isid/ms/2017/01
April 7, 2017
http://www.isid.ac.in/~statmath/index.php?module=Preprint

Construction of some new families of nested orthogonal arrays

Tian-fang Zhang, Guobin Wu and Aloke Dey

Indian Statistical Institute, Delhi Centre
7, SJSS Marg, New Delhi–110 016, India

Construction of some new families of nested orthogonal arrays

Tian-fang Zhang, Guobin Wu
College of Mathematical and Informational Sciences,
Jiangxi Normal University, Nanchang 330022, China
and
Aloke Dey
Indian Statistical Institute, New Delhi 110016, India

Abstract

Nested orthogonal arrays have been used in the design of an experimental setup consisting of two experiments, the expensive one of higher accuracy being nested in a larger and relatively less expensive one of lower accuracy. In this paper, we provide new methods of construction of two types of nested orthogonal arrays.

MSC: 62K15

Keywords: Galois field; Modulus projection; Nested orthogonal arrays.

---

1 Introduction and Preliminaries

Computer models are widely used in business, engineering and sciences to study real-world systems, especially when the corresponding physical experiment might be time consuming, costly or even infeasible to conduct. Space-filling designs are desirable for conducting computer experiments. Nested orthogonal arrays have practical use in the construction of space-filling designs when an experimental endeavour consists of two experiments, the expensive one of higher accuracy to be nested in a larger and relatively inexpensive one of lower accuracy. For example, the higher and lower accuracy experiments can correspond to a physical versus a computer experiment, or a detailed versus an approximate computer experiment, respectively. Experimental setups of this kind were considered, among others, by Qian and Wu (2008), Qian, Tang and Wu (2009) and Sun, Liu and Qian (2014). There are two types of nested orthogonal arrays available in the literature. The purpose of this paper is to provide new methods of construction of such arrays.

For completeness, we recall the definitions of a symmetric orthogonal array and the two types of nested orthogonal arrays considered in this paper. A (symmetric) orthogonal array $OA(N, k, s, g)$ with N rows, k columns, s (≥ 2) symbols and strength g is an $N \times k$ matrix with symbols from a finite set of s symbols, in which all possible combinations of symbols appear equally often as rows in every $N \times g$ submatrix, for $2 \leq g \leq k$.

Definition 1. (Type I NOAs): *A (symmetric) nested orthogonal array, $NOA_I((N, M), k, (s, r), g)$, where $M < N$ and $r < s$, is an orthogonal array $OA(N, k, s, g)$ which contains an $OA(M, k, r, g)$ as a subarray.*

In order to define nested orthogonal arrays of type II, we need to define a kind of projection considered by Qian, Ai and Wu (2009).

Let $s_1 = p^{m_1}$, $s_2 = p^{m_2}$, where p is a prime and m_1, m_2 ($m_1 > m_2$) are positive integers. Suppose $g_1(x)$ is an irreducible polynomial of $GF(s_1)$ and $g_2(x)$ is an irreducible polynomial of $GF(s_2)$, where $GF(\cdot)$ stands for a Galois field. We are now in a position to describe a projection from $GF(s_1)$ to $GF(s_2)$, called the *modulus projection* by Qian, Ai and Wu (2009) and denoted by φ. This is defined as

$$\varphi\left(f(x)\right) = f(x)\left(mod\ g_2(x)\right),$$

for any $f(x) \in GF(s_1)$.

Since φ works by taking residues modulo $g_2(x)$, this projection is called *modulus projection*. We now have the following definition.

Definition 2.(Type II NOAs): *Let A_1 be an $OA(n_1, k, s_1, g)$. Suppose there is a subarray A_2 of A_1 of size n_2, and there is a modulus projection φ that collapses the s_1 levels of A_1 into s_2 levels. Further suppose A_2 becomes an $OA(n_2, k, s_2, g)$ after the levels of its entries are collapsed accord-*

ing to φ. Then (A_1, A_2), is a type II nested orthogonal array, denoted by $NOA_{II}(A_1, A_2)$.

Sun, Liu and Qian (2014) extended the definition of Type II nested orthogonal array with two layers (as given in Definition 2) to a more general case having more than two layers.

Definition 3. *Suppose A_i is an $OA(n_i, k, s_i, g)$ and φ_i for $i = 1, \cdots, I$ ($I \geq 2$) are a series of projections satisfying the condition that $\varphi_i(\alpha) = \varphi_i(\beta)$ implies $\varphi_j(\alpha) = \varphi_j(\beta)$ for $i \leq j$. Then $(A_1, \cdots, A_I; \varphi_1, \cdots, \varphi_I)$ is called a nested orthogonal array with I layers, denoted by $NOA_{II}(A_1, \cdots, A_I)$ or $NOA_{II}(A_1, \cdots, A_I;\ \varphi_1, \cdots, \varphi_I)$, if:*

(i) A_i is nested within A_{i-1} for $2 \leq i \leq I$, that is, $A_1 \supset A_2 \supset \cdots \supset A_I$;

(ii) $\varphi_j(A_i)$ is an $OA(n_i, k, s_j, g)$, for $i \leq j$,

where $n_1 > n_2 > \cdots > n_I$ and $s_1 > s_2 > \cdots > s_I$.

The question of existence of symmetric nested orthogonal arrays of type I has been examined by Mukerjee, Qian and Wu (2008), who proved that for the existence of a type I nested orthogonal array $NOA_I((N, M), k, (s, r), g)$, $g \leq k$, it is necessary that:

$$N \geq M \sum_{j=0}^{u} \binom{k}{j} (r^{-1}s - 1)^j, \text{ if } g\ (= 2u, u \geq 1) \text{ is even}, \tag{1}$$

$$\begin{aligned} N \ \geq \ & M \left[\sum_{j=0}^{u} \binom{k}{j} (r^{-1}s - 1)^j + \binom{k-1}{u} (r^{-1}s - 1)^{u+1} \right], \\ & \text{if } g(= 2u + 1, u \geq 1) \text{ is odd}. \end{aligned} \tag{2}$$

Construction methods of some families of symmetric type I NOAs were provided by Dey (2010, 2012). Some methods of construction of type II $NOAs$ were discussed among others, by Qian, Ai and Wu (2009), Qian, Tang and Wu (2009) and Sun, Liu and Qian (2014). The present article aims at constructing new families of type I and type II NOAs. Throughout, for a positive integer t, $\mathbf{0}_t$, $\mathbf{1}_t$, I_t will respectively, denote a $t \times 1$ null vector, a $t \times 1$ vector of all ones and an identity matrix of order t. A' denotes the transpose of a matrix (or, vector) A.

2 Construction of type I NOAs

We first have the following result.

Theorem 1. *Let p be a prime, s_1, s_2 $(s_1 > s_2)$ be powers of p and A be a $t \times k$ matrix over $GF(p)$. If any $t \times g$ $(t \geq g)$ submatrix of A has full column rank over $GF(p)$, then an $NOA_I((s_1^t, s_2^t), k, (s_1, s_2), g)$ exists.*

Proof. Suppose B_1 is $s_1^t \times t$ matrix having rows as all possible t-plets with entries from $GF(s_1)$. Since any $t \times g$ submatrix of A has full column rank over $GF(p)$, by Bose and Bush (1952), $C_1 = B_1 A$ is an $OA(s_1^t, k, s_1, g)$. It is easy to see that B_1 has a $s_2^t \times t$ submatrix B_2 having rows as all possible t-plets with entries from $GF(s_2)$, for $s_1 > s_2$. Then $C_2 = B_2 A$ is an $OA(s_2^t, k, s_2, g)$. Hence, C_2 is a $s_2^t \times k$ submatrix of C_1, with elements over $GF(s_2)$ and thus an $NOA_I((s_1^t, s_2^t), k, (s_1, s_2), g)$ exists. □

Invoking Theorem 1, one gets the following results.

Theorem 2. *Let s_1, s_2 $(s_1 > s_2)$ be powers of a prime p and t, g $(t \geq g)$ be integers. Then the following families of symmetric type I NOAs exist:*

(a) $NOA_I((s_1^t, s_2^t), \frac{p^t-1}{p-1}, (s_1, s_2), 2)$,

(b) $NOA_I((s_1^t, s_2^t), 2^{t-1}, (s_1, s_2), 3)$, for $p = 2$,

(c) $NOA_I((s_1^t, s_2^t), 2(t-1), (s_1, s_2), 3)$, for $p \neq 2$,

(d) $NOA_I((s_1^g, s_2^g), g+1, (s_1, s_2), g)$.

Proof. (a) Let A be a $t \times \frac{p^t-1}{p-1}$ matrix with entries from $GF(p)$ having columns as all possible nonzero t-tuples, such that no two columns are proportional to each other. Then, it is easy to see that any $t \times 2$ submatrix of A has full column rank over $GF(p)$. Invoking Theorem 1, we obtain an $NOA_I((s_1^t, s_2^t), \frac{p^t-1}{p-1}, (s_1, s_2), 2)$.

(b) Let $A = \begin{bmatrix} \mathbf{1}'_{2^{t-1}} \\ A_1 \end{bmatrix}$, where A_1 is a $(t-1) \times 2^{t-1}$ matrix having columns as all possible $(t-1)$-tuples with entries from $GF(2)$. As shown by Zhang, Deng and Dey (2017), any three columns of A are linearly independent. We can now obtain an $NOA_I((s_1^t, s_2^t), 2^{t-1}, (s_1, s_2), 3)$ using Theorem 1.

(c) Let $A = \begin{bmatrix} \mathbf{1}'_{t-1} & \mathbf{1}'_{t-1} \\ I_{t-1} & 2I_{t-1} \end{bmatrix}$. By Lemma 3 in Zhang, Deng and Dey (2017), any three columns of A are linearly independent. The result now follows by invoking Theorem 1.

(d) Let $A = [\ I_g \quad \mathbf{1}_g\ ]$. Then any g columns of A are linearly independent. Using Theorem 1, we obtain an $NOA_I((s_1^g, s_2^g), g+1, (s_1, s_2), g)$. □

Remark 1. (i) When $t > 4$, Theorem 2 (b) accommodates more columns than the array reported in Theorem 1 of Dey (2010). Also, the number of symbols in the smaller array is not restricted to 2.
(ii) In part (a) and (b) of Theorem 2, when $s_1 = 2s_2$, $2^t - 1$ and 2^{t-1} are the maximum number of columns that a nested orthogonal array can accommodate, respectively, as, these arrays attain the upper bound on the number of columns given by (1) and (2) respectively.
(iii) For $s_2 \leq g$, $g+1$ is the maximum number of columns that can be accommodated in an orthogonal array $OA(s_2^g, k, s_2, g)$ (see Hedayat, Sloane and Stufken (1999)) and hence in that case, the $NOA_I((s_1^g, s_2^g), g+1, (s_1, s_2), g)$ in part (d) of Theorem 2 cannot have more than $g+1$ columns.

The following example illustrates Theorem 2.

Example 1. (i) Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$. Also, let B_1, B_2 be 16×2 and 4×2 matrices having rows as all possible 2-plets with entries from $GF(4)$ and $GF(2)$, respectively. Then an $NOA_I((4^2, 2^2), 3, (4, 2), 2)$ is obtained, displayed below in transposed form:

$$\begin{bmatrix} 0011 & 0011 & 2233 & 2233 \\ 0101 & 2323 & 0101 & 2323 \\ 0110 & 2332 & 2332 & 0110 \end{bmatrix}'.$$

The first 4 rows of the above array constitute an $OA(4, 3, 2, 2)$ (obtained by B_2A), while all the 16 rows form an $OA(16, 3, 4, 2)$ (obtained by B_1A).

(ii) Let

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}.$$

It can be seen that any three columns of A are linearly independent over $GF(3)$. Let B_1 be an $9^3 \times 3$ matrix having rows as all possible 3-tuples with entries from $GF(9)$. Then $C_1 = B_1 A$ is an $OA(9^3, 4, 9, 3)$. Let B_2 be a $3^3 \times 3$ submatrix of B_1 with elements over $GF(3)$. Then $C_2 = B_2 A$ is an $OA(3^3, 4, 3, 3)$. Hence we obtain an $NOA_I((9^3, 3^3), 4, (9, 3), 3)$.

3 Construction of type II NOAs

In this section, we construct a family of type II $NOAs$. As before, let p be a prime and A be a $t \times k$ matrix, $s_1 = p^{m_1}$, $s_2 = p^{m_2}$, where m_1, m_2 ($m_1 > m_2$) are integers. Let $g_1(x)$ and $g_2(x)$ be irreducible polynomials of $GF(s_1)$ and $GF(s_2)$ respectively, and φ_1, φ_2 respectively, be modulus $g_1(x)$, $g_2(x)$ projections. We then have the following result.

Theorem 3. *If any $t \times g$ $(t \geq g)$ submatrix of A has full column rank over $GF(s_2)$ and $2m_2 \leq m_1 + 1$, then an $NOA_{II}(A_1, A_2)$ exists, where $A_1 = OA(s_1^t, k, s_1, g)$, $A_2 = OA(s_2^t, k, s_2, g)$.*

Proof. Suppose B_1 is a $s_1^t \times t$ matrix having rows as all possible t-plets with entries from $GF(s_1)$. It is easy to see that B_1 has a $s_2^t \times t$ submatrix B_2 having rows as all possible t-plets with entries from $GF(s_2)$, for $s_1 > s_2$. Since any $t \times g$ submatrix of A has full column rank over $GF(s_2)$, $A_2 = \varphi_2(B_2 A)$ is an $OA(s_2^t, k, s_2, g)$. And A is a $t \times k$ matrix over $GF(s_2)$, for $s_1 > s_2$, so the elements of A can be considered as over $GF(s_1)$. It follows then that $A_1 = \varphi_1(B_1 A)$ is an $OA(s_1^t, k, s_1, g)$. If we can show that

$$\varphi_2(\varphi_1(B_2 A)) = \varphi_2(B_2 A),$$

then A_2 is nested in A_1. Since $2m_2 \leq m_1 + 1$, by the result of Qian, Tang and Wu (2009), the above holds. □

Theorem 3 is an improvement of the result of Qian, Tang and Wu (2009), who considered arrays of strength *two*, while Theorem 3 is true for arrays of any strength.

Example 2. Suppose $s_1 = 2^{m_1}$, $s_2 = 2^{m_2}$, m_1, m_2 ($m_1 > m_2$) are integers, and $2m_2 \le m_1 + 1$. Then an $NOA_{II}(OA(s_1^3, s_2+2, s_1, 3), OA(s_2^3, s_2+2, s_2, 3))$ exists.

Let $g_1(x) = x^{m_1} + x + 1$, $g_2(x) = x^{m_2} + x + 1$. Next set the first column of A as $A_1 = [1, 0, 0]'$, the second column of A as $A_2 = [0, 1, 0]'$, and the third to the $(s_2 + 2)$th columns of the form

$$A_i = [x^2, x, 1]',$$

where $x \in GF(s_2)$, $3 \le i \le s_2 + 2$. The matrix $[A_{j_1}\, A_{j_2}\, A_{j_3}]$, $1 \le j_1 < j_2 < j_3 \le s_2 + 2$, has full column rank. The result now follows.

Theorem 3 can be modified to generate $NOAs$ with more than two layers. Suppose $s_i = p^{m_i}$ where p is a prime and for $2 \le i \le I$, m_i's are integers satisfying $m_i < m_{i-1}$. Let $g_i(x)$ be an irreducible polynomial of $GF(s_i)$, and φ_i be a modulus $g_i(x)$ projection. We then have the following result whose proof is similar to that of Theorem 3 and is therefore omitted. As before, let A be a $t \times k$ matrix.

Theorem 4. *If any $t \times g$ ($t \ge g$) submatrix of A has full column rank over $GF(s_I)$ and $2m_i \le m_{i-1} + 1$ for $2 \le i \le I$, then an $NOA_{II}(A_1, \cdots, A_I)$ exists, where $A_i = OA(s_i^t, k, s_i, g)$, $1 \le i \le I$.*

Remark 2. The construction of NOAs with more than two layers given by Sun, Liu and Qian (2014) requires that the conditions (i) $m_i < m_{i-1}$ and (ii) $m_i | m_{i-1}$, $2 \le i \le I$ hold. Our result requires condition (i) and (ii$'$) $2m_i \le m_{i-1} + 1$ to hold. Thus, the requirements of our construction are less stringent than those of Sun, Liu and Qian (2014).

Example 3. Construction of $NOA_{II}(A_1, A_2, A_3)$, where $A_1 = (32^2, 5, 32, 2)$, $A_2 = OA(8^2, 5, 8, 2)$, $A_3 = OA(4^2, 5, 4, 2)$.

Since $s_1 = 2^5$, $s_2 = 2^3$ and $s_3 = 2^2$, we have $m_1 = 5$, $m_2 = 3$ and $m_3 = 2$. The condition $2m_i \le m_{i-1} + 1$ is satisfied. Note that here the condition $m_i | m_{i-1}$ does not hold. Let $g_1(x) = x^5 + x + 1$, $g_2(x) = x^3 + x + 1$, $g_3(x) = x^2 + x + 1$ and B_i be an $s_i^2 \times 2$ matrix having rows as all possible

2-plets with entries from $GF(s_i)$ $(i = 1, 2, 3)$. Define

$$A = \begin{bmatrix} 1 & 0 & 1 & x & x+1 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Then any two columns of A are linearly independent. By Theorem 4, $NOA_{II}(A_1, A_2, A_3)$ can now be constructed.

Acknowledgements

The work of first two authors was supported by the National Natural Science Foundation of China (Grant Nos. 11601201, 11661076). A. Dey's work was supported by the National Academy of Sciences, India under the Senior Scientist program of the Academy.

References

Bose, R.C. and Bush, K.A. (1952). Orthogonal arrays of strength two and three. *Ann. Math. Statist* **23** 508–524.

Dey, A. (2010). Construction of nested orthogonal arrays. *Discrete Math.* **310** 2831–2834.

Dey, A. (2012). On the construction of nested orthogonal arrays. *Australas. J. Combin.* **54** 37–48.

Hedayat, A.S., Sloane, N.J.A. and Stufken, J. (1999). *Orthogonal Arrays: Theory and Applications*, Springer, New York.

Mukerjee, R., Qian, P.Z.G. and Wu, C.F.J. (2008). On the existence of nested orthogonal arrays. *Discrete Math.* **308** 4635–4642.

Qian, P.Z.G., Ai, M. and Wu, C.F.J. (2009). Construction of nested space-filling designs. *Ann. Statist.* **37(6A)** 3616–3643.

Qian, P.Z.G., Tang, B. and Wu, C.F.J. (2009). Nested space-filling designs for computer experiments with two levels of accuracy. *Statist. Sinica* **19** 287–300.

Qian, P.Z.G., and Wu, C.F.J. (2008). Bayesian hierarchial modeling for integrating low-accuracy and high-accuracy experiments. *Technometrics* **50** 192–204.

Sun, F.S., Liu, M.Q. and Qian, P.Z.G. (2014). On the construction of nested space-filling designs. *Ann. Statist.* **42** 1394–1425.

Zhang, T.F., Deng, Q.L. and Dey, A. (2017). Construction of asymmetric orthogonal arrays of strength three via a replacement method. *J. Combin. Designs.* to appear.