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# Cubes in products of terms from an arithmetic progression

Pranabesh Das, Shanta Laishram and N. Saradha

Indian Statistical Institute, Delhi Centre
7, SJSS Marg, New Delhi–110 016, India

# CUBES IN PRODUCTS OF TERMS FROM AN ARITHMETIC PROGRESSION

PRANABESH DAS, SHANTA LAISHRAM AND N. SARADHA

*Dedicated to Professor R. Tijdeman on his 75th birthday*

Abstract. We show that there are no cubes in a product with at least $k-(1-\epsilon)k\frac{\log\log k}{\log k}, \epsilon > 0$, terms from a set of $k(\geq 2)$ successive terms in an arithmetic progression having common difference $d$ if either $k$ is sufficiently large or $3^{\omega(d)} \gg k\frac{\log\log k}{\log k}$. Here $\omega(d)$ denotes the number of distinct prime divisors of $d$. This result improves an earlier result of Shorey and Tijdeman.

## 1. Introduction

Let $b, d, \ell > 1, m, k \geq 2, t \geq 2$ be positive integers. Further let $d_1, \cdots, d_t \in [0, k)$ be integers with $d_1 < d_2 < \cdots < d_k$. For any integer $n > 1$, let $P(n)$ denote the greatest prime factor on $n$ and put $P(1) = 1$. We consider solutions to the equation

$$\Delta_d = (m + d_1 d) \cdots (m + d_t d) = by^\ell, P(b) < k \tag{1}$$

i.e., we are looking for perfect powers or almost perfect powers in a product of $t$ terms which are taken from $k$ terms in an arithmetic progression. When $t = k$, all the $k$ terms in the arithmetic progression are present in $\Delta_d$. When $t < k$, then $k-t$ terms are missing in $\Delta_d$. This equation has been intensely studied and several papers are available in the literature since the remarkable work of Erdős and Selfridge [3] in 1975 where they showed that

*a product of two or more consecutive positive integers is never a perfect power.*

In other words, equation (1) with $b = d = 1, t = k$ has no solution. Fundamental contributions were made by Shorey and Tijdeman towards (1). We refer to the expository articles of Shorey [9] and [10] for a detailed account of many of the results. In this paper, we improve a result of Shorey and Tijdeman [11] on the value of $t$ in the case of cubes i.e. when $\ell = 3$.

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## 2. **Results-old and new**

In 1955, Erdős [2] proved the following result when $d = 1$.

*Suppose* (1) *with* $b = d = 1, \ell \geq 3, m > k^\ell$ *hold and*

$$t \geq k - (1-\epsilon)\frac{k \log\log k}{\log k}, \epsilon > 0.$$

*Then $k$ is bounded by an absolute constant.*

This result was considerably sharpened by Shorey [7, 8] and later by Nesterenko & Shorey [5]. As a result of [5], one obtains that $k$ is bounded by an absolute constant whenever

$$t > .4832k,\ \ell \geq 7 \text{ and } P(b) \leq k.$$

The results for $\ell > 2$ depend on the theory of linear forms in logarithms,irrationality measures of Baker based on hypergeometric method and estimates of Halberstam and Roth on difference between consecutive $k$ free integers.

When $\ell = 2$, in the above result of Erdős, $t$ was taken to be $t \geq k - c_1\frac{k}{\log k}$, where $c_1$ is an absolute constant. This was sharpened by Shorey [8] to $t \geq k - (1-\epsilon)k\frac{\log\log k}{\log k}$ and later relaxed further by Balasubramanian and Shorey [1].

From now on we assume that $d > 1$. Shorey and Tijdeman combined several elementary arguments of Erdős with the application of box principle on numerous occassions in a beautiful paper to show the following result (see [11, p.343]). Throughout the paper, we denote by $c(\epsilon) > 0$ an effectively computable, sufficiently large number depending on $\epsilon > 0$.

*Let $\epsilon > 0$. Equation* (1) *with $d > 1$ and $P(y) > k$ implies that there exists an absolute constant $c_2$ such that either*

$$k \leq c(\epsilon) \text{ or } \ell^{\omega(d)} \geq c_2 k\frac{h(k)}{\log k}$$

*provided*

$$t > k - (1-\epsilon)k\frac{h(k)}{\log k} \tag{2}$$

*where*

$$h(k) = \begin{cases} \log\log k \ if\ \ell \geq 5 \\ \log\log\log k \ if\ \ell = 3 \\ 1 \ if\ \ell = 2. \end{cases}$$

Recently, Saradha and Shorey [6] improved the above result for $\ell = 2$ as follows.

*Suppose equation (1) holds with $d > 1$ and $P(y) > k$. Then there exist positive numbers $c_3$ and $c_4$ depending only on $d$ such that if*

$$t \geq k - k\left(\frac{\log\log k}{\log k} - \frac{\log\log\log k}{\log k} - \frac{c_3}{\log k}\right) \tag{3}$$

*then $k \leq c_4$.*

We observe from the above two results that (2) is weaker for the values of $t$ when $\ell = 3$. In this paper we bring the case of cubes on par with other values of $\ell$ as far as $t$ is concerned.

**Theorem 2.1.** *Equation (1) with $d > 1, P(y) > k$ and $\ell = 3$ implies that there exists an absolute constant $c_5$ such that either $k \leq c(\epsilon)$ or $3^{\omega(d)} \geq c_5 k^{\frac{\log\log k}{\log k}}$ provided*

$$t > k - (1-\epsilon)k\frac{\log\log k}{\log k}.$$

The above improvement is due to a better box principle that is applied while counting the number of distinct Thue equations. See Section 7 below.

## 3. **Preliminaries**

Suppose equation (1) holds with $P(y) > k$. Then

$$m + d_t d \geq (k+1)^3 \tag{4}$$

implying that

$$m + d > k^2. \tag{5}$$

Further we may write

$$m + d_i d = A_i X_i^3, P(A_i) < k, \gcd(X_i, \prod_{p<k} p) = 1, 1 \leq i \leq t$$

where the product is taken over all primes $< k$. Note that since $\gcd(m,d) = 1$, we have

$$\gcd(A_i, d) = \gcd(X_i, d) = 1 \text{ for } 1 \leq i \leq t$$

and

$$X_i \geq k \text{ if } X_i \neq 1 \text{ and } \gcd(X_i, X_j) = 1 \text{ if } i \neq j. \tag{6}$$

Let

$$S_1 = \{A_1, \cdots, A_t\}.$$

By deleting a term corresponding to every prime $< k$, we find that there exists a set $S_2$ with

$$S_2 \subseteq S_1 \text{ and } \prod_{A_i \in S_2} A_i \Big| k!. \tag{7}$$

Let $K = \mathbb{Q}(\rho)$ where $\rho = e^{2\pi i/3}$. Then $[K : \mathbb{Q}] = 2$ and disc$(K)$=3. For any prime $p \neq 3$, we have the following possibilities for the ideal $[p]$ in $K$.

$$[p] = \wp \textit{ or } \wp^2 \textit{ or } \wp_1\wp_2$$

where $\wp, \wp_1, \wp_2$ are prime ideals in $K$. Thus for any integer $d > 1$, we can write $d' = d/3^{\text{ord}_3(d)}$ as

$$[d'] = D_1 D_2 D_3 \tag{8}$$

where $D_1, D_2, D_3$ are coprime ideals of $K$ in at most $9^{\omega(d')}$ ways.

## 4. On Equal $A_i$'s

Although many of the facts below are true for any prime $\ell \geq 3$, we restrict to $\ell = 3$. We refer to [11, pages 326-336] for the various arguments given in the lemmas below. We have put them in lemmas to make the exposition more lucid.

**Lemma 4.1.** *Suppose* (1) *holds with* $\ell = 3$ *with* $A_\mu = A_\nu$ *for some* $A_\mu, A_\nu$ *in* $S_1$ *with* $\mu < \nu$. *Then* $\frac{X_\nu}{X_\mu}$ *belongs to at most* $3^{\omega(d)}$ *residue classes (mod d).*

*Proof.* We have the identity

$$(\nu - \mu)d = A_\mu(X_\nu^3 - X_\mu^3).$$

It follows that $A_\mu < k, A_\nu < k$ and by (5), $X_\mu \neq 1, X_\nu \neq 1$. Hence by (6), they are distinct. Also

$$X_\nu^3 - X_\mu^3 \equiv 0 (\text{mod } d).$$

Let $R(d)$ denote the number of residue classes $z$ such that $z^3 \equiv 1($ mod $d)$. Then a result of Evertse [4] gives that $R(d) \leq 3^{\omega(d)}$ which implies the assertion of the lemma. □

In the next lemma we find out many $X_\nu/X_\mu$ belonging to distinct residue classes.

**Lemma 4.2.** *Suppose* (1) *holds with* $\ell = 3$ *and* $m + (k-1)d \geq k^{4-\epsilon}, \epsilon > 0$. *Assume that*

$$|S_1| < t - (1 - \epsilon/2)k \frac{\log \log k}{\log k}$$

*Then for $k \geq c(\epsilon)$, there are at least*

$$\lceil (1-\epsilon)k\frac{\log\log k}{\log k}\rceil$$

*distinct pairs $(\mu, \nu)$ with $X_\nu/X_\mu$ belonging to distinct residue classes.*

*Proof.* Let

$$R = \left\{ i : t \geq i \geq \epsilon k\frac{\log\log k}{4\log k} \right\} \text{ and } S_3 = \{A_i \in S_1 : i \in R\}.$$

For $A_i \in S_3$, let $\chi(A_i) = |\{j \in R : A_j = A_i\}|$ and for $h \geq 1$, let

$$C_h = |\{A_i : \chi(A_i) = h\}|.$$

Then clearly

$$\sum hC_h = |R| \geq t - \epsilon k\frac{\log\log k}{4\log k} \text{ and } \sum C_h \leq |S_3| \leq |S_1|.$$

Let $\chi(A_{i_1}) = h$ and $i_1 < i_2 < \cdots < i_h$ be such that $A_{i_1} = A_{i_2} = \cdots = A_{i_h}$. Then there are $\frac{h(h-1)}{2}$ distinct pairs $(\mu, \nu)$ with $\mu > \nu$ and $A_\mu = A_\nu = A_{i_1}$. Hence the number of distinct pairs $(\mu, \nu)$ with

$$k > \nu > \mu \geq \epsilon k\frac{\log\log k}{4\log k} \text{ and } A_\mu = A_\nu \tag{9}$$

is

$$\sum \frac{h(h-1)}{2}C_h \geq \sum hC_h - \sum C_h \geq \left(t - \frac{\epsilon k\log\log k}{4\log k}\right) - |S_1|$$
$$\geq (1-3\epsilon/4)k\frac{\log\log k}{\log k}.$$

Hence we can find at least

$$\lceil (1-\epsilon)k\frac{\log\log k}{\log k}\rceil$$

distinct pairs $(\mu, \nu)$ with (9).

Thus the conclusion of the lemma is true if we show that

$$X_{\nu_1}/X_{\mu_1} \text{ and } X_{\nu_2}/X_{\mu_2}$$

belong to distinct residue classes (mod $d$) for two distinct pairs $(\mu_1, \nu_1)$ and $(\mu_2, \nu_2)$ satisfying (9). We now proceed to prove this claim. Suppose there exist pairs $(\mu_1, \nu_1) \neq (\mu_2, \nu_2)$, with

$$\frac{X_{\mu_1}}{X_{\nu_1}} \equiv \frac{X_{\mu_2}}{X_{\nu_2}} (\text{mod } d). \tag{10}$$

Consider

$$\delta = (m + \mu_1 d)(m + \nu_2 d) - (m + \nu_1 d)(m + \mu_2 d)$$

$$= A_{\mu_2}A_{\nu_1}\left((X_{\mu_1}X_{\nu_2})^3 - (X_{\nu_1}X_{\mu_2})^3\right).$$

Assume $X_{\mu_1}X_{\nu_2} > X_{\nu_1}X_{\mu_2}$. Then

$$|\delta| \geq (A_{\mu_2}A_{\nu_1})^{1/3} 3d\left((A_{\mu_2}X_{\mu_2}^3)(A_{\nu_1}X_{\nu_1}^3)\right)^{2/3}$$

$$\geq 3d\left(m + \epsilon kd\frac{\log\log k}{2\log k}\right)^{4/3} \geq 3d\frac{\epsilon^2}{4}\left(\frac{(m+(k-1)d)\log\log k}{\log k}\right)^{4/3}.$$

On the other hand,

$$|\delta| \leq 2kd(m+(k-1)d).$$

Comparing the lower and upper bounds for $|\delta|$ we get

$$m+(k-1)d < \left(\frac{8}{3\epsilon^2}\right)^3 k^3 \left(\frac{\log k}{\log\log k}\right)^4$$

giving

$$m+(k-1)d < k^{4-\epsilon}$$

since $k$ is sufficiently large. This is a contradiction. □

Putting together the above two lemmas we get the following proposition.

**Proposition 4.3.** *Suppose equation* (1) *holds with* $\ell = 3$. *For* $\epsilon > 0$, *let*

$$m+(k-1)d \geq k^{4-\epsilon};\ 3^{\omega(d)} < \lceil (1-\epsilon)k\frac{\log\log k}{\log k}\rceil$$

*and*

$$t > k - (1-\epsilon)k\frac{\log\log k}{\log k}.$$

*Then for* $k > c(\epsilon)$ *we have*

$$|S_1| \geq t - (1-\epsilon/2)k\frac{\log\log k}{\log k}.$$

**Lemma 4.4.** *Suppose equation* (1) *holds with* $\ell = 3$. *Then the number of equal* $A_i$*'s do not exceed* $9^{\omega(d)}+1$.

*Proof.* Let $Z = 9^{\omega(d)}+2$. Suppose there exist indices $1 \leq \mu_1 < \mu_2 < \cdots < \mu_Z \leq t$ such that

$$A_{\mu_1} = \cdots = A_{\mu_Z}.$$

For any $\mu_r$ with $2 \leq r \leq Z$, we have

$$(\mu_r - \mu_1)d = A_{\mu_1}(X_{\mu_r}^3 - X_{\mu_1}^3).$$

Since $\gcd(A_{\mu_1}, d) = 1$ the ideal $[d]$ divides $P_r = [X_{\mu_1} - X_{\mu_r}][X_{\mu_1} - \rho X_{\mu_r}][X_{\mu_1} - \rho^2 X_{\mu_r}]$ in $K$. There are $Z - 1 > 9^{\omega(d)}$ products $P_r$. Hence there exist coprime ideals $D_1, D_2, D_3$ as in (8) and indices $r, s$ such that $2 < \mu_r < \mu_s < Z$ with

$$D_h\Big|[X_{\mu_1} - \rho^h X_{\mu_r}]; D_h\Big|[X_{\mu_1} - \rho^h X_{\mu_s}], h = 1, 2, 3.$$

Hence $D_h\Big|[X_{\mu_r} - X_{\mu_s}]$ implying

$$d'\Big|(X_{\mu_r} - X_{\mu_s}).$$

Since $A_{\mu_r} = A_{\mu_s}$, we have $d\Big|(X^3_{\mu_r} - X^3_{\mu_s})$. Further it is well known that

$$9 \nmid \frac{X^3_{\mu_r} - X^3_{\mu_s}}{X_{\mu_r} - X_{\mu_s}}.$$

Thus we get

$$3^{\text{ord}_3(d)-1}\Big|(X_{\mu_r} - X_{\mu_s}) \text{ if } 3\Big|d.$$

Putting together the above facts, we find that

$$|X_{\mu_r} - X_{\mu_s}| > d/3. \tag{11}$$

From the identity

$$(\mu_r - \mu_s)d = A_{\mu_r}(X^3_{\mu_r} - X^3_{\mu_s})$$

and (11) it follows that

$$kd \geq d(A_{\mu_r} X^3_{\mu_r})^{2/3} \geq d\left(\frac{(m + (k-1)d)}{k}\right)^{2/3}$$

which simplifies to

$$m + (k-1)d \leq k^{5/2}.$$

This is a contradiction since $m + (k-1)d > k^3$ by (4). □

## 5. Large number of small $A_i$'s

For the following lemma we refer to [11, Lemma 6].

**Lemma 5.1.** *Let $0 < \eta < 1/2$. Let $S' \subseteq S_1$ such that*

$$\prod_{A_i \in S'} A_i | k!$$

*Suppose $g$ is a positive number such that $g \leq (\eta \log k)/8$ and*

$$|S'| \geq t - \frac{gk}{\log k} \tag{12}$$

*Then there exists a set $S'' \subseteq S'$ with at least $\eta k/2$ elements satisfying*

$$A_i \leq 4e^{(1+\eta)g}k.$$

We apply the above lemma to get the following result.

**Lemma 5.2.** *Suppose equation* (1) *holds with* $\ell = 3$. *For* $\epsilon > 0$ *let*

$$m + (k-1)d \geq k^{4-\epsilon};\ 3^{\omega(d)} < \lceil (1-\epsilon)k\frac{\log\log k}{\log k}\rceil$$

*and*

$$t > k - (1-\epsilon)k\frac{\log\log k}{\log k}.$$

*Then for* $k > c(\epsilon)$, *there exists* $S_4 \subseteq S_1$ *with* $|S_3| \geq \epsilon k/20$ *such that*

$$A_i \leq ke^{(1-\epsilon/4)\log\log k}\ for\ A_i \in S_4.$$

*Proof.* As noted in the preliminaries, there is a set $S_2 \subseteq S_1$ with $|S_2| \geq |S_1| - \pi(k)$ and such that

$$\prod_{A_i \in S_2} A_i | k!.$$

By Proposition 4.3, we have therefore

$$|S_2| \geq t - (1-\epsilon/3)k\frac{\log\log k}{\log k}.$$

In Lemma 5.1, we take $S' = S_2, g = (1-\epsilon/3)\log\log k, \eta = \epsilon/10$. Then there exists a set $S_4 \subseteq S_2$ satisfying the properties of the lemma. □

Another instance when $A_i$ may be small is when $X_i \neq 1$. We shall show that there are few terms in $\Delta_d$ with $X_i = 1$.

**Lemma 5.3.** *Suppose equation* (1) *holds with* $\ell = 3$ *and for* $\epsilon > 0$, *let*

$$t > k - (1-\epsilon)k\frac{\log\log k}{\log k}.$$

*Suppose*

$$T_1 = \{\mu : 2 \leq \mu \leq t\ and\ X_\mu = 1\}.$$

*Then for* $k > c(\epsilon)$, *we have* $|T_1| \leq 2k/3$.

*Proof.* For any $\mu \in T_1$, we have $A_\mu = m + d_\mu d \geq m + d$. From $T_1$, for every prime $p < k$ delete one $\mu$ such that $p$ appears to the maximum power in $A_\mu$. Let $T_2$ be the set of remaining $\mu$'s. Then $|T_2| \geq |T_1| - \pi(k)$ and

$$\prod_{\mu \in T_2} A_\mu | k!$$

This implies

$$(m+d)^{|T_1| - \pi(k)} \leq k^k$$

giving by (5) that

$$|T_1| \leq k/2 + \pi(k) \leq 2k/3$$

since $k$ is large. □

**REMARK.** By the above lemma, we therefore get that the complement set $T_3$ of $T_1$ in $[2, t]$ satisfies $|T_3| \geq t - 1 - 2k/3 \geq k/4$ and for each $\mu \in T_3$, we have $X_\mu \neq 1$ and hence

$$A_\mu \leq (m + (k-1)d)/k^3 \text{ for } \mu \in T_3. \tag{13}$$

## 6. Many pairs of $A_i$'s with large gcd

We use an argument of Erdős to show that there is a subset of $S_3$ in which gcd of any two elements is large.

**Lemma 6.1.** *Suppose equation* (1) *holds with* $\ell = 3$. *For* $\epsilon > 0$, *let*

$$m + (k-1)d \geq k^{3-\epsilon};\ 3^{\omega(d)} < \lceil (1-\epsilon)k \frac{\log \log k}{\log k} \rceil$$

*and*

$$t > k - (1-\epsilon)k \frac{\log \log k}{\log k}.$$

*Then for* $k > c(\epsilon)$, *there exists a set* $S_5 \subseteq S_1$ *with* $|S_4| \geq \epsilon k/80$ *and such that* $\gcd(A_\mu, A_\nu) \geq k^{1-\theta}$ *where* $\theta < 1$.

*Proof.* We take the set $S_4$ given by Lemma 5.2. Let

$$S_6 = \{A_i \in S_4 : A_i \geq k^{1-\theta}\}.$$

Then

$$|S_6| \geq \epsilon k/40.$$

Denote by $b_1, \cdots, b_s$ all integers between $k^{1-\theta}$ and $ke^{(1-\epsilon/4)\log\log k}$ such that every proper divisor of $b_i$ is $\leq k^{1-\theta}$. Observe that if $b_i \geq k^{1-\theta/2}$, every prime divisor of $b_i$ is $\geq k^{\theta/2}$. Hence by Brun's sieve,

$$s \ll k^{1-\theta/2} + \frac{2ke^{(1-\epsilon/4)\log\log k}}{\theta \log k}. \tag{14}$$

Every element of $S_6$ is divisible by at least one $b_i$. Let

$$S_7 = \{A_i \in S_6 : \exists\ b_i \text{ dividing } A_i \text{ and no other } A_j \text{ in } S_6\}.$$

Then

$$|S_7| \leq s$$

and hence by (14), taking $S_5$, as the complement of $S_7$ in $S_6$ we get

$$|S_5| \geq \epsilon k/80.$$

Further, for any $A_\mu, A_\nu \in S_5$ we have

$$\gcd(A_\mu, A_\nu) \geq b_i$$

for some $b_i$ which gives the assertion of the lemma. □

## 7. Proof of Theorem 2.1 when $m+(k-1)d \geq k^{4-\epsilon}$

Suppose equation (1) with $d>1, P(y)>k$ and $\ell=3$ holds. Let $\epsilon>0$. Suppose

$$m+(k-1)d \geq k^{4-\epsilon}$$

and

$$t > k-(1-\epsilon)k\frac{\log\log k}{\log k}.$$

Also assume that $k>c_1(\epsilon)$ where $c_1(\epsilon)$ is sufficiently large. We shall show that

$$3^{\omega(d)} \geq (1-\epsilon)k\frac{\log\log k}{\log k}.$$

Suppose not. Then we take $S_5$ as in Lemma 6.1 with $\theta<1/3$. For any pair $(A_\mu, A_\nu)$ with $A_\mu, A_\nu \in S_5$, form the cubic equation

$$A_\mu X_\mu^3 - A_\nu X_\nu^3 = (\mu-\nu)d.$$

This reduces to

$$LX^3 - MY^3 = Nd \tag{15}$$

where

$$L=\frac{A_\mu}{\gcd(A_\mu, A_\nu)}, M=\frac{A_\nu}{\gcd(A_\mu, A_\nu)}, N=\frac{\mu-\nu}{\gcd(A_\mu, A_\nu)}$$

and $L, M, N$ are co-prime and bounded by $k^\theta e^{(1-\epsilon/4)\log\log k}$ and (15) has a solution $(X_\mu, X_\nu)$. The number of distinct pairs $(A_\mu, A_\nu)$ is at least $(\epsilon k/80)^2/2$ and the number of distinct triples $(L, M, N)$ is at most $k^{3\theta}e^{3(1-\epsilon/4)\log\log k}$. Hence there are at least

$$K=\frac{1}{2}(\epsilon/80)^2 k^{2-3\theta} e^{-3(1-\epsilon/4)\log\log k}$$

pairs $(A_\mu, A_\nu)$ which give rise to the same cubic equation as in (15). Further this cubic equation has at least $K$ distinct solutions $(X_\mu, X_\nu)$. By a result of Evertse [4] and our supposition we get

$$K \leq 3^{\omega(d)} < (1-\epsilon)k\frac{\log\log k}{\log k}$$

which gives $k \leq c_1(\epsilon)$. Taking $c(\epsilon)>c_1(\epsilon)$ we get a contradiction. □

## 8. Proof of Theorem 2.1 when $m+(k-1)d<k^{4-\epsilon}$

Suppose equation (1) with $d>1, P(y)>k$ and $\ell=3$ holds. Let

$$m+(k-1)d<k^{4-\epsilon}$$

and

$$t>k-(1-\epsilon)k\frac{\log\log k}{\log k}.$$

Also assume that $k>c(\epsilon)$. We take the set $T_3$ as given in the Remark after Lemma 5.3. Thus by (13) there are at least $k/4$ number of $A_\mu$'s with

$$A_\mu\le k^{1-\epsilon}.$$

Hence there are at least $k^\epsilon/4$ number of $A_\mu$'s which are equal. By Lemma 4.4,

$$k^\epsilon/4\le 9^{\omega(d)}+1.$$

It is well known that $\omega(d)\le 4\log d/\log\log d$. Since $d<k^{3-\epsilon}$,we have

$$9^{\omega(d)}+1<k^{5(3-\epsilon)/\log\log k}<k^\epsilon/4$$

for $k>c(\epsilon)$. This is a contradiction. □

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Stat-Math Unit, Indian Statistical Institute Delhi Centre, New Delhi 110016, INDIA

*E-mail address*: pranabesh.math@gmail.com

*E-mail address*: shanta@isid.ac.in

School of Mathematics, Tata Institute of Fundamental Research, Mumbai, 400005, INDIA

*E-mail address*: saradha@math.tifr.res.in