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Pranabesh Das, Shanta Laishram and N. Saradha

Indian Statistical Institute, Delhi Centre
7, SJSS Marg, New Delhi–110 016, India

VARIATIONS OF ERDŐS- SELFRIDGE SUPERELLIPTIC CURVES AND THEIR RATIONAL POINTS

PRANABESH DAS, SHANTA LAISHRAM AND N. SARADHA

ABSTRACT. For the superelliptic curves of the form
$$(x+1)\cdots(x+i-1)(x+i+1)\cdots(x+k)=y^\ell$$
with $k\geq 3$ and $\ell\geq 2$, a prime, for some values of i in the range $2\leq i\leq k$, we bound $\ell<e^{3^k}$, as in a recent paper of Bennett and Siksek.

1. INTRODUCTION

Let $x\in\mathbb{Q}$ and $k\geq 2$ be an integer. For any integer $n\geq 1$, let $P(n)$ denote the greatest prime factor of n and take $P(1)=1$. Put
$$\Delta_0=(x+1)\cdots(x+k)$$
and for $1\leq i\leq k$, let
$$\Delta_i=(x+1)\cdots(x+i-1)(x+i+1)\cdots(x+k).$$
In a recent paper, Bennett and Siksek [2] considered rational solutions of
$$\Delta_0=y^\ell \tag{1}$$
in x and y with $\ell\geq 2$, a prime. They showed that if (1) holds, then
$$\ell\leq e^{3^k}. \tag{2}$$
This can be considered as a rational analogue to the Schinzel-Tijdeman theorem on integral solutions to the superelliptic equation $f(x)=y^l$ where $f(x)$ is a polynomial. In this paper, we extend the result of Bennett and Siksek to the equation
$$\Delta_i=y^\ell, 1\leq i\leq k.$$

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Note that $\Delta_1 = y^\ell$ and $\Delta_k = y^\ell$ are equations similar to (1) with k replaced by $k-1$ and so (2) holds for these equations. Hence we will consider

$$\Delta_i = y^\ell, 2 \le i \le k-1. \tag{3}$$

Let

$$\theta = \begin{cases} \pi(\frac{k-1}{2}) + 1 & \text{if } k \text{ is odd} \\ \pi(\frac{k}{2}) & \text{if } k \text{ is even and } \frac{k}{2} \text{ is prime} \\ \pi(\frac{k}{2}) + 1 & \text{if } k \text{ is even and } \frac{k}{2} \text{ is not prime.} \end{cases} \tag{4}$$

Let $2 = p_1 < p_2 < \cdots$ denote the sequence of all primes. Then p_θ is the least prime $\ge k/2$. We show the following result.

Theorem 1.1. *Let equation* (3) *be valid with* $2 \le i \le k - p_\theta$ *or* $p_\theta < i < k$. *Then* (2) *holds.*

When $k = 2q$ where q is a prime, we have $p_\theta = q$. Thus we get the following corollary.

Corollary 1.2. *Let equation* (3) *be valid with* $k = 2q$ *where* q *is a prime. Then* (2) *holds.*

Write $\Delta_i = N_i/D_i$ with $\gcd(N_i, D_i) = 1$. The proof of Theorem 1.1 depends on the fact that $P(N_i) \ge k/2$. On the other hand, when $k/3 < P(N_i) < k/2$ and k small, we can show using combinatorial arguments that $\ell < k$. More precisely, we have

Theorem 1.3. *Suppose* (3) *holds with* $9 \le k \le 26$ *and* $P(N_i) < p_\theta$. *Then* $\ell \le k-1$.

From the works of Sander [10], Lakhal and Sander [7], Bennett et al [1] and Győry et al [5], we know that all the rational solutions of (1) can be determined for $k \le 34$. Here we completely solve $k = 3$ and remove the condition on $P(N_i)$ in Theorem 1.3 for $4 \le k \le 8$.

Theorem 1.4. *Suppose* (3) *holds. Then*

$$\ell \le k - 1 \ for\ 4 \le k \le 8.$$

Further, if $k = 3$, *then* $\ell = 2$ *and all the rational solutions are given by*

$$i = 2,\ (x, y) \in \left\{ \left(\frac{(r^2 - 2s^2)^2}{4r^2s^2} - 1, \pm\frac{(r^4 - 4s^4)}{4r^2s^2} \right) \right\}$$

$$i = 2,\ (x, y) \in \left\{ \left(\frac{8r^2s^2}{(r^2 - s^2)^2} - 1, \pm\frac{4rs(r^2 + s^2)}{(r^2 - s^2)^2} \right) \right\}$$

where r, s *are co-prime integers* $r > s > 0$ *of opposite parity.*

As a consequence of the above two theorems, we are able to remove the restriction on i in Theorem 1.1 for small values of k.

Corollary 1.5. *Let equation* (3) *be valid with* $3 \leq k \leq 26$. *Then* (2) *holds.*

By the remarkable result of Erdős and Selfridge [4], it is known that (1) has no integral solutions. Since their result in 1975, several variations of the equation have been considered and integral solutions were investigated. Further, rational solutions to (1) leads to finding perfect powers in certain products in arithmetic progression (see (6) below). We refer to the survey articles of Shorey [13] and [14] for various results in this direction.

2. Preliminaries

While proving Theorems 1.1,1.3 and 1.4, we may assume that ℓ is a prime and

$$\ell > k-1. \tag{5}$$

Write $x = \frac{n}{s}$ and $y = \frac{m}{t}, m \neq 0$ with s, t positive integers and $\gcd(n,s) = \gcd(m,t) = 1$. Then (3) becomes

$$(n+s)\cdots(n+(i-1)s)(n+(i+1)s)\cdots(n+ks) = \frac{s^{k-1}m^\ell}{t^\ell}.$$

Since the left hand side is an integer and $\gcd(n,s) = \gcd(m,t) = 1$, we get $s^{k-1} = t^\ell$. As ℓ is a prime $> k-1$, there is a positive integer d such that $s = d^\ell$ and $t = d^{k-1}$. Thus (3) gives rise to the equation

$$(n+d^\ell)\cdots(n+(i-1)d^\ell)(n+(i+1)d^\ell)\cdots(n+kd^\ell) = m^\ell \tag{6}$$

with $\gcd(n,d) = 1$. We denote by $\Delta_i^{(0)}$ the product on the left hand side of (6) and put

$$\Delta_{i,1}^{(0)} = (n+d^\ell)\cdots(n+(i-1)d^\ell); \Delta_{i,2}^{(0)} = (n+(i+1)d^\ell)\cdots(n+(k-1)d^\ell).$$

An empty product is taken as equal to 1. Further we can write each term

$$n + jd^\ell = a_j x_j^\ell, 1 \leq j \leq k, j \neq i$$

with $P(a_j) < k$ and $a_j - \ell$ th power free. We use the notation $a_j^{(i)} = a_j$ if the role of i is necessary. Equation (6), when d^ℓ is replaced by any common difference D has been the subject of study in the papers [11] and [12]. This is an equation dealing with perfect powers in a product with terms in arithmetic progression and one term missing. It has been shown in [11] and [12] that such an equation with $D > 1$ implies that there exists a prime $\equiv 1 (\bmod\ \ell)$ dividing D. In particular this implies that if a rational solution (x, y) exists for (1), then the denominators of x and y exceed $(2\ell)^\ell$ and $(2\ell)^{k-1}$, respectively. Note that $(a_1^{(i)}, \cdots, a_{i-1}^{(i)}, a_{i+1}^{(i)}, \cdots, a_k^{(i)})$ of Δ_i and $(a_1^{(k-i)}, \cdots, a_{k-i-1}^{(k-i)}, a_{k-i+1}^{(k-i)}, \cdots, a_k^{(k-i)})$ of Δ_{k-i} are mirror images of each other. In other words,

$$(a_1^{(k-i)}, \cdots, a_{k-i-1}^{(k-i)}, a_{k-i+1}^{(k-i)}, \cdots, a_k^{(k-i)}) = (a_k^{(i)}, \cdots, a_{i+1}^{(i)}, a_{i-1}^{(i)}, \cdots, a_1^{(i)}).$$

Hence it is enough to consider (6) with $1 < i \leq \lceil \frac{k}{2} \rceil$ which we assume from now onwards. Here for any $x \geq 0, \lceil x \rceil$ denotes the least integer $\geq x$.

3. Lemmas

As in [2] it is clear that one needs to derive a suitable ternary form from (6). Towards this we prove the following lemma which is similar to Lemma 2.1 of [2].

Lemma 3.1. *Let $k \geq 9$. Suppose (3) has a rational point (x, y) with $y \neq 0$. Let p be a prime dividing at most two terms in $\Delta_i^{(0)}$. Then there are non-zero integers a, b, c, u, v, w satisfyinng*

$$au^\ell + bv^\ell + cw^\ell = 0$$

such that

(1) a, b, c are ℓ-th power free integers
(2) $P(abc) \leq k$
(3) $p \nmid abc$
(4) p divides precisely one of u, v, w.

Proof. Note that $p \geq 3$ since $k \geq 9$. Suppose $p|d$. Then $p \nmid (n + jd^\ell)$ for any $1 \leq j \leq k, j \neq i$ since $\gcd(n, d) = 1$. Then the equations

$$n + 2d^\ell - (n + d^\ell) = d^\ell \ if\ i \neq 2; n + d^\ell - (n + 3d^\ell) = -2d^\ell \ if\ i = 2$$

satisfies (1)-(4) of the lemma.

Let $p \nmid d$. Suppose p divides only one term of $\Delta_i^{(0)}$, say, $n + jd^\ell$. Then p occurs to an ℓ-th power in this term. Hence $p \nmid a_j$ and $p|x_j$. Form equations as follows.

$$n+jd^\ell-(n+(j-1)d^\ell) = d^\ell \text{ if } i \leq j-2;\ n+jd^\ell-(n+(j+1)d^\ell) = -d^\ell \text{ if } i \geq j+2;$$

$$n+jd^\ell-(n+(j-2)d^\ell) = 2d^\ell \text{ if } i = j-1;\ n+jd^\ell-(n+(j+2)d^\ell) = -2d^\ell \text{ if } i = j+1.$$

These equations satisfy (1)-(4) of the lemma.

Suppose p divides exactly two terms $n+jd^\ell$ and $n+(j+p)d^\ell$. Form an equation as follows:

$$(7)\quad (n+jd^\ell)(n+(j+p)d^\ell)-(n+(j+1)d^\ell)(n+(j+p-1)d^\ell) = -(p-1)d^{2\ell}$$

if $i \notin \{j+1, j+p-1\}$. This satisfies (1)-(4) of the lemma. Thus we need to consider $i \in \{j+1, j+p-1\}$. For $p \geq 5$, we take

$$(n+jd^\ell)(n+(j+p)d^\ell)-(n+(j+2)d^\ell)(n+(j+p-2)d^\ell) = -2(p-2)d^{2\ell}$$

which satisfies (1)-(4) of the lemma. Let $p=3$. Since p divides exactly two terms, $k \leq 11$ and further the deleted term $n+id^\ell$ is divisible by 3. Then $3|(i-j)$ contradicting $i \in \{j+1, j+2\}$. □

From Lemma 3.1 and the discussions in the Proof of Theorem 1 of [2], we obtain the following lemma as an easy consequence.

Lemma 3.2. *Let $k \geq 9$. Suppose* (3) *has a rational point (x, y) with $y \neq 0$. Let $p \leq k$ be a prime dividing at most two terms in $\Delta_i^{(0)}$. Then*

$$\log \ell \leq \frac{16(\prod_{q<k, q\neq p} q)+1}{6}\log(\sqrt{p}+1) \leq 3^k.$$

Remark 3.1. *We would like to note that the above bound for ℓ is already known in* [1, Theorem 1.4].

For the proof of Theorems 1.3 and 1.4, we need to be more precise while forming the ternary equations. This involves locating the primes dividing the a_j's. We also need the following three lemmas for excluding several cases.

Lemma 3.3. *Let $\ell \geq 3, \alpha \geq 0, \beta \geq 0$ be integers. Then the equation*

$$x^\ell + y^\ell = 2^\alpha z^\ell$$

in relatively prime integers $x, y, z \geq 1$ has no solution for $\alpha \neq 1$, and for $\alpha = 1$ the equation has only the trivial solution $x = y = z = 1$. Further, the equations

$$x^\ell - y^\ell = 2^\alpha z^\ell \text{ and } x^\ell + y^\ell = 3^\beta z^\ell$$

have no solution in relatively prime integers $x, y, z \geq 1$.

The results in the first two equations were established by Wiles[15] for $\alpha = 0$, by Darmon and Merel [3] for $\alpha = 1$, and by Ribet [8] for $\alpha > 1$. The result in the third equation is due to Serre [9]. The next lemma is [11, Lemma 13].

Lemma 3.4. *Let $\ell \geq 5$. Let a, b, c be non-zero integers such that either $P(abc) \leq 3$ or a,b,c are composed of 2 and 5. Then the equation*

$$ax^\ell - by^\ell = cz^\ell \text{ in nonzero integers } x, y, z \text{ with}$$

$$\gcd(ax^\ell, by^\ell, cz^\ell) = 1,\ \mathrm{ord}_2(by^\ell) \geq 4$$

has no solution.

As an easy consequence of the above two lemmas, we obtain the following result.

Corollary 3.5. *Suppose* (6) *is valid. Let* $1 \leq j_1 < j_2 \leq k$ *and none of them equal to* i.

(1) Suppose

$$P(a_{j_1}a_{j_2}) \leq 2 \text{ and } |j_1 - j_2| = 2^\delta \text{ for some integer } \delta \text{ with } \delta \geq 0.$$

Then $\ell = 2$.

(2) Suppose

$$P(a_{j_1}a_{j_2}) \leq 3 \text{ and } |j_1 - j_2| = 2^{\delta_1}3^{\delta_2} \text{ for some integers } \delta_1, \delta_2 \text{ with } \delta_1 \geq 0,\ \delta_2 \geq 0.$$

Then in the ternary equation

$$a_{j_1}x_{j_1}^\ell - a_{j_2}x_{j_2}^\ell = (j_1 - j_2)d^\ell$$

after cancelling common factors, 2 divides only one term and ord$_2$ *in that term is* ≤ 3.

In the following discussion we shall put the above corollay differently which will be useful in the proof of Theorem 1.3. Let k be fixed and p be a prime dividing $\Delta_i^{(0)}$. Suppose j_p is the first j such that $p \mid (n + j_p d^\ell)$. Further assume that

$$j_p + 2p \leq k < j_p + 3p. \tag{8}$$

Note that i belongs to either $[j_p + 1, j_p + p]$ or $[j_p + p + 1, j_p + 2p]$. From now on, without loss of generality, we shall assume that

$$i \notin [j_p + 1, j_p + p]. \tag{9}$$

Definition 3.1. *We say that property* $\mathbf{T}$ *holds if there exists a prime* p *satisfying* (8) *and we can find two indices* $\mu, \nu \in [j_p + 1, j_p + p)$ *such that the ternary equation*

$$a_\mu {x_\mu}^\ell + a_\nu {x_\nu}^\ell = (\mu - \nu)d^\ell$$

can be reduced to an equation as in Corollary 3.5 (1).

Thus if property $\mathbf{T}$ holds, then we conclude that $\ell = 2$. Let us denote

$$(b)_p = a_{j_p+b}\ ;\ (b_1, b_2, \cdots)_p = a_{j_p+b_1}a_{j_p+b_2}\cdots$$

for any $b, b_1, b_2, \cdots \in [j_p + 1, j_p + p)$.

Lemma 3.6. *Suppose* (6) *holds. Let* p *be a prime dividing* $\Delta_i^{(0)}$ *and satisfying* (8)*. Suppose any of the following conditions hold.*
(i) There exist integers $a \geq 0, 2^a < p, b \in [j_p + 1, j_p + p - 2^a)$ *with* $P((b, b + 2^a)_p) \leq 2$.
(ii) There exist integers $a \geq 0, 2^a 3 < p, b \in [j_p + 1, j_p + p - 2^a 3)$, *with* $3 \mid\mid (b)_p, 3 \mid\mid (b + 2^a 3)_p$ *and* $P((b, b + 2^a 3)_p) \leq 3$.
Then property $\mathbf{T}$ *holds and hence* $\ell = 2$.

Proof. Suppose (i) holds. Then we consider the ternary equation

$$a_{j_p+b+2^a} {x_{j_p+b+2^a}}^{\ell} - a_{j_p+b} {x_{j_p+b}}^{\ell} = 2^a d^{\ell} \tag{10}$$

After cancelling the powers of 2, we conclude that $\ell = 2$ from Corollary 3.5(1).

The proof, when (ii) holds is similar. □

As a consequence of the above lemma, we get

Corollary 3.7. *Suppose* (6) *holds with* $\ell \geq 3$. *Let* p *be a prime dividing* $\Delta_i^{(0)}$ *and satisfying* (8)*. Then for any integer* $a \geq 0$ *with* $2^a < p$ *and for any integer* b *with* $b \in [j_p + 1, j_p + p - 2^a)$ *we have*
($\mathbf{A_1}$) $P((b, b + 2^a)_p) > 2$.
Also for any integer $a \geq 0$ *with* $2^a 3 < p$ *and for any integer* b *with* $b \in [j_p + 1, j_p + p - 2^a 3), 3 \mid\mid (b)_p$ *and* $3 \mid\mid (b + 2^a 3)_p$ *we have*
($\mathbf{A_2}$) $P((b, b + 2^a 3)_p) > 3$.

The next lemma is part of [1, Proposition 3.1].

Lemma 3.8. *Let* $\ell \geq 7$ *be prime and* A, B *co-prime positive integers. Then the following equations have no solution in non-zero co-prime integers* (x, y, z) *with* $xy \neq \pm 1$*:*

(i) $Ax^{\ell} + By^{\ell} = z^2, P(AB) \leq 3,\ p \mid xy$ *for each* $p \in \{5, 7\}$.
(ii) $Ax^{\ell} + By^{\ell} = z^2, P(AB) \leq 5,\ 7 \mid xy$ *and* $\ell \geq 11$
(iii) $x^{\ell} + 2^{\alpha} y^{\ell} = 3z^2$ *with* $p \mid xy$ *for each* $p \in \{5, 7\}$ *and* $\alpha \geq 0$.

We will use the above lemma by forming an identity as

$$(n + i_1 d^{\ell})(n + i_2 d^{\ell}) - (n + j_1 d^{\ell})(n + j_2 d^{\ell}) = (i_1 i_2 - j_1 j_2) d^{2\ell} \tag{11}$$

for $1 \leq i_1 < i_2 \leq k, 1 \leq j_1 < j_2 \leq k, (i_1, i_2) \neq (j_1, j_2)$ *with* $i_1 + i_2 = j_1 + j_2$. We end this section with the following well known result on Pythogorean triples and a related result.

Lemma 3.9. *Positive integral solutions of the equation*

$$x^2 + y^2 = z^2, y \text{ even}$$

are given by $x = r^2 - s^2, y = 2rs, z = r^2 + s^2$ *with* $r > s > 0$ *co-prime integers of opposite parity.*

Positive integral solutions of the equation

$$x^2 + 2y^2 = z^2$$

are given by $x = |u^2 - 2v^2|, y = 2uv, z = u^2 + 2v^2$ *with* u, v *co-prime positive integers with* u *odd.*

4. Proof of Theorem 1.1

For $1 \leq i \leq k$, let $\Delta_{i,1}^{(0)}$ and $\Delta_{i,2}^{(0)}$ be as defined in Section 2. Suppose $2 \leq i \leq k - p_\theta$. Then length of $\Delta_{i,2}^{(0)}$ is $k - i \geq p_\theta$. Hence $\Delta_i^{(0)}$ is divisible by a prime $\geq k/2$ by the definition of θ. This prime can divide at most two terms of $\Delta_i^{(0)}$. Hence by Lemma 3.2, inequality (2) is valid.

Suppose $p_\theta < i < k$. Then length of $\Delta_{i,1}^{(0)}$ is $i - 1 \geq p_\theta \geq k/2$. Now the conclusion follows as in the previous case. □

5. Proof of Theorem 1.4

First we take $k = 3$. We assume that $\ell > 2$. Then (6) is

$$(n + d^\ell)(n + 3d^\ell) = m^\ell.$$

Thus

$$a_3 x_3^\ell - a_1 x_1^\ell = 2d^\ell.$$

Since $P(a_1 a_3) \leq 2$, the above equation gives rise to a ternary equation as in Corollary 3.5 (1), by which we conclude that $\ell = 2$. Further $(a_1, a_3) \in \{(1, 1), (2, 2)\}$. These choices lead to

$$x_1^2 + 2d^2 = x_3^2 \text{ or } x_1^2 + d^2 = x_3^2.$$

We apply Lemma 3.9 to get the assertions of the theorem.

Let $4 \leq k \leq 8$. We assume that (5) holds. Thus $\ell \geq k$ and (6) is valid. Suppose $2|d$. Then $2 \nmid a_j$ for any j with $1 \leq j \leq k, j \neq i$. Hence we can find $1 \leq j_1 < j_2 \leq k, \{j_1, j_2\} \not\ni i$ such that $j_2 - j_1 = 1$ or 2 and $P(a_{j_1} a_{j_2}) \leq 3$. Thus the ternary equation

$$a_{j_2} x_{j_2}^\ell - a_{j_1} x_{j_1}^\ell = (j_2 - j_1) d^\ell$$

satisfies Corollary 3.5 (2), giving $\ell \leq 3$.

From now on we shall assume that $2 \nmid d$. Then 2 divides terms of $\Delta_i^{(0)}$. Let 2^μ be the maximum power of 2 appearing in any term of $\Delta_i^{(0)}$ and let

$$n + j^{(\mu)}d^\ell = 2^\mu a^{(\mu)}(x_{j^{(\mu)}})^\ell, \quad where\ a^{(\mu)} = \frac{a_{j^{(\mu)}}}{2^\mu}$$

be the term divisible by 2^μ. We discuss each value of k now.

Let $k = 4$. Then $i = 2$ and $P(a_j) \le 3$. Since a_j's are $\ell-$ th power free, we have either $3|a_1, a_4$ or $P(a_j) \le 2$. In the latter case, we apply Corollary 3.5(1) with $(j_1, j_2) = (1, 3)$ or $(3, 4)$ to conclude that $\ell = 2$. Let $3|a_1, a_4$. We have $j^{(\mu)} \in \{1, 3\}, \mu \ge l - 1$ or $j^{(\mu)} = 4, \mu \ge \ell$. Form ternary equations

$$2^\mu a^{(\mu)}(x_{j^{(\mu)}})^\ell - a_4 x_4^\ell = (j^{(\mu)} - 4)d^\ell$$

or

$$2^\mu a^{(\mu)}(x_{j^{(\mu)}})^\ell - a_3 x_3^\ell = (j^{(\mu)} - 3)d^\ell,$$

respectively to conclude from Corollary 3.5(2) that $\ell \le 3$.

Let $k = 5$. Then $i = 2$ or 3, $P(a_j) \le 3$ and $j^{(\mu)} \in \{1, 3, 5\}$ or $j^{(\mu)} = 4$ if $i = 2$; $j^{(\mu)} \in \{1, 5\}$ or $j^{(\mu)} \in \{2, 4\}$ if $i = 3$. Thus $\mu \ge \ell - 3$. As shown in the case $k = 4$ we can form a ternary equation as in Corollary 3.5(2) to conclude that $\ell \le 5$.

Let $k = 6$. Then $i = 2$ or 3 and $P(a_j) \le 5$. Then $\mu \ge \ell - 3$. Suppose 5 does not divide any of the a_j's. Then we form a ternary equation as in Corollary 3.5(1) to conclude that $\ell = 2$. Thus we may assume that $5|a_1$ and $5|a_6$. Observe that $\mathrm{ord}_5(a_1 a_6) \equiv 0 (\mathrm{mod}\ l)$. We form the identities

$$(n + d^\ell)(n + 6d^\ell) - (n + 3d^\ell)(n + 4d^\ell) = -6d^{2\ell}\ if\ i = 2; \tag{12}$$

$$(n + d^\ell)(n + 6d^\ell) - (n + 2d^\ell)(n + 5d^\ell) = -4d^{2\ell}\ if\ i = 3 \tag{13}$$

These are reduced to equations of the form given in Lemma 3.8 since $P(a_3 a_4) \le 3$ if $i = 2$ and $P(a_2 a_5) \le 3$ if $i = 3$. Hence we conclude that $\ell \le 5$.

Let $k = 7$. Then $i \in \{2, 3, 4\}$ and $P(a_j) \le 5$. Suppose $P(a_j) \le 3$ for $1 \le j \le 7, j \ne i$ we can form a ternary equation as in Corollary 3.5(1) to get $\ell = 2$. Thus we may suppose that $5|a_j$ and $5|a_{j+5}$ with $j = 1$ when $i = 2$ and $j \in \{1, 2\}$ when $i = 3, 4$. Then we form the identity (12) if $i = 2$ and identity (13) if $i \in \{3, 4\}$ and $5|a_1, a_6$ to conclude $\ell \le 5$. When $i \in \{3, 4\}$ and $5|a_2, a_7$, we form the identities

$$(n + 2d^\ell)(n + 7d^\ell) - (n + 4d^\ell)(n + 5d^\ell) = -6d^{2\ell} \text{ if } i = 3$$
$$(n + 2d^\ell)(n + 7d^\ell) - (n + 3d^\ell)(n + 6d^\ell) = -4d^{2\ell} \text{ if } i = 4$$

to conclude that $\ell \le 5$.

Let $k=8$. Then $i\in\{2,3,4\}$ and $P(a_j)\le 7$. If $P(a_j)\le 5$, we argue as in $k=7$ to get $\ell\le 5$. So we may suppose that $7|a_1,a_8$. Also we may assume that $5|a_ja_{j+5}$ with $j\in\{1,2,3\}, j\ne i$. Otherwise, $P(a_j)\le 3$ for $1<j<8, j\ne i$, and it can be seen easily that we can form a ternary equation as in Corollary 3.5 (1) to get $\ell=2$. Now we form the identities

$$(n+d^\ell)(n+8d^\ell)-(n+3d^\ell)(n+6d^\ell)=-10d^{2\ell} \text{ if } i=2,4,\ 5|a_1a_6 \text{ or } 5|a_3a_8$$

$$(n+d^\ell)(n+8d^\ell)-(n+2d^\ell)(n+7d^\ell)=-6d^{2\ell} \text{ if } i=3,4,\ 5|a_2a_7$$

$$(n+d^\ell)(n+8d^\ell)-(n+4d^\ell)(n+5d^\ell)=-12d^{2\ell} \text{ if } i=3,\ 5|a_1a_6$$

to conclude that $\ell\le 5$. □

6. Proofs of Theorem 1.3 and Corollary 1.5

Proof of Theorem 1.3. Note that either $\Delta_{i,1}^{(0)}$ or $\Delta_{i,1}^{(0)}$ is divisible by a prime $>k/3$. Hence by hypothesis, if $p=P(\Delta_i^{(0)})$, then $j_p+3p>k\ge j_p+2p$ and without loss of generality $i\notin[j_p+1,j_p+p]$. We restrict to the interval $[j_p+1,j_p+p)$. We have

$$P(a_j)\le\begin{cases}3 & \text{if } k=9,10\\ 5 & \text{if } 11\le k\le 14\\ 7 & \text{if } 15\le k\le 22\\ 11 & \text{if } 23\le k\le 26.\end{cases}$$

The assertion is true for $k=10,14,22,26$ by Corollary 3. Let $k=9$. Take $p=3$. Then Corollary 3.7 ($\mathbf{A_1}$) does not hold with $a=0, b=j_3+1$. Hence by (5), $\ell<k$.

Let $11\le k\le 13$. Take $p=5$. Then $P((j_5+1,\cdots,j_5+4)_5)\le 3$. Since 3 can divide at most two terms of $(j_5+1)_5,\cdots,(j_5+4)_5$, there exists $a,b\in[j_5+1,j_5+4)$ with $P((b,b+1)_5)\le 2$, which is a contradiction to Corollary 3.7 ($\mathbf{A_1}$) with $a=0$. Thus by (5), $\ell<k$.

Let $15\le k\le 21$. Take $p=7$. Then $P((j_7+1,\cdots,j_7+6)_7)\le 5$. Here 3 divides exactly two terms, say $(b)_7$ and $(b+3)_7$ for some $b\in[j_7+1,j_7+2,j_7+3]$. By Corollary 3.7 ($\mathbf{A_1}$), $P((b+1,b+2)_7)=5$ and $b\ne j_7+1,j_7+3$. If $b=j_7+2$, then $P((j_7+1,j_7+3)_7)\le 2$ or $P((j_7+4,j_7+6)_7)\le 2$ according as $5\mid(j_7+4)_7$ or $5\mid(j_7+3)_7$ which contradicts Corollary 3.7 ($\mathbf{A_1}$) and we get $\ell\le k-1$.

Let $23\le k\le 25$. Take $p=11$. Then $P((j_{11}+1,\cdots,j_{11}+10)_{11})\le 7$. Here 3 divides at least three terms. Let 3 divide say $(b)_{11}$, $(b+3)_{11}$, $(b+6)_{11}$. By Corollary 3.5 ($\mathbf{A_1}$), $7\ge P((b+1,b+2)_{11})\ge 5$; $7\ge P((b+4,b+5)_{11})\ge 5$; $7\ge P((b+1,b+5)_{11})\ge 5$; $7\ge P((b+2,b+4)_{11})\ge 5$. This implies $5\mid(b+1)_{11}$ and $7\mid(b+4)_{11}$ or $5\mid(b+2)_{11}$ and $7\mid(b+5)_{11}$.

Hence $P((b, b+3, b+6)_{11}) \leq 3$. This is a contradiction to Corollary 3.7 ($\mathbf{A_2}$), since there exists at least two of $(b)_{11}, (b+3)_{11}$ and $(b+6)_{11}$ which are exactly divisible by 3. Thus $\ell \leq k-1$. □

Proof of Corollary 1.5. By Theorems 1.4 and 1.3, we may assume that $9 \leq k \leq 26$ and $P(\Delta_i^{(0)}) > p_\theta$. As $p_\theta \geq k/2$, the result follows from Lemma 3.2 by taking $p = P(\Delta_i^{(0)})$. □

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Stat-Math Unit, Indian Statistical Institute Delhi Centre, New Delhi 110016, INDIA
E-mail address: pranabesh.math@gmail.com
E-mail address: shanta@isid.ac.in

School of Mathematics, Tata Institute of Fundamental Research, Mumbai, 400005, INDIA
E-mail address: saradha@math.tifr.res.in