Construction of some new families of nested orthogonal arrays

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Abstract

Nested orthogonal arrays have been used in the design of an experimental setup consisting of two experiments, the expensive one of higher accuracy being nested in a larger and relatively less expensive one of lower accuracy. In this paper, we provide new methods of construction of two types of nested orthogonal arrays.

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1 Introduction and Preliminaries

Computer models are widely used in business, engineering and sciences to study real-world systems, especially when the corresponding physical experiment might be time consuming, costly or even infeasible to conduct. Space-filling designs are desirable for conducting computer experiments. Nested orthogonal arrays have practical use in the construction of space-filling designs
when an experimental endeavour consists of two experiments, the expensive one of higher accuracy to be nested in a larger and relatively inexpensive one of lower accuracy. For example, the higher and lower accuracy experiments can correspond to a physical versus a computer experiment, or a detailed versus an approximate computer experiment, respectively. Experimental setups of this kind were considered, among others, by Qian and Wu (2008), Qian, Tang and Wu (2009) and Sun, Liu and Qian (2014). There are two types of nested orthogonal arrays available in the literature. The purpose of this paper is to provide new methods of construction of such arrays.

For completeness, we recall the definitions of a symmetric orthogonal array and the two types of nested orthogonal arrays considered in this paper. A (symmetric) orthogonal array $OA(N, k, s, g)$ with $N$ rows, $k$ columns, $s (\geq 2)$ symbols and strength $g$ is an $N \times k$ matrix with symbols from a finite set of $s$ symbols, in which all possible combinations of symbols appear equally often as rows in every $N \times g$ submatrix, for $2 \leq g \leq k$.

**Definition 1.** (Type I NOAs): A (symmetric) nested orthogonal array, $NOA_I((N, M), k, (s, r), g)$, where $M < N$ and $r < s$, is an orthogonal array $OA(N, k, s, g)$ which contains an $OA(M, k, r, g)$ as a subarray.

In order to define nested orthogonal arrays of type II, we need to define a kind of projection considered by Qian, Ai and Wu (2009).

Let $s_1 = p^{m_1}$, $s_2 = p^{m_2}$, where $p$ is a prime and $m_1, m_2$ ($m_1 > m_2$) are positive integers. Suppose $g_1(x)$ is an irreducible polynomial of $GF(s_1)$ and $g_2(x)$ is an irreducible polynomial of $GF(s_2)$, where $GF(\cdot)$ stands for a Galois field. We are now in a position to describe a projection from $GF(s_1)$ to $GF(s_2)$, called the modulus projection by Qian, Ai and Wu (2009) and denoted by $\varphi$. This is defined as

$$\varphi(f(x)) = f(x) \mod g_2(x),$$

for any $f(x) \in GF(s_1)$.

Since $\varphi$ works by taking residues modulo $g_2(x)$, this projection is called modulus projection. We now have the following definition.

**Definition 2.** (Type II NOAs): Let $A_1$ be an $OA(n_1, k, s_1, g)$. Suppose there is a subarray $A_2$ of $A_1$ of size $n_2$, and there is a modulus projection $\varphi$ that collapses the $s_1$ levels of $A_1$ into $s_2$ levels. Further suppose $A_2$ becomes an $OA(n_2, k, s_2, g)$ after the levels of its entries are collapsed accord-
ing to \( \varphi \). Then \((A_1, A_2)\), is a type II nested orthogonal array, denoted by \(\text{NOA}_{II}(A_1, A_2)\).

Sun, Liu and Qian (2014) extended the definition of Type II nested orthogonal array with two layers (as given in Definition 2) to a more general case having more than two layers.

**Definition 3.** Suppose \(A_i\) is an \(\text{OA}(n_i, k, s_i, g)\) and \(\varphi_i\) for \(i = 1, \cdots, I\) \((I \geq 2)\) are a series of projections satisfying the condition that \(\varphi_i(\alpha) = \varphi_i(\beta)\) for \(i \leq j\). Then \((A_1, \cdots, A_I; \varphi_1, \cdots, \varphi_I)\) is called a nested orthogonal array with \(I\) layers, denoted by \(\text{NOA}_{II}(A_1, \cdots, A_I)\) or \(\text{NOA}_{II}(A_1, \cdots, A_I; \varphi_1, \cdots, \varphi_I)\), if:

(i) \(A_i\) is nested within \(A_{i-1}\) for \(2 \leq i \leq I\), that is, \(A_1 \supset A_2 \supset \cdots \supset A_I\);

(ii) \(\varphi_j(A_i)\) is an \(\text{OA}(n_i, k, s_j, g)\), for \(i \leq j\),

where \(n_1 > n_2 > \cdots > n_I\) and \(s_1 > s_2 > \cdots > s_I\).

The question of existence of symmetric nested orthogonal arrays of type I has been examined by Mukerjee, Qian and Wu (2008), who proved that for the existence of a type I nested orthogonal array \(\text{NOA}_I((N, M), k, (s, r), g)\), \(g \leq k\), it is necessary that:

\[
N \geq M \sum_{j=0}^{u} \binom{k}{j} (r^{-1}s - 1)^j, \quad \text{if } g = 2u, u \geq 1 \text{ is even,} \tag{1}
\]

\[
N \geq M \left[ \sum_{j=0}^{u} \binom{k}{j} (r^{-1}s - 1)^j + \binom{k-1}{u} (r^{-1}s - 1)^{u+1} \right],
\]

\[\text{if } g = 2u + 1, u \geq 1 \text{ is odd.} \tag{2}\]

Construction methods of some families of symmetric type I \(\text{NOA}s\) were provided by Dey (2010, 2012). Some methods of construction of type II \(\text{NOA}s\) were discussed among others, by Qian, Ai and Wu (2009), Qian, Tang and Wu (2009) and Sun, Liu and Qian (2014). The present article aims at constructing new families of type I and type II \(\text{NOA}s\). Throughout, for a positive integer \(t\), \(0_t, 1_t, I_t\) will respectively, denote a \(t \times 1\) null vector, a \(t \times 1\) vector of all ones and an identity matrix of order \(t\). \(A'\) denotes the transpose of a matrix (or, vector) \(A\).
2 Construction of type I NOAs

We first have the following result.

**Theorem 1.** Let $p$ be a prime, $s_1, s_2$ ($s_1 > s_2$) be powers of $p$ and $A$ be a $t \times k$ matrix over $GF(p)$. If any $t \times g$ ($t \geq g$) submatrix of $A$ has full column rank over $GF(p)$, then an NOA$_I((s'_1, s'_2), k, (s_1, s_2), g)$ exists.

**Proof.** Suppose $B_1$ is a $s'_1 \times t$ matrix having rows as all possible $t$-plets with entries from $GF(s_1)$. Since any $t \times g$ submatrix of $A$ has full column rank over $GF(p)$, by Bose and Bush (1952), $C_1 = B_1A$ is an OA$(s'_1, k, s_1, g)$. It is easy to see that $B_1$ has a $s'_2 \times t$ submatrix $B_2$ having rows as all possible $t$-plets with entries from $GF(s_2)$, for $s_1 > s_2$. Then $C_2 = B_2A$ is an $OA(s'_2, k, s_2, g)$. Hence, $C_2$ is a $s'_2 \times k$ submatrix of $C_1$, with elements over $GF(s_2)$ and thus an NOA$_I((s'_1, s'_2), k, (s_1, s_2), g)$ exists. □

Invoking Theorem 1, one gets the following results.

**Theorem 2.** Let $s_1, s_2$ ($s_1 > s_2$) be powers of a prime $p$ and $t$, $g$ ($t \geq g$) be integers. Then the following families of symmetric type I NOAs exist:

(a) NOA$_I((s'_1, s'_2), \frac{p^t - 1}{p - 1}, (s_1, s_2), 2)$,

(b) NOA$_I((s'_1, s'_2), 2^{t-1}, (s_1, s_2), 3)$, for $p = 2$,

(c) NOA$_I((s'_1, s'_2), 2(t - 1), (s_1, s_2), 3)$, for $p \neq 2$,

(d) NOA$_I((s'_1, s'_2), g + 1, (s_1, s_2), g)$.

**Proof.** (a) Let $A$ be a $t \times \frac{p^t - 1}{p - 1}$ matrix with entries from $GF(p)$ having columns as all possible nonzero $t$-tuples, such that no two columns are proportional to each other. Then, it is easy to see that any $t \times 2$ submatrix of $A$ has full column rank over $GF(p)$. Invoking Theorem 1, we obtain an NOA$_I((s'_1, s'_2), \frac{p^t - 1}{p - 1}, (s_1, s_2), 2)$.

(b) Let $A = \begin{bmatrix} 1_{2t-1}^t & A_1 \end{bmatrix}$, where $A_1$ is a $(t - 1) \times 2^{t-1}$ matrix having columns as all possible $(t - 1)$-tuples with entries from $GF(2)$. As shown by Zhang,
Deng and Dey (2017), any three columns of \( A \) are linearly independent. We can now obtain an \( NOA_I((s_1^t, s_2^t), 2^{t-1}, (s_1, s_2), 3) \) using Theorem 1.

(c) Let \( A = \begin{bmatrix} 1_{t-1} & 1_{t-1} \\ I_{t-1} & 2I_{t-1} \end{bmatrix} \). By Lemma 3 in Zhang, Deng and Dey (2017), any three columns of \( A \) are linearly independent. The result now follows by invoking Theorem 1.

(d) Let \( A = [I_g \ 1_g] \). Then any \( g \) columns of \( A \) are linearly independent. Using Theorem 1, we obtain an \( NOA_I((s_1^g, s_2^g), g + 1, (s_1, s_2), g) \). \( \square \)

**Remark 1.** (i) When \( t > 4 \), Theorem 2 (b) accommodates more columns than the array reported in Theorem 1 of Dey (2010). Also, the number of symbols in the smaller array is not restricted to 2.

(ii) In part (a) and (b) of Theorem 2, when \( s_1 = 2s_2 \), \( 2^t - 1 \) and \( 2^{t-1} \) are the maximum number of columns that a nested orthogonal array can accommodate, respectively, as, these arrays attain the upper bound on the number of columns given by (1) and (2) respectively.

(iii) For \( s_2 \leq g \), \( g + 1 \) is the maximum number of columns that can be accommodated in an orthogonal array \( OA(s_2^g, k, s_2, g) \) (see Hedayat, Sloane and Stufken (1999)) and hence in that case, the \( NOA_I((s_1^g, s_2^g), g + 1, (s_1, s_2), g) \) in part (d) of Theorem 2 cannot have more than \( g + 1 \) columns.

The following example illustrates Theorem 2.

**Example 1.** (i) Let \( A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \). Also, let \( B_1, B_2 \) be \( 16 \times 2 \) and \( 4 \times 2 \) matrices having rows as all possible 2-plets with entries from \( GF(4) \) and \( GF(2) \), respectively. Then an \( NOA_I((4^2, 2^2), 3, (4, 2), 2) \) is obtained, displayed below in transposed form:

\[
\begin{bmatrix}
0011 & 0011 & 2233 & 2233 \\
0101 & 2323 & 0101 & 2323 \\
0110 & 2332 & 2332 & 0110
\end{bmatrix}.
\]

The first 4 rows of the above array constitute an \( OA(4, 3, 2, 2) \) (obtained by \( B_2A \)), while all the 16 rows form an \( OA(16, 3, 4, 2) \) (obtained by \( B_1A \)).
Let

\[
A = \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 0 & 2 & 0 \\
0 & 1 & 0 & 2
\end{bmatrix}.
\]

It can be seen that any three columns of \(A\) are linearly independent over \(GF(3)\). Let \(B_1\) be an \(9^3 \times 3\) matrix having rows as all possible 3-tuples with entries from \(GF(9)\). Then \(C_1 = B_1 A\) is an \(OA(9^3, 4, 9, 3)\). Let \(B_2\) be a \(3^3 \times 3\) submatrix of \(B_1\) with elements over \(GF(3)\). Then \(C_2 = B_2 A\) is an \(OA(3^3, 4, 3, 3)\). Hence we obtain an \(NOA_I((9^3, 3^3), 4, (9, 3), 3)\).

3 Construction of type II \(NOAs\)

In this section, we construct a family of type II \(NOAs\). As before, let \(p\) be a prime and \(A\) be a \(t \times k\) matrix, \(s_1 = p^{m_1}, s_2 = p^{m_2}\), where \(m_1, m_2\) are integers. Let \(g_1(x)\) and \(g_2(x)\) be irreducible polynomials of \(GF(s_1)\) and \(GF(s_2)\) respectively, and \(\varphi_1, \varphi_2\) respectively, be modulus \(g_1(x), g_2(x)\) projections. We then have the following result.

**Theorem 3.** If any \(t \times g\) \((t \geq g)\) submatrix of \(A\) has full column rank over \(GF(s_2)\) and \(2m_2 \leq m_1 + 1\), then an \(NOA_{II}(A_1, A_2)\) exists, where \(A_1 = OA(s_1^t, k, s_1, g), A_2 = OA(s_2^t, k, s_2, g)\).

**Proof.** Suppose \(B_1\) is a \(s_1^t \times t\) matrix having rows as all possible \(t\)-plets with entries from \(GF(s_1)\). It is easy to see that \(B_1\) has a \(s_2^t \times t\) submatrix \(B_2\) having rows as all possible \(t\)-plets with entries from \(GF(s_2)\), for \(s_1 > s_2\). Since any \(t \times g\) submatrix of \(A\) has full column rank over \(GF(s_2)\), \(A_2 = \varphi_2(B_2 A)\) is an \(OA(s_2^t, k, s_2, g)\). And \(A\) is a \(t \times k\) matrix over \(GF(s_2)\), for \(s_1 > s_2\), so the elements of \(A\) can be considered as over \(GF(s_1)\). It follows then that \(A_1 = \varphi_1(B_1 A)\) is an \(OA(s_1^t, k, s_1, g)\). If we can show that

\[
\varphi_2(\varphi_1(B_2 A)) = \varphi_2(B_2 A),
\]

then \(A_2\) is nested in \(A_1\). Since \(2m_2 \leq m_1 + 1\), by the result of Qian, Tang and Wu (2009), the above holds. \(\square\)
Theorem 3 is an improvement of the result of Qian, Tang and Wu (2009), who considered arrays of strength two, while Theorem 3 is true for arrays of any strength.

**Example 2.** Suppose \( s_1 = 2m_1, s_2 = 2m_2, m_1, m_2 \ (m_1 > m_2) \) are integers, and \( 2m_2 \leq m_1 + 1 \). Then an \( NOA_{II}(OA(s_1^3, s_2 + 2, s_1, 3), OA(s_2^3, s_2 + 2, s_2, 3)) \) exists.

Let \( g_1(x) = x^{m_1} + x + 1, g_2(x) = x^{m_2} + x + 1 \). Next set the first column of \( A \) as \( A_1 = [1, 0, 0]' \), the second column of \( A \) as \( A_2 = [0, 1, 0]' \), and the third to the \((s_2 + 2)\)th columns of the form

\[
A_i = [x^j, x, 1]',
\]

where \( x \in GF(s_2), 3 \leq i \leq s_2 + 2 \). The matrix \( [A_1 A_2 A_3] \), \( 1 \leq j_1 < j_2 < j_3 \leq s_2 + 2 \), has full column rank. The result now follows.

Theorem 3 can be modified to generate NOAs with more than two layers. Suppose \( s_i = p^{m_i} \) where \( p \) is a prime and for \( 2 \leq i \leq I \), \( m_i \)'s are integers satisfying \( m_i < m_{i-1} \). Let \( g_i(x) \) be an irreducible polynomial of \( GF(s_i) \), and \( \varphi_i \) be a modulus \( g_i(x) \) projection. We then have the following result whose proof is similar to that of Theorem 3 and is therefore omitted. As before, let \( A \) be a \( t \times k \) matrix.

**Theorem 4.** If any \( t \times g \ (t \geq g) \) submatrix of \( A \) has full column rank over \( GF(s_i) \) and \( 2m_i \leq m_{i-1} + 1 \) for \( 2 \leq i \leq I \), then an \( NOA_{II}(A_1, \cdots, A_I) \) exists, where \( A_i = OA(s_i^t, k, s_i, g), 1 \leq i \leq I \).

**Remark 2.** The construction of NOAs with more than two layers given by Sun, Liu and Qian (2014) requires that the conditions (i) \( m_i < m_{i-1} \) and (ii) \( m_i | m_{i-1}, 2 \leq i \leq I \) hold. Our result requires condition (i) and (ii') \( 2m_i \leq m_{i-1} + 1 \) to hold. Thus, the requirements of our construction are less stringent than those of Sun, Liu and Qian (2014).

**Example 3.** Construction of \( NOA_{II}(A_1, A_2, A_3) \), where \( A_1 = (32^2, 5, 32, 2), A_2 = OA(8^2, 5, 8, 2), A_3 = OA(4^2, 5, 4, 2) \).

Since \( s_1 = 2^5, s_2 = 2^5 \) and \( s_3 = 2^2 \), we have \( m_1 = 5, m_2 = 3 \) and \( m_3 = 2 \). The condition \( 2m_i \leq m_{i-1} + 1 \) is satisfied. Note that here the condition \( m_i | m_{i-1} \) does not hold. Let \( g_1(x) = x^5 + x + 1, g_2(x) = x^3 + x + 1, g_3(x) = x^2 + x + 1 \) and \( B_i \) be an \( s_i^t \times 2 \) matrix having rows as all possible
2-plets with entries from $GF(s_i)$ ($i = 1, 2, 3$). Define

$$A = \begin{bmatrix} 1 & 0 & 1 & x & x+1 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}.$$ 

Then any two columns of $A$ are linearly independent. By Theorem 4, $NOA_{II}(A_1, A_2, A_3)$ can now be constructed.

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**References**


