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Irreducibility and Galois Groups of Generalized Laguerre Polynomials $L_n^{(-1-n-r)}(x)$

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IRREDUCIBILITY AND GALOIS GROUPS OF GENERALIZED LAGUERRE POLYNOMIALS $L_n^{(-1-n-r)}(x)$

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Dedicated to Professor T. N. Shorey on his 70th birthday

Abstract. We study the algebraic properties of Generalized Laguerre polynomials for negative integral values of a given parameter which is $L_n^{(-1-n-r)}(x) = \sum_{j=0}^{n} \binom{n-j+r}{n-j} \frac{x^j}{j!}$ for integers $r \geq 0, n \geq 1$. For different values of parameter r, this family provides polynomials which are of great interest. Hajir conjectured that for integers $r \geq 0$ and $n \geq 1$, $L_n^{(-1-n-r)}(x)$ is an irreducible polynomial whose Galois group contains A_n, the alternating group on n symbols. Extending earlier results of Schur, Hajir, Sell, Nair and Shorey, we confirm this conjecture for all $r \leq 60$. We also prove that $L_n^{(-1-n-r)}(x)$ is an irreducible polynomial whose Galois group contains A_n whenever $n > e^{r\left(1+\frac{1.2762}{\log r}\right)}$.

1. Introduction

For an arbitrary real number α and a positive integer n, the Generalized Laguerre Polynomials (GLP) is a family of polynomials defined by

$$L_n^{(\alpha)}(x) = (-1)^n \sum_{j=0}^{n} \binom{n+\alpha}{n-j} \frac{(-x)^j}{j!}.$$

The inclusion of the sign $(-1)^n$ is not standard. The corresponding monic polynomial is obtained as $\mathcal{L}_n^{(\alpha)}(x) = n! L_n^{(\alpha)}(x)$. These classical orthogonal polynomials play an important role in various branches of analysis and mathematical physics and has been well studied. Schur [15], [16] was the first to study the algebraic properties of these polynomials by proving that $L_n^{(\alpha)}(x)$ where $\alpha \in \{0, 1, -n-1\}$ are irreducible. For an account of results obtained on GLP, we refer to Hajir [10] and Filaseta, Kidd and Trifonov [6].

In this paper, we study α at negative integral values via a parameter r. For integer $r \geq 0$, we consider

$$\begin{aligned} L_n^{\langle r \rangle}(x) &:= L_n^{(-1-n-r)}(x) \\ &= (-1)^n \sum_{j=0}^{n} \binom{-1-r}{n-j} \frac{(-x)^j}{j!} \\ &= \sum_{j=0}^{n} \binom{n-j+r}{n-j} \frac{x^j}{j!}. \end{aligned}$$

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By a factor of a polynomial, we always mean its factor over $\mathbb{Q}$. We observe that $\mathcal{L}_n^{\langle r\rangle}(x) := n!L_n^{\langle r\rangle}(x) = \sum_{j=0}^{n} \binom{n}{j}(r+1)\dots(r+n-j)x^j$ is a monic polynomial with integer coefficients and $L_n^{\langle r\rangle}(x)$ is irreducible if and only if $\mathcal{L}_n^{\langle r\rangle}(x)$ is irreducible. Schur [16] computed the discriminant of $\mathcal{L}_n^{\langle r\rangle}(x)$ which is

$$\Delta_n^{\langle r\rangle} = \prod_{j=2}^{n} j^j(-1-n-r+j)^{j-1}.$$

Let $G_n(r)$ denote the Galois group of $\mathcal{L}_n^{\langle r\rangle}(x)$ over $\mathbb{Q}$. Let S_n denote the symmetric group on n symbols and A_n, the alternating group on n symbols. Schur [15, 16] and Coleman [2] used two different techniques to prove that $L_n^{\langle 0\rangle}(x)$ is irreducible and $G_n(0) = S_n$ for every n. Hajir [8] proved that $L_n^{\langle 1\rangle}(x)$ is irreducible and $G_n(1)$ is A_n if $n \equiv 1 (\text{mod } 4)$ and is S_n, otherwise. Sell [14] proved that $L_n^{\langle 2\rangle}(x)$ is irreducible and $G_n(2)$ is A_n if $n+1$ is an odd square and is S_n, otherwise.

The irreducibility of $L_n^{\langle n\rangle}(x)$, also known as Bessel polynomials, was conjectured for all n by Grosswald [7] and assuming his conjecture he proved that the Galois group is S_n for every n. The irreducibility of all Bessel polynomials was proved, first for all but finitely many n by Filaseta [4] and later for all n by Filaseta and Trifonov [5].

Hajir [10] conjectured that for integers $r \geq 0$, $n \geq 1$, $L_n^{\langle r\rangle}(x)$ is irreducible and $G_n(r)$ contains A_n. It was also proved in [10] that if r is a fixed integer in the range $0 \leq r \leq 8$, then for all $n \geq 1$, $L_n^{\langle r\rangle}(x)$ is irreducible and has Galois group containing A_n. This was extended by Nair and Shorey [13] who proved the following.

Theorem A. *For $n \geq 1$,*

(i) *$L_n^{\langle r\rangle}(x)$ is irreducible for $3 \leq r \leq 22$.*

(ii) *For $9 \leq r \leq 22$, $G_n(r) = S_n$ unless $(n,r) \in \{(8,9),(12,13),(13,16),(16,17),(17,18),(20,21)\}$ in which case $G_n(r) = A_n$. For $3 \leq r \leq 8$, $G_n(r) = S_n$ unless $(n,r) \in \{(2,3),(24,4),(4,5),(6,7),(7,8),(9,8),(2,8)\}$ or*
$r = 3$; $n \equiv 1(\text{mod } 24)$ and $\frac{n+2}{3}$ is a square
$r = 4$; $n+2$ is a rational part of $(2+\sqrt{3})^{2k+1}$ where $k \geq 0$ is an integer
$r = 5$; $n+3$ is a rational part of $(4+\sqrt{15})^{2k+1}$ where $k \geq 0$ is an integer
in which case $G_n(r) = A_n$.

We further extend this work to confirm the conjecture of Hajir for all $r \leq 60$. We prove

Theorem 1.1. *For $n \geq 1$ and $23 \leq r \leq 60$, we have*

(i) *$L_n^{\langle r\rangle}(x)$ is irreducible.*

(ii) *$G_n(r) = S_n$ unless $(n,r) \in \{(4,24),(5,28),(24,25),(25,24),(28,23),(28,29),(32,33),(33,36),(36,37),(40,41),(44,45),(48,49),(48,51),(49,48),(49,50),(52,53),(56,57)\}$ in which case $G_n(r) = A_n$.*

The proof of Theorem 1.1 is given in Sections 4 and 5. We see that Theorem 1.1 considerably extends earlier results of [10] and [13]. The new ingredients in the proof are Lemma 3.1 which arise from clever and important observations on prime divisors of n and $\binom{n+r}{r}$ and Lemmas 3.5-3.7 which arise from an application of p-adic Newton polygons. These results are general in nature and make our computations much less. In fact, for checking irreducibility of $L_n^{\langle r\rangle}(x)$, we need to exclude factors of degrees up to 3 which can be handled easily. The observations also imply the following result which improves the bound for n given by Hajir [10] and Nair and Shorey [13].

Theorem 1.2. *$L_n^{\langle r\rangle}(x)$ is irreducible and $G_n(r)$ contains A_n if*

$$n > e^{r\left(1+\frac{1.2762}{\log r}\right)}.$$

We prove Theorem 1.2 in Section 6.

The computations in this paper are carried out with SAGE except for computing a few Galois groups in Section 5 for which MAGMA online is used.

2. Preliminaries

Henceforth, we always use p for a prime and n, r for integers with $r \geq 0$, $n \geq 1$ unless otherwise specified.

Definition 1. *The p-adic valuation of an integer m with respect to p, denoted by $\nu_p(m)$, is defined as*

$$\nu_p(m) = \begin{cases} \max\{k : p^k | m\} & \text{if } m \neq 0, \\ \infty & \text{if } m = 0. \end{cases}$$

Definition 2. *Let m be a positive integer. Let $m = m_0 + m_1 p + \cdots + m_t p^t$ with $m_t \neq 0$ be the p-adic representation of m. We define $\sigma_p(m) := m_0 + m_1 + \cdots + m_t$.*

For integers $m \geq 1$ and $t \geq 0$, we have

$$\begin{aligned} \nu_p(m!) &= \frac{m - \sigma_p(m)}{p-1}, \\ \text{and } \nu_p\left(\binom{m}{t}\right) &= \frac{\sigma_p(t) + \sigma_p(m-t) - \sigma_p(m)}{p-1}. \end{aligned}$$

These are well known results of Legendre [12].

Definition 3. *Let $f(x) = \sum_{j=0}^{n} a_j x^j \in \mathbb{Z}[x]$ with $a_o a_n \neq 0$. We consider the set*

$$S = \{(0, \nu_p(a_n)), (1, \nu_p(a_{n-1})), \ldots, (n, \nu_p(a_0))\}$$

consisting of points in the extended plane $\mathbb{R}^2 \cup \{\infty\}$. The polygonal path formed by the lower edges along the convex hull of S is called the Newton polygon associated to $f(x)$ with respect to prime p and is denoted by $NP_p(f)$.

It can be observed that the left-most edge has one end-point being $(0, \nu_p(a_n))$ and the right-most edge has $(n, \nu_p(a_0))$ as an end point. The end points of every edge belong to the set S. Thus every point in S lies either on or above the line obtained by extending such an edge. In particular, if $(i, \nu_p(a_{n-i}))$ and $(j, \nu_p(a_{n-j}))$ are the two end-points of such an edge, then every point $(u, \nu_p(a_{n-u}))$ with $i < u < j$ lies on or above the line passing through $(i, \nu_p(a_{n-i}))$ and $(j, \nu_p(a_{n-j}))$. Also the slopes of the edges are always increasing when calculated from the left- most edge to the right-most edge.

We need the following result due to Filaseta [4, Lemma 2] which is an application of Newton polygons.

Lemma 2.1. *Let k and l be integers with $k > l \geq 0$. Suppose $g(x) = \sum_{j=0}^{n} b_j x^j \in \mathbb{Z}[x]$ and p is a prime such that $p \nmid b_n$, $p | b_j$ for all $j \in \{0, 1, \ldots, n-l-1\}$ and the right-most edge of the Newton polygon for $g(x)$ with respect to p has slope $< \frac{1}{k}$. Then for any integers $a_0, a_1, \ldots, a_n$ with $|a_0| = |a_n| = 1$, the polynomial $f(x) = \sum_{j=0}^{n} a_j b_j x^j$ cannot have a factor with degree in the interval $[l+1, k]$.*

In this paper, we use Lemma 2.1 with $a_0 = a_1 = \cdots = a_n = 1$ always.

Definition 4. *Given $f \in \mathbb{Q}[x]$, we define the Newton Index of f, denoted by $\mathcal{N}_f$, to be the least common multiple of the denominators (in lowest terms) of all slopes of $NP_p(f)$ as p ranges over all primes.*

The following results by Hajir [9, Theorem 2.2] are used for calculating the Galois groups of polynomials.

Lemma 2.2. *Given an irreducible polynomial $f \in \mathbb{Q}[x]$, $\mathcal{N}_f$ divides the order of the Galois group of f. Moreover, if $\mathcal{N}_f$ has a prime divisor q in the range $\frac{n}{2} < q < n-2$, where n is the degree of f, then the Galois group of f contains A_n.*

As a consequence of Lemma 2.2, Hajir [10, Theorem 5.4] proved the following result.

Lemma 2.3. *Let $L_n^{\langle r \rangle}(x)$ be irreducible.*

(*i*) *If there exists a prime p satisfying $\frac{n+r}{2} < p < n-2$, then $G_n(r)$ contains A_n.*
(*ii*) *If $n \geq \max\{48 - r, 8 + \frac{5r}{3}\}$, then $G_n(r)$ contains A_n.*
(*iii*) *If $G_n(r)$ contains A_n, then*

$$G_n(r) = \begin{cases} A_n & \text{if } \Delta_n^{\langle r \rangle} \text{ is a square,} \\ S_n & \text{otherwise.} \end{cases}$$

If $\mathcal{L}_n^{\langle r \rangle}(x)$ is reducible, it has one factor with degree $\in [1, \frac{n}{2}]$. Thus from now onwards, whenever we consider a factor of degree k of $\mathcal{L}_n^{\langle r \rangle}(x)$, we mean a factor of degree k with $1 \leq k \leq \frac{n}{2}$.

For fixed integers $r \geq 0$ and $n \geq 1$, we write $n = n_0 n_1$ where

$$n_0 := \prod_{p | n, \; p \nmid \binom{n+r}{r}} p^{\nu_p(n)} \text{ and } n_1 := \prod_{p | \gcd(n, \binom{n+r}{r})} p^{\nu_p(n)}.$$

The following result is contained in the first line of the proof of Hajir [10, Lemma 4.1]

Lemma 2.4. *Every factor of $L_n^{\langle r \rangle}(x)$ has degree divisible by n_0.*

Next three results are due to Nair and Shorey [13, Corollary 3.2, Corollary 3.3 and Lemma 2.10].

Lemma 2.5. *Assume that $L_n^{\langle r \rangle}(x)$ has a factor of degree $k \geq 2$. Then $r > 1.63k$.*

Lemma 2.6. *Assume that $L_n^{\langle r \rangle}(x)$ has a factor of degree $k \geq 2$. Then*

$$r > \min\{104, 3.42k + 1\}.$$

Lemma 2.7. *For $n \leq 127$ and $r \leq 103$, $L_n^{\langle r \rangle}(x)$ is irreducible.*

We also need the following statement used in [13] and we give a proof here.

Lemma 2.8. *For $p|n_1$, we have $p^{\nu_p(n)} \le r$.*

Proof. Write $n = p^e d$, where d is coprime to p such that $p^e > r$. We will show that $\nu_p\left(\binom{n+r}{r}\right) = 0$.

Let $r = r_{e-1}p^{e-1} + \cdots + r_1 p + r_0$ be the p-adic representation of r. Then $n + r = dp^e + r_{e-1}p^{e-1} + \cdots + r_1 p + r_0$. So we have $\sigma_p(n) = \sigma_p(d)$, $\sigma_p(r) = r_{e-1} + \cdots + r_1 + r_0$ and $\sigma_p(n+r) = \sigma_p(d) + r_{e-1} + \cdots + r_1 + r_0$. Thus $\nu_p\left(\binom{n+r}{r}\right) = \frac{\sigma_p(n)+\sigma_p(r)-\sigma_p(n+r)}{p-1} = 0$. □

The following result is due to Harborth and Kemnitz [11].

Lemma 2.9. *There exists a prime p satisfying :*
(*a*) $x < p < \frac{6}{5}x$ *for* $x \ge 25$,
(*b*) $x < p \le \frac{11}{10}x$ *for* $x \ge 116$.

For real number $x > 1$, we denote

$$\pi(x) = \sum_{p \le x} 1.$$

We need the following result due to Dusart [3] for the proof of Theorem 1.2.

Lemma 2.10. *We have*

$$\pi(x) \le \frac{x}{\log x}\left(1 + \frac{1.2762}{\log x}\right) \quad \textit{for } x > 1.$$

3. Lemmas for the proof of Theorem 1.1

For the proof of Theorem 1.1, we use a number of results which we record here as lemmas and corollaries. These results are general in nature and valid for any positive integers n and r.

Lemma 3.1. *Let $p|n_1$ and $r < p^2$. Then*

$$\frac{n}{p} \equiv -j (\mathrm{mod}\ p) \textit{ for some } j \textit{ with } 1 \le j \le \left\lfloor \frac{r}{p} \right\rfloor.$$

Proof. Since $p|n_1$ and $r < p^2$, $\nu_p(n_1) = 1$. We can write $n = pd$, where d is coprime to p and $r = r_1 p + r_0$, where $0 \le r_1, r_0 < p$. Then $n + r = p(d + r_1) + r_0$. So we have $\sigma(n) = \sigma(d)$, $\sigma(r) = r_1 + r_0$ and $\sigma(n+r) = \sigma(d + r_1) + r_0$. Therefore

$$\begin{aligned}
1 \le \nu_p\left(\binom{n+r}{r}\right) &= \frac{\sigma_p(n) + \sigma_p(r) - \sigma_p(n+r)}{p-1} \\
&= \frac{\sigma_p(d) + r_1 - \sigma_p(d + r_1)}{p-1} \\
&= \nu_p\left(\binom{d+r_1}{r_1}\right) \\
&= \nu_p\left(\frac{(d+1)(d+2)\cdots(d+r_1)}{r_1!}\right) \\
&= \nu_p((d+1)(d+2)\cdots(d+r_1)) \text{ (since } r_1 < p) \\
&= \nu_p(d+j) \text{ for exactly one } j \text{ with } 1 \le j \le r_1.
\end{aligned}$$

Since $r_1 = \left\lfloor \frac{r}{p} \right\rfloor < p$, we have $\frac{n}{p} \equiv -j (\mathrm{mod}\ p)$, for some $1 \le j \le \left\lfloor \frac{r}{p} \right\rfloor$. □

Corollary 3.2. *If $p|n_1$ and $r < p^2$, then $d + \left\lfloor \frac{r}{p} \right\rfloor \geq p$ where $d \equiv \frac{n}{p} (\text{mod } p)$ with $1 \leq d < p$.*

For the remaining part of this paper, we need the following notation and remark.

Remark 3.3. *For $1 \leq j \leq n$, we define $b_j := \binom{n}{j}(r+1)\cdots(r+j)$. The Newton polygon for $\mathcal{L}_n^{\langle r \rangle}(x) = \sum_{j=0}^{n} b_{n-j}x^j$ with respect to p is given by the lower edges along the convex hull of the points $(j, \nu_p(b_j))$ for $1 \leq j \leq n$. Thus the slope of the right-most edge of $NP_p(\mathcal{L}_n^{\langle r \rangle}(x))$ is at most $M_p = \max_{1 \leq j \leq n}\{\mu_j\}$ where*

$$\begin{aligned}
\mu_j &:= \frac{\nu_p(b_n) - \nu_p(b_{n-j})}{j} \\
&= \frac{\nu_p((r+n)!) - \nu_p((r+n-j)!) - \nu_p(\binom{n}{j})}{j} \\
&= \frac{j - \sigma_p(r+n) + \sigma_p(r+n-j)}{(p-1)j} - \frac{\sigma_p(j) + \sigma_p(n-j) - \sigma_p(n)}{(p-1)j} \\
&= \frac{j - \sigma_p(j)}{(p-1)j} + \frac{\sigma_p(r) + \sigma_p(n) - \sigma_p(r+n)}{(p-1)j} - \frac{\sigma_p(n-j) + \sigma_p(r) - \sigma_p(r+n-j)}{(p-1)j} \\
&= \frac{j - \sigma_p(j)}{(p-1)j} + \frac{1}{j}\nu_p\left(\binom{n+r}{r}\right) - \frac{1}{j}\nu_p\left(\binom{r+n-j}{r}\right) \\
&\leq \frac{j - \sigma_p(j)}{(p-1)j} + \frac{1}{j}\nu_p\left(\binom{n+r}{r}\right) (\text{since } \nu_p\left(\binom{r+n-j}{r}\right) \geq 0).
\end{aligned}$$

Lemma 3.4. *Let $p = p_{\pi(n)} = n - k_n$ be the largest prime less than or equal to n with $r + k_n < p$. Then $\mathcal{L}_n^{\langle r \rangle}(x)$ cannot have a factor with degree $> k_n$.*

Proof. Clearly $p \nmid b_0$. Since $p \mid n(n-1)\cdots(n-k_n)$, $p|\binom{n}{j}$ for $k_n + 1 \leq j < p$. Also, $p \mid (r+1)\cdots(r+j)$ for $j \geq p$. Thus $p|b_j$ for $k_n + 1 \leq j \leq n$.

Note that $r + k_n < p$ implies $p \nmid (r+1)\cdots(r+k_n)$ and $p \nmid n(n-1)\cdots(n-k_n+1)$. Thus $p \nmid (r+1)\cdots(r+j)$ and $p \nmid \binom{n}{j}$ for $1 \leq j \leq k_n$. Therefore $p \nmid b_j$ for $1 \leq j \leq k_n$.

Next $r + n = r + k_n + p < 2p$ implies $\nu_p(b_n) = \nu_p((r+1)\cdots(r+n)) = 1$. Hence the vertices of first edge of the Newton polygon are $(0,0)$ and $(k_n, 0)$ and the slope of the right-most edge is at most

$$\max_{k_n \leq j < n}\left\{\frac{\nu_p(b_n) - \nu_p(b_j)}{n-j}\right\}.$$

For $k_n < j < n$, we have $p|b_j$ implying $\nu_p(b_j) \geq 1$. Hence $\nu_p(b_n) - \nu_p(b_j) \leq 1 - 1 = 0$ for $k_n < j < n$. For $j = k_n$, we have

$$\frac{\nu_p(b_n) - \nu_p(b_{k_n})}{n - k_n} = \frac{1}{n - k_n} = \frac{1}{p}.$$

Thus we have

$$\max_{k_n \leq j < n}\left\{\frac{\nu_p(b_n) - \nu_p(b_j)}{n-j}\right\} \leq \frac{1}{p} < \frac{2}{n}$$

since $p > \frac{n}{2}$. Therefore, by Lemma 2.1, $\mathcal{L}_n^{\langle r \rangle}(x)$ cannot have a factor with degree in the interval $[k_n + 1, \frac{n}{2}]$ and the assertion follows. $\square$

Lemma 3.5. *Let $l_n \in [1, k_n]$ be the least positive integer such that there exists p with $p|(n-l_n)$, $p > k_n$ and $\nu_p\left(\binom{n+r}{r}\right) = 0$. Then $\mathcal{L}_n^{\langle r\rangle}(x)$ cannot have a factor with degree in the interval $[l_n+1, k_n]$.*

Proof. Clearly $p \nmid b_0$. Since $p \mid n(n-1)\cdots(n-l)$, $p|\binom{n}{j}$ for $l+1 \le j < p$. Also $p \mid (r+1)\cdots(r+j)$ for $j \ge p$. Thus $p|b_j$ for $l_n+1 \le j \le n$.

From Remark 3.3, the slope of the right-most edge of $NP_p(L_n^{\langle r\rangle}(x))$ is less than equal to $M_P \le \max\limits_{1\le j\le n}\left\{\frac{j-\sigma_p(j)}{(p-1)j} + \frac{1}{j}\nu_p\left(\binom{n+r}{r}\right)\right\}$.

Note that $\frac{j-\sigma_p(j)}{(p-1)j} \le 0$ if $j \le p-1$ and $\frac{j-\sigma_p(j)}{(p-1)j} < \frac{1}{p-1}$ if $j \ge p$. Since $p > k_n$ and $\nu_p\left(\binom{n+r}{r}\right) = 0$, we have

$$M_p < \frac{1}{k_n}.$$

Therefore, by Lemma 2.1, $\mathcal{L}_n^{\langle r\rangle}(x)$ cannot have a factor with degree in the interval $[l_n+1, k_n]$. □

Lemma 3.6. *Let i be a positive integer such that $p|n(n-1)\cdots(n-i+1)(r+1)\cdots(r+i)$ and let $\nu_p\left(\binom{n+r}{r}\right) = u$. Then $\mathcal{L}_n^{\langle r\rangle}(x)$ cannot have a factor of degree equal to i if any one of the following conditions holds:*

(*a*) *$u = 0$ and $p > i$,*
(*b*) *$u > 0$, $p > 2$ and $\max\{\frac{u+1}{p}, \frac{\nu_p(n+r-z_0)-\nu_p(n)}{z_0+1}\} < \frac{1}{i}$, where $z_0 \equiv n + r (mod\ p)$ with $1 \le z_0 < p$.*

Proof. Clearly $p \nmid b_0$. If $p|(r+1)\cdots(r+i)$, then $p|b_j$ for $j \ge i$. If $p \nmid (r+1)\cdots(r+i)$, then $p|n(n-1)\cdots(n-i+1)$ implies $p|\binom{n}{j}$ for $i \le j < p$. Also $p|(r+1)\cdots(r+j)$ for $j \ge p$. Thus $p|b_j$ for $i \le j \le n$.

From Remark 3.3, the slope of the right-most edge of $NP_p(L_n^{\langle r\rangle}(x))$ is at most $M_p = \max\limits_{1\le j\le n}\{\mu_j\}$ where

$$\mu_j \le \frac{j-\sigma_p(j)}{(p-1)j} + \frac{u}{j}.$$

(*a*) $u = 0$ and $p > i$. For $1 \le j \le n$, we have

$$\mu_j \le \frac{j-\sigma_p(j)}{(p-1)j} < \frac{1}{p-1} \le \frac{1}{i}.$$

(*b*) $u > 0$ and $p > 2$. We have

$$\begin{aligned}\mu_j &= \frac{\nu_p((r+n)!) - \nu_p((r+n-j)!) - \nu_p(\binom{n}{j})}{j} \\ &= \frac{\nu_p((r+n)\cdots((r+n-j+1)) - \nu_p(\binom{n}{j})}{j}.\end{aligned}$$

For $1 \le j \le p$, we have

$$\begin{aligned}\mu_j &\le \begin{cases} 0 & \text{if } j \le z_0 \\ \frac{\nu_p(n+r-z_0)-\nu_p(n)}{j} & \text{if } j > z_0 \end{cases} \\ &\le \frac{\nu_p(n+r-z_0)-\nu_p(n)}{z_0+1}.\end{aligned}$$

For $p \le j \le p^2$, we have

$$\mu_j \le \frac{j-\sigma_p(j)}{(p-1)j} + \frac{u}{j} \le \frac{1}{p} + \frac{u}{p} = \frac{u+1}{p}.$$

For $j \ge p^2$, since $p > 2$, we have

$$\mu_j \le \frac{j-\sigma_p(j)}{(p-1)j} + \frac{u}{j} < \frac{1}{p-1} + \frac{u}{p^2} < \frac{u+1}{p}.$$

Thus, by the assumption on (b), for $1 \le j \le n$,

$$\mu_j \le \max\left\{\frac{u+1}{p}, \frac{\nu_p(n+r-z_0)-\nu_p(n)}{z_0+1}\right\} < \frac{1}{i}.$$

Hence $M_p < \frac{1}{i}$ and therefore, by Lemma 2.1, $\mathcal{L}_n^{\langle r\rangle}(x)$ cannot have a factor of degree i. □

The following lemma is more of general nature which will be useful for higher values of r when l_n, defined in Lemma 3.5, is large. In our proof of Theorem 1.1, $l_n \le 3$ and Lemma 3.6 suffices.

Lemma 3.7. *Let $l > 0$ and let $p | n(r+1)$ and $\nu_p\left(\binom{n+r}{r}\right) = u$. Then $\mathcal{L}_n^{\langle r\rangle}(x)$ cannot have a factor with degree in the interval $[1, l]$ if any one of the following conditions hold:*

(*a*) $u = 0$ *and* $p > l$,
(*b*) $u = 1, p > 2l+1$ *and* $\mu_j < \frac{1}{l}$ *for* $1 \le j \le l$,
(*c*) $u > 1, p = l+1$ *and* $\mu_j < \frac{1}{l}$ *for* $1 \le j \le u - \frac{1}{l}$,
(*d*) $u > 1, p \ne l+1$ *and* $\mu_j < \frac{1}{l}$ *for* $1 \le j \le ul + \frac{(ul-1)l}{p-l-1}$,

where $\mu_j = \frac{\nu_p((r+n)!)-\nu_p((r+n-j)!)-\nu_p(\binom{n}{j})}{j}$ *(as defined in Remark 3.3).*

Proof. Clearly $p \nmid b_0$. If $p|(r+1)$, then $p|b_j$ for all $1 \le j \le n$. If $p \nmid (r+1)$, then $p|n$ implies $p|\binom{n}{j}$ for $1 \le j < p$. Also $p|(r+1)\cdots(r+j)$ for $j \ge p$. Thus $p|b_j$ for all $1 \le j \le n$.

From Remark 3.3, the slope of the right-most edge of $NP_p(L_n^{\langle r\rangle}(x))$ is at most $M_p = \max_{1\le j\le n}\{\mu_j\}$, where

$$\mu_j \le \frac{j-\sigma_p(j)}{(p-1)j} + \frac{u}{j}.$$

(*a*) $u = 0$ and $p > l$. For $1 \le j \le n$, we have

$$\mu_j \le \frac{j-\sigma_p(j)}{(p-1)j} < \frac{1}{p-1} \le \frac{1}{l}.$$

(*b*) $u = 1$ and $p > 2l+1$. For $1 \le j \le l$, we have

$$\mu_j \le \frac{1}{l}.$$

For $l < j < p$, we have

$$\mu_j \le \frac{j-\sigma_p(j)}{(p-1)j} + \frac{1}{j} = \frac{1}{j} < \frac{1}{l}.$$

For $j \geq p$, we have

$$\begin{aligned}\mu_j &\leq \frac{j-\sigma_p(j)}{(p-1)j} + \frac{1}{j} < \frac{1}{p-1} + \frac{1}{j}\\ &< \frac{1}{2l} + \frac{1}{2l} \text{ (since } p-1 \geq 2l \text{ and } j \geq p > 2l)\\ &= \frac{1}{l}.\end{aligned}$$

(c) $u > 1$ and $p \neq l+1$. For $1 \leq j \leq n$, we have

$$\mu_j \leq \frac{j-\sigma_p(j)}{(p-1)j} + \frac{u}{j} \leq \frac{1}{p-1} - \frac{1}{(p-1)j} + \frac{u}{j} = \frac{1}{p-1} + \frac{u(p-1)-1}{(p-1)j}.$$

Thus $\mu_j < \frac{1}{l}$, if

$$\frac{u(p-1)-1}{(p-1)j} < \frac{p-l-1}{(p-1)l} \text{ or } j > ul + \frac{(ul-1)l}{p-l-1}.$$

(d) $u > 1$ and $p = l+1$. For $1 \leq j \leq n$, we have

$$\mu_j \leq \frac{j-\sigma_p(j)}{lj} + \frac{u}{j} \leq \frac{1}{l} - \frac{1}{lj} + \frac{u}{j} = \frac{1}{l} + \frac{ul-1}{lj}.$$

Thus $\mu_j < \frac{1}{l}$, if $\frac{ul-1}{lj} < 0$ or $j > u - \frac{1}{l}$.

Therefore the slope of the right-most edge is less than $\frac{1}{l}$ and hence, by Lemma 2.1, $\mathcal{L}_n^{\langle r\rangle}(x)$ cannot have a factor with degree in the interval $[1, l]$. □

We need the following three lemmas for describing the Galois groups of $L_n^{\langle r\rangle}(x)$. The third lemma is computational.

Lemma 3.8. *Given that $\mathcal{L}_n^{\langle r\rangle}(x)$ is irreducible, if there is a prime p with $\frac{n}{2} < p < n-2$ and $r < p$, then $G_n(r)$ contains A_n.*

Proof. Let $n_0 = n-p$ and $r_0 = p-r$. For $1 \leq j \leq n$, we have

$$\nu_p\left(\binom{n}{j}\right) = \nu_p\left(\frac{n(n-1)\cdots(n-j+1)}{j!}\right) = \begin{cases}1 & \text{if } n_0 < j < p,\\ 0 & \text{otherwise.}\end{cases}$$

First assume that $r+n < 2p$. Note that $r_0 > n_0$ and $r_0 + p = r_0 + n - n_0 > n$. Thus $r + r_0 = p$ is the only multiple of p in the product $(r+1)(r+2)\cdots(r+n)$. So for $1 \leq j \leq n$, we have

$$\nu_p((r+1)(r+2)\cdots(r+j)) = \begin{cases}0 & \text{if } j < r_0,\\ 1 & \text{otherwise.}\end{cases}$$

Therefore $NP_p(\mathcal{L}_n^{\langle r\rangle}(x))$ is given by the lower edges along the convex hull of the points:

$$(0,0), \ldots, (n_0, 0), (n_0+1, 1), \ldots, (r_0-1, 1), (r_0, 2), \ldots, (p-1, 2), (p, 1), \ldots, (n, 1).$$

Thus the vertices of $NP_p(\mathcal{L}_n^{\langle r\rangle}(x))$ are $(0,0), (n_0, 0)$ and $(n, 1)$. Hence $\frac{1}{p}$ is a slope of $NP_p(\mathcal{L}_n^{\langle r\rangle}(x))$ and it follows from Lemma 2.2 that $G_n(r)$ contains A_n.

Next assume that $r+n \geq 2p$. Since $r+n < 3p$, $r+r_0 = p$ and $r+r_0+p = 2p$ are the only multiples of p in the product $(r+1)(r+2)\cdots(r+n)$. So for $1 \leq j \leq n$, we have

$$\nu_p((r+1)(r+2)\cdots(r+j)) = \begin{cases} 0 & \text{if } j < r_0, \\ 1 & \text{if } r_0 \le j < r_0 + p, \\ 2 & \text{if } j \ge r_0 + p. \end{cases}$$

Therefore in this case $NP_p(\mathcal{L}_n^{\langle r\rangle}(x))$ is given by the lower edges along the convex hull of the points:

$$(0,0),\ldots,(r_0-1,0),(r_0,1),\ldots,(r_0+p-1,1),(r_0+p,2),\ldots,(n_0,2),(n_0+1,3),\ldots,$$
$$(p-1,3),(p,2),\ldots,(n,2).$$

Thus the vertices of $NP_p(\mathcal{L}_n^{\langle r\rangle}(x))$ are $(0,0),(r_0-1,0),(r_0+p-1,1)$ and $(n,2)$. Hence $\frac{1}{p}$ is one of the slopes of $NP_p(\mathcal{L}_n^{\langle r\rangle}(x))$ and it follows from Lemma 2.2 that $G_n(r)$ contains A_n. $\square$

Lemma 3.9. *Let $m \ge 197$ be an odd integer and let $k \le 60$ be an even integer. Then product of any two distinct terms in the set $\{m+2, m+4, \ldots, m+k\}$ cannot be a square.*

Proof. Suppose $(m+2i)(m+2j)$ is a square with $1 \le i < j \le \frac{k}{2}$. We may assume $m+2i = ax^2$ and $m+2j = ay^2$ where $y-x \ge 2$. Then $k-2 \ge 2(j-i) = a(y-x)(y+x) \ge 2a(y+x) \ge 4ax$. Therefore $x \le \lfloor \frac{k-2}{4a} \rfloor \le \lfloor \frac{58}{4} \rfloor = 14$ which implies $m \le 195$, a contradiction. $\square$

Lemma 3.10. *There is a prime in every set of 20 consecutive positive integers each ≤ 1129.*

4. Irreducibility of $L_n^{\langle r\rangle}(x)$: Proof of Theorem 1.1($i$)

In this section, we give proof of Theorem 1.1(i) by showing that $L_n^{\langle r\rangle}(x)$ is irreducible for each $23 \le r \le 60$ and $n \ge 1$. Recall that for fixed integers $r \ge 0$ and $n \ge 1$, $n = n_0 n_1$ where

$$n_0 := \prod_{p|n,\ p\nmid\binom{n+r}{r}} p^{\nu_p(n)} \text{ and } n_1 := \prod_{p|\gcd(n,\binom{n+r}{r})} p^{\nu_p(n)}.$$

Let $23 \le r \le 60$ and $n \ge 1$ be integers. Suppose $L_n^{\langle r\rangle}(x)$ has a factor of degree k. By Lemma 2.4, we have $n_0 | k$. So if $n_0 \ge 2$, then $k \ge 2$ and thus Lemma 2.6 implies $r > 3.42k + 1$, i.e., $n_0 \le k < \frac{r-1}{3.42}$. Therefore we have $1 \le n_0 \le \lfloor \frac{r-1}{3.42} \rfloor$ for each value of r.

Fix r with $23 \le r \le 60$. For each n_0, we have

$$\{n = n_0 n_1 : p^{\nu_p(n_1)} \le r\} \subseteq \{n : p^{\nu_p(n)} \le r\}.$$

Since $\lfloor \frac{r-1}{3.42} \rfloor \ge \max\{n_0, \sqrt{r}\}$, if $p|n$ with $p > \lfloor \frac{r-1}{3.42} \rfloor$, then $p | n_1$ and $r < p^2$. Thus, by Lemma 2.7, Lemma 2.8 and Corollary 3.2, it is enough to check irreducibility of $L_n^{\langle r\rangle}(x)$ for $n \in H_r$ where

$$H_r = \{n \in \mathbb{N} : n > 127 \text{ and for each } p|n,\ p^{\nu_p(n)} \le r \text{ and if } p > \left\lfloor \frac{r-1}{3.42} \right\rfloor \text{ then } d + \left\lfloor \frac{r}{p} \right\rfloor \ge p\}$$

where d denotes the remainder of $\frac{n}{p}$ modulo p.

For each $n \in H_r$, we compute k_n and l_n (defined respectively in Lemma 3.4 and Lemma 3.5). We find that $l_n \le 3$ for each $n \in H_r$ and it follows that $k \le l_n \le 3$. For

$1 \le i \le 3$, we define $H_{i,r} = \{n \in H_r : l_n \ge i\}$. To obtain a contradiction, we need to prove non-existence of a factor of degree i for each $n \in H_{i,r}$. For this we use Lemma 3.6 and we are left with $(n, r) \in T$ for which $L_n^{\langle r \rangle}(x)$ may have a factor of degree 1, where T is given by

$$\begin{aligned} T = \{&(144,23),(144,25),(144,26),(144,51),(144,53),(216,29),(216,31),(216,42),\\ &(216,44),(216,47),(216,49),(216,53),(216,59),(240,35),(288,40),(288,41),\\ &(288,47),(288,48),(288,51),(288,53),(312,26),(600,26),(720,31),(1440,35),\\ &(4320,55)\}. \end{aligned}$$

Observe that $p|n$ implies $p|b_j$ for $1 \le j \le n$ (see the first paragraph in the proof of Lemma 3.7). Since $2|n$ and $3|n$ for each n given in T, to remove the existence of a factor of degree 1, by Lemma 2.1 and Remark 3.3, it suffices to show that $\mu_j < 1$ for each $1 \le j \le n$, for either $p = 2$ or $p = 3$, where

$$\begin{aligned} \mu_j &= \frac{\nu_p((r+n)(r+n-1)\cdots(r+n-j+1)) - \nu_p(\binom{n}{j})}{j} \\ &\le \frac{j - \sigma_p(j)}{(p-1)j} + \frac{1}{j}\nu_p\left(\binom{n+r}{r}\right). \end{aligned} \tag{1}$$

It can be easily observed that

$$\frac{j - \sigma_p(j)}{(p-1)j} + \frac{1}{j}\nu_p\left(\binom{n+r}{r}\right) < 1,$$

if and only if,

$$(p-1)\nu_p\left(\binom{n+r}{r}\right) < (p-2)j + \sigma_p(j). \tag{2}$$

For $(n, r) \in T \setminus \{(216, 29), (4320, 55)\}$ and $p = 3$, we find the least positive integer j_0 such that (2) holds for $j \ge j_0$, so that $\mu_j < 1$ for $j \ge j_0$. For $j < j_0$, we verify that $\mu_j < 1$ by using (1). Hence $\mathcal{L}_n^{\langle r \rangle}(x)$ does not have factor of degree 1.

For $(n, r) \in \{(216, 29), (4320, 55)\}$, we take $p = 2$ and proceed as above to verify that $\mathcal{L}_n^{\langle r \rangle}(x)$ does not have a factor of degree 1. $\square$

5. Galois groups of $L_n^{\langle r \rangle}(x)$: Proof of Theorem 1.1($ii$)

In this section, we prove Theorem 1.1(ii) by describing the Galois groups of $L_n^{\langle r \rangle}(x)$ for $23 \le r \le 60$, $n \ge 1$. From Section 4, we have $L_n^{\langle r \rangle}(x)$ is irreducible for each $23 \le r \le 60$ and $n \ge 1$.

For $23 \leq r \leq 60$, let B_r be given by

$$\begin{aligned}
&B_{23} = B_{24} = \cdots = B_{28} = \{1, 2, \ldots, 31\},\\
&B_{29} = B_{30} = \{1, 2, \ldots, 33\},\\
&B_{31} = B_{32} = \cdots = B_{36} = \{1, 2, \ldots, 39\},\\
&B_{37} = B_{38} = \cdots = B_{40} = \{1, 2, \ldots, 43\},\\
&B_{41} = B_{42} = \{1, 2, \ldots, 45\},\\
&B_{43} = B_{44} = \cdots = B_{46} = \{1, 2, \ldots, 49\},\\
&B_{47} = B_{48} = \cdots = B_{52} = \{1, 2, \ldots, 55\},\\
&B_{53} = B_{54} = \cdots = B_{58} = \{1, 2, \ldots, 61\},\\
&B_{59} = B_{60} = \{1, 2, \ldots, 63\}.
\end{aligned}$$

For each $23 \leq r \leq 60$ and $n \in B_r$, we compute $G_n(r)$ using MAGMA online, and in fact, $G_n(r) = A_n$ for $(n, r) \in \{(4, 24), (5, 28), (24, 25), (25, 24), (28, 23), (28, 29), (32, 33), (33, 36), (36, 37), (40, 41), (44, 45), (48, 49), (48, 51), (49, 48), (49, 50), (52, 53), (56, 57)\}$ and $G_n(r) = S_n$ otherwise.

From now onwards, we assume that $n \notin B_r$. We first show that $G_n(r)$ contains A_n.

Fix r with $23 \leq r \leq 60$. We have $\max\{48 - r, 8 + \frac{5r}{3}\} = 8 + \frac{5r}{3}$. Let

$$C_r = \{n \in \mathbb{N} : n < 8 + \frac{5r}{3} \text{ and } \nexists \text{ a prime } p \text{ with } \frac{n+r}{2} < p < n - 2\}.$$

Observe that C_r is finite and $B_r \subseteq C_r$. By Lemma 2.3 (i) and (ii), we have $G_n(r)$ contains A_n for each $n \notin C_r$. For $n \in C_r$, we now apply Lemma 3.8 to get $G_n(r)$ contains A_n for each $n \in C_r$, $n \notin B_r$. Hence $G_n(r)$ contains A_n for $n \notin B_r$.

Thus, by Lemma 2.3(iii), we have

$$G_n(r) = \begin{cases} A_n & \text{if } \Delta_n^{\langle r \rangle} \text{is a square,} \\ S_n & \text{otherwise.} \end{cases}$$

Therefore to complete the proof of Theorem 1.1(ii), it suffices to check if $\Delta_n^{\langle r \rangle}$ is a square or not. In fact, we show that for each $23 \leq r \leq 60$ and $n \notin B_r$, $\Delta_n^{\langle r \rangle}$ is never a square.

For integers a and b, we write $a \sim b$ if $a = bc^2$ for some integer $c > 0$. We consider the following cases:

Case 1. n is odd: We have

$$\Delta_n^{\langle r \rangle} \sim (-1)^{n(n-1)/2}(1 \cdot 3 \cdot 5 \cdots n)(n + r - 1)(n + r - 3) \cdots (r + 2).$$

If $n \equiv 3 \pmod 4$, then $\Delta_n^{\langle r \rangle}$ is not a square. Thus assume $n \equiv 1 \pmod 4$.

Subcase 1(a). r is even: By re-arranging the factors, we see that

$$\Delta_n^{\langle r \rangle} \sim (1 \cdot 3 \cdot 5 \cdots (r - 1))((r + 1)(r + 2) \cdots n)(n + 1)(n + 3) \cdots (n + r - 1).$$

For $n > \frac{3(r-1)}{2}$, we have

$$\frac{n + r - 1}{2} < \frac{5}{6}n.$$

By Lemma 2.9 with $x = \frac{5}{6}n$, there is a prime p satisfying

$$\frac{n + r - 1}{2} < p < n$$

so that $\nu_p(\Delta_n^{\langle r\rangle})$ is odd, and hence $\Delta_n^{\langle r\rangle}$ is not a square.

For $n \le \frac{3(r-1)}{2}$ with $n \notin B_r$, we check directly that $\Delta_n^{\langle r\rangle}$ is not a square.

Subcase 1(b). r is odd: By re-arranging the factors, we see that

$$\Delta_n^{\langle r\rangle} \sim (1\cdot 3\cdot 5\cdots r)(n+2)(n+4)\cdots(n+r-1).$$

If $n \le 1070$, then $n+r-1 \le 1129$ and since there are at least 10 consecutive odd integers in $\{n+2, n+4, \ldots, n+r-1\}$, it follows from Lemma 3.10 that there is a prime p in this set. For $\frac{r-3}{2} \le n \le 1070$, we have

$$\frac{r-3}{2} \le n \le p-2 < p \le n+r-1 < 3p.$$

Since $n+2, n+4, \ldots, n+r-1$ are all odd, $2p$ is not in the set $\{n+2, n+4, \ldots, n+r-1\}$ and hence we get $\nu_p(\Delta_n^{\langle r\rangle})$ is odd. Therefore $\Delta_n^{\langle r\rangle}$ is not a square.

For $n < \frac{r-3}{2}$ with $n \notin B_r$, we check directly that $\Delta_n^{\langle r\rangle}$ is not a square. Now suppose that $n > 1070$ and $\Delta_n^{\langle r\rangle}$ is a square.

Let $r = 23$. Then

$$\Delta_n^{\langle r\rangle} \sim (3\cdot 11\cdot 13\cdot 17\cdot 19\cdot 23)(n+2)(n+4)\cdots(n+22).$$

There are at most 5 terms in $\{n+2, n+4, \ldots, n+22\}$ which are divisible by $11, 13, 17, 19$ or 23. After removing these terms, we are left with at least 6 terms each of which is either a square or 3 times a square. Therefore there are two distinct terms in $\{n+2, n+4, \ldots, n+22\}$ whose product is a square. This contradicts Lemma 3.9 for $m = n$ and $k = r-1$. Therefore $\Delta_n^{\langle r\rangle}$ is not a square.

Similarly, for $r \in \{25, 33, 35, 51, 53, 55\}$, we get a contradiction using Lemma 3.9 as above.

Let $r = 27$. Then

$$\Delta_n^{\langle r\rangle} \sim (11\cdot 13\cdot 17\cdot 19\cdot 23)(n+2)(n+4)\cdots(n+26).$$

There are at most 4 terms in $\{n+2, n+4, \ldots, n+26\}$ which are divisible by $13, 17, 19$ or 23 and further 11 divides at most 2 terms of this set. After removing these terms, we are left with 7 terms in this set which are squares. This contradicts Lemma 3.9 for $m = n$ and $k = r-1$. Thus $\Delta_n^{\langle r\rangle}$ is not a square.

For $r \in \{29, 31, 39, 41, 43, 45, 47, 49, 57, 59\}$, we proceed as in the case of $r = 27$ and get a contradiction using Lemma 3.9.

Let $r = 37$. Then

$$\Delta_n^{\langle r\rangle} \sim (3\cdot 5\cdot 7\cdot 13\cdot 17\cdot 19\cdot 23\cdot 29\cdot 31\cdot 37)(n+2)(n+4)\cdots(n+36).$$

The number of terms in $\{n+2, n+4, \ldots, n+36\}$ divisible by $7, 13$ and 17 are at most $3, 2$ and 2 respectively. Also each of $19, 23, 29, 31$ and 37 divides at most one term in this set. After removing these terms, we are left with 6 terms in the set $\{n+2, n+4, \ldots, n+36\}$ each of which is either a square or of the form ax^2 with $a \in \{3, 5, 15\}$ and it follows that there are two distinct terms in $\{n+2, n+4, \ldots, n+36\}$ whose product is a square. We get a contradiction using Lemma 3.9 as above.

Case 2. n is even: We have

$$\Delta_n^{\langle r\rangle} \sim (-1)^{n(n-1)/2}(1\cdot 3\cdot 5\cdots(n-1))(n+r-1)(n+r-3)\cdots(r+1).$$

If $n \equiv 2 \pmod 4$, then $\Delta_n^{\langle r\rangle}$ is not a square. Thus assume $n \equiv 0 \pmod 4$.

Subcase 2(a). r is odd: By re-arranging the factors, we see that

$$\Delta_n^{\langle r\rangle} \sim (1\cdot 3\cdot 5\cdots(r-2))(r(r+1)\cdots n)(n+2)(n+4)\cdots(n+r-1).$$

For $n > \frac{3(r-1)}{2}$, we have

$$\frac{n+r-1}{2} < \frac{5}{6}n.$$

By Lemma 2.9 with $x = \frac{5}{6}n$, there is a prime p satisfying

$$\frac{n+r-1}{2} < \frac{5}{6}n < p < n$$

so that $\nu_p(\Delta_n^{\langle r\rangle})$ is odd, and hence $\Delta_n^{\langle r\rangle}$ is not a square.

For $n \le \frac{3(r-1)}{2}$ with $n \notin B_r$, we check directly that $\Delta_n^{\langle r\rangle}$ is not a square.

Subcase 2(b). r is even: By re-arranging the factors, we see that

$$\Delta_n^{\langle r\rangle} \sim (1\cdot 3\cdot 5\cdots(r-1))(n+1)(n+3)\cdots(n+r-1).$$

If $n \le 1070$, then $n+r-1 \le 1129$ and since there are at least 10 consecutive odd integers in $\{n+1, n+3, \dots, n+r-1\}$, it follows from Lemma 3.10 that there is a prime p in this set. For $\frac{r-2}{2} \le n \le 1070$, we have

$$\frac{r-2}{2} \le n \le p-1 < p \le n+r-1 < 3p.$$

Since $n+1, n+3, \dots, n+r-1$ are all odd, we get $\nu_p(\Delta_n^{\langle r\rangle})$ is odd. Hence $\Delta_n^{\langle r\rangle}$ is not a square.

For $n < \frac{r-2}{2}$ with $n \notin B_r$, we check directly that $\Delta_n^{\langle r\rangle}$ is not a square. Now we suppose that $n > 1070$ and $\Delta_n^{\langle r\rangle}$ is a square.

Let $r = 24$. Then

$$\Delta_n^{\langle r\rangle} \sim (3\cdot 11\cdot 13\cdot 17\cdot 19\cdot 23)(n+1)(n+3)\cdots(n+23).$$

There are at most 4 terms in $\{n+1, n+3, \dots, n+23\}$ which are divisible by $13, 17, 19$ or 23 and further 11 divides at most 2 terms of this set. After removing these terms, we are left with 6 terms each of which is either a square or 3 times a square. Thus there are two distinct terms in $\{n+1, n+3, \dots, n+23\}$ whose product is a square. This contradicts Lemma 3.9 for $m = n-1$ and $k = r$. Therefore $\Delta_n^{\langle r\rangle}$ is not a square.

Similarly, for $r \in \{26, 34, 36, 38, 52, 54, 56\}$, we get a contradiction using Lemma 3.9 as above.

Let $r = 28$. Then

$$\Delta_n^{\langle r\rangle} \sim (11\cdot 13\cdot 17\cdot 19\cdot 23)(n+1)(n+3)\cdots(n+27).$$

There are at most 3 terms in $\{n+1, n+3, \dots, n+27\}$ which are divisible by $17, 19$ or 23 and further each of 11 and 13 divides at most 2 terms of this set. After removing these terms, we are left with 7 terms in this set which are squares. This contradicts Lemma 3.9 for $m = n-1$ and $k = r$. Thus $\Delta_n^{\langle r\rangle}$ is not a square.

For $r \in \{30, 32, 40, 42, 44, 46, 48, 50, 58, 60\}$, we proceed as in the case of $r = 36$ and get a contradiction using Lemma 3.9. $\square$

6. Proof of Theorem 1.2

Suppose that $L_n^{\langle r\rangle}(x)$ has a factor of degree k. Then by Lemma 2.5, $k < \frac{r}{1.63}$. By Lemma 2.4, we have $n_0 \le k < \frac{r}{1.63}$. Thus if $p|n_0$, then $p^{\nu_p(n_0)} < r$ and in fact $p^{\nu_p(n)} = p^{\nu_p(n_0)} < r$. Also by Lemma 2.8, if $p|n_1$, then $p^{\nu_p(n)} \le r$. Hence

$$n = n_0 n_1 = \prod_{p|n} p^{\nu_p(n)} \le \prod_{p\le r} r = r^{\pi(r)} = e^{\pi(r)\ \log\ r} \le e^{r\left(1+\frac{1.2762}{\log r}\right)}$$

by Lemma 2.10. This proves Theorem 1.2. □

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