$-M \stackrel{d}{=} M^*\,. \qquad (6)$$

Therefore, for all $x \in [0, \infty)$,

$$\begin{aligned} P(M^- > x) &= P(M < -x)\\ \text{(by (6))} &= P(M^* > x)\\ \text{(by (5))} &\ge P(M > x)\\ &= P(M^+ > x)\,. \end{aligned}$$

Thus (2) follows. □

For the next step which is the strict inequality we shall need the following lemma.

Lemma 1. *Let α be a drifted Brownian motion as in (3). Then for all $b > 0$ and real u, v such that $-\infty < u < \min(0, v)$,*

$$P\left(\min_{t\ge b} \alpha(t) - \alpha(b) \in (u, v)\right) > 0\,. \qquad (7)$$

Furthermore, for all $\varepsilon > 0$ *and* $0 < s \le r < \infty$,

$$P\left(\min_{s\le t\le r}\alpha(t)-\alpha(s) > -\varepsilon\right) > 0\,, \tag{8}$$

and

$$P\left(\min_{t\le s}\alpha(t)-\alpha(s) > -\varepsilon\right) > 0\,. \tag{9}$$

Proof. We start with proving (7). Fix b, u, v as required. Without loss of generality, we assume that $-\infty < u < v \le 0$. Let $\varepsilon = v-u$. From continuity of the drift term $d(t) := \psi(t) + \mathcal{A}(t)$, it follows that there exists $\delta > 0$ such that

$$\max_{b\le t\le b+\delta}|d(t)-d(b)| < \frac{\varepsilon}{3}\,.$$

Since $\alpha(t)\to\infty$ a.s. as $t\to\infty$, it is immediate that

$$\inf_{t\ge b+\delta}\alpha(t) > -\infty \text{ a.s.}\,,$$

which is why the infimum is achieved, and therefore can be replaced by minimum. Thus,

$$\min_{t\ge b+\delta}\alpha(t)-\alpha(b+\delta) > -\infty \text{ a.s.}\,,$$

and hence there exists $K > 0$ such that

$$P\left(\min_{t\ge b+\delta}\alpha(t)-\alpha(b+\delta) > -K\right) > 0\,. \tag{10}$$

The choice of δ implies that

$$\begin{aligned}
& P\left(\min_{b\le t\le b+\delta}\alpha(t)-\alpha(b)\in(u,v), \alpha(b+\delta)-\alpha(b) > K\right)\\
\ge\; & P\Big(\min_{b\le t\le b+\delta}W(t)-W(b)\in(u+\frac{\varepsilon}{3}, v-\frac{\varepsilon}{3}), W(b+\delta)-W(b) > K+\varepsilon\Big)\\
=\; & P\left(\min_{0\le t\le\delta}W(t)\in(u+\frac{\varepsilon}{3}, v-\frac{\varepsilon}{3}),\, W(\delta) > K+\varepsilon\right)\\
=\; & P\left(\min_{0\le t\le\delta}W(t) < v-\frac{\varepsilon}{3},\, W(\delta) > K+\varepsilon\right)\\
& -P\left(\min_{0\le t\le\delta}W(t) \le u+\frac{\varepsilon}{3},\, W(\delta) > K+\varepsilon\right)\\
=\; & P\left(M-2v+\frac{5}{3}\varepsilon \le W(\delta) \le K-2u+\frac{\varepsilon}{3}\right) > 0\,,
\end{aligned}$$

the equality in the last line following from the fact below, which can be proved by the reflection principle. For $x, y\in\mathbb{R}$ such that $x < \min\{y, 0\}$, it holds that

$$P\left(\min_{0\le t\le\delta}W(t) < x, W(\delta) > y\right) = P\left(W(\delta) > y-2x\right)\,.$$

Notice that

$$
\begin{aligned}
&\left[\min_{t\ge b}\alpha(t)-\alpha(b)\in(u,v)\right]\\
\supset\ &\left[\min_{b\le t\le b+\delta}\alpha(t)-\alpha(b)\in(u,v),\ \min_{t\ge b+\delta}\alpha(t)-\alpha(b)>0\right]\\
\supset\ &\Bigg[\min_{b\le t\le b+\delta}\alpha(t)-\alpha(b)\in(u,v),\ \alpha(b+\delta)-\alpha(b)>K,\\
&\quad\min_{t\ge b+\delta}\alpha(t)-\alpha(b+\delta)>-K\Bigg]\,.
\end{aligned}
$$

Since $b+\delta\ge 0$, the σ-fields $\sigma(\alpha(t):t\le b+\delta)$ and $\sigma(\alpha(t)-\alpha(b+\delta):t\ge b+\delta)$ are independent, and therefore,

$$
\begin{aligned}
&P\left(\min_{t\ge b}\alpha(t)-\alpha(b)\in(u,v)\right)\\
\ge\ &P\left(\min_{b\le t\le b+\delta}\alpha(t)-\alpha(b)\in(u,v),\alpha(b+\delta)-\alpha(b)>K\right)\\
&\quad\times P\left(\min_{t\ge b+\delta}\alpha(t)-\alpha(b+\delta)>-K\right)\\
>\ &0\,,
\end{aligned}
$$

the last line following from (10). This establishes (7).

The claim (8) follows from (7) by taking $u=-\varepsilon$ and $v=0$. For (9), note that for $s>0$, it follows by conditioning on $\sigma(\alpha(t):0\le t\le s)$ that

$$
\begin{aligned}
&P\left(\min_{t\le s}\alpha(t)-\alpha(s)>-\varepsilon\right)\\
=\ &P\left(\min_{0\le t\le s}\alpha(t)-\alpha(s)>-\varepsilon\right)\int_{-\infty}^{\varepsilon}P\left(\min_{t\le 0}\alpha(t)>x-\varepsilon\right)P(\alpha(s)\in dx)\\
>\ &0\,,
\end{aligned}
$$

the last line following from the fact that the integrand is positive for every $x<\varepsilon$ which essentially is a restatement of (7). This completes the proof. □

Now we shall complete the proof of Theorem 1 by proving the strict inequality in the following proposition.

Proposition 2. *The inequality* (2) *is strict for at least one* x.

Proof. Proposition 1 and its proof show that it suffices to show that (5) is strict with positive probability, that is,

$$P(M\neq M^*)>0\,.$$

To that end, let α and β be as in (3) and (4) respectively. Since A is not identically zero, β is non-constant on $[0,\infty)$. Continuity implies that there exist $0<a<b<\infty$ such that $\beta(a)<\beta(b)$. Denoting

$$\varepsilon=\beta(b)-\beta(a)\,,$$

our first claim is that

$$P(M \neq M^*) \quad \geq \quad P\left(\alpha(a) - \min_{t\in\mathbb{R}} \alpha(t) < \varepsilon,\, M^* \geq b\right). \tag{11}$$

In order to see this, assuming that the event in the right hand side above occurs, observe that

$$\begin{aligned} \alpha(a) - \alpha(M^*) &< \varepsilon \\ &= \beta(b) - \beta(a) \\ &\leq \beta(M^*) - \beta(a). \end{aligned}$$

Thus,

$$\alpha(a) + \beta(a) < \alpha(M^*) + \beta(M^*),$$

showing that $M \neq M^*$. Thus, (11) follows.

Now we proceed towards proving that the right hand side of (11) is positive. To that end, notice that

$$\begin{aligned} & P\left(\alpha(a) - \min_{t\in\mathbb{R}} \alpha(t) < \varepsilon,\, M^* \geq b\right) \qquad (12) \\ = \ & P\Bigg(\alpha(a) < \min_{t\leq a} \alpha(t) + \varepsilon,\ \min_{a\leq t\leq b} \alpha(t) > \alpha(a) - \varepsilon, \\ & \min_{t\geq b} \alpha(t) \in \left(\alpha(a) - \varepsilon, \min_{t\leq b} \alpha(t)\right)\Bigg). \end{aligned}$$

For calculating the right hand side, we recall that

$$\sigma\left(\alpha(t) - \alpha(s) : s \leq t < \infty\right) \text{ and } \sigma\left(\alpha(t) : -\infty < t \leq s\right)$$

are independent for all $s \geq 0$. Using this independence for $s = a$ and $s = b$, and noting that $0 < a < b$, it follows that the probability in (12) equals

$$P\left(\alpha(a) < \min_{t\leq a} \alpha(t) + \varepsilon\right) P\left(\min_{a\leq t\leq b} \big(\alpha(t) - \alpha(a)\big) > -\varepsilon\right) \tag{13}$$

$$\begin{aligned} \times \int_{[\alpha(a)<\min_{t\leq b}\alpha(t)+\varepsilon]} & P\Bigg(\min_{t\geq b} \big(\alpha(t) - \alpha(b)\big) \in \qquad (14) \\ & \left(\alpha(a) - \alpha(b) - \varepsilon, \min_{t\leq b} \alpha(t) - \alpha(b)\right) \Bigg| \sigma\big(\alpha(t) : t \leq b\big)\Bigg)(\omega) P(d\omega). \end{aligned}$$

The positivity of the two terms in (13) follow from (9) and (8) respectively. To show that the integral in (14) is positive, fix $\omega \in [\alpha(a) < \min_{t\leq b} \alpha(t) + \varepsilon]$. Defining

$$\begin{aligned} u &= \alpha(a) - \alpha(b) - \varepsilon, \\ v &= \min_{t\leq b} \alpha(t) - \alpha(b), \end{aligned}$$

it is immediate that u, v are measurable w.r.t. $\sigma\big(\alpha(t) : t \le b\big)$, and that $u(\omega) < \min\big(0, v(\omega)\big)$. Using once again the independence of the above σ-field and $\sigma(\alpha(t) - \alpha(b) : t \ge b)$, it follows that

$$\begin{aligned}
& P\left(\min_{t \ge b} \big(\alpha(t) - \alpha(b)\big) \in \left(\alpha(a) - \alpha(b) - \varepsilon, \min_{t \le b} \alpha(t) - \alpha(b)\right) \Big| \right. \\
& \qquad \left. \sigma\big(\alpha(t) : t \le b\big) \right)(\omega) \\
= \; & P\left(\min_{t \ge b} \big(\alpha(t) - \alpha(b)\big) \in (u(\omega), v(\omega)) \right) \\
> \; & 0\,,
\end{aligned}$$

the inequality in the last line being implied by (7). Therefore, the integral in (14) is positive, and hence so is the left hand side of (12). This along with (11) completes the proof. □

Clearly, Propositions 1 and 2 complete the proof of Theorem 1.

References

[1] J. Kim and D. Pollard. Cube root asymptotics. *The Annals of Statistics*, 18(1):191–219, 1990.

[2] V. N. Kulikov and H. P. Lopuhaa. The behavior of the NPMLE of a decreasing density near the boundaries of the support. *The Annals of Statistics*, 34(2):742–768, 2006.

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