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# Characterization of probability distributions via binary associative operation

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CHARACTERIZATION OF PROBABILITY DISTRIBUTIONS VIA BINARY ASSOCIATIVE OPERATION

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**Abstract**

A binary operation * over real numbers is said to be associative if $(x*y)*z = x*(y*z)$ and it is said to be reducible if $x*y = x*z$ or $y*w = z*w$ if and only if $z = y$. The operation * is said to have an identity element $\tilde{e}$ if $x*\tilde{e} = x$. We characterize different classes of probability distributions under binary operations between random variables. Further more we characterize distrbutions with the almost lack of memory property or with strong Markov property or with periodic failure rate under such a binary operation extending the results on exponential distributions under addition operation as binary operation.

## 1 Introduction

Summarization of statistical data without loosing information is one of the fundamental objectives of statistical data analysis. More precisely the problem is to determine whether the knowledge of a possibly smaller set of functions of several random variables is sufficient to determine the behaviour of individual random components. For instance, if $X, Y$, and $Z$ are three independent random variables, we would like to know sufficient conditions under which the joint distribution of $U = g(X, Y, Z)$ and $V = h(X, Y, Z)$ determine either the individual distributions of $X, Y$ and $Z$, or the family to which they belong when $g(.)$ and $h(.)$ are specified. The functions $g(.)$ and $h(.)$ could be linear or nonlinear functions or they could be the maximum or minimum functions etc. Problems of this nature were discussed in Prakasa Rao (1992). We now study such characterization problems when the operation between the variables is a binary operation which is associative.

A binary operation * over real numbers is said to be associative if

$$(x*y)*z = x*(y*z) \tag{1.1}$$

for all real numbers $x, y, z$. The binary operation * is said to be reducible if $x*y = x*z$ if and only if $y = z$ and if $y*w = z*w$ if and only if $y = z$. It is known that the general reducible continuous solution of the functional equation (1.1) is

$$x*y = g^{-1}(g(x) + g(y)) \tag{1.2}$$

where $g(.)$ is a continuous and strictly monotone function provided $x, y, x*y$ belong to a fixed (possibly infinite) interval $A$ (cf. Aczel (1966)). The function $g$ in (1.2) is determined up to a multiplicative constant, that is,

$$g_1^{-1}(g_1(x) + g_1(y)) = g_2^{-1}(g_2(x) + g_2(y))$$

for all $x, y$ in a fixed interval $A$ implies $g_2(x) = \alpha g_1(x)$ for all $x$ in that interval for some $\alpha \neq 0$. We assume here after that the binary operation is reducible and associative with the function $g(.)$ continuous and strictly increasing. Further assume that there exists an identity element $\tilde{e} \in \bar{R}$ such that

$$x * \tilde{e} = x, x \in A.$$

It is also known that every continuous, reducible and associative operation defined on an interval $A$ in the real line is commutative (cf. Aczel (1966), p.267). Let $X$ be a random variable with the distribution function $F(x)$ having support $A$. Define

$$\phi_X^*(s) = \int_A \exp\{isg(x)\} \ F(dx), -\infty < s < \infty. \tag{1. 3}$$

Note that the function $\phi_X^*(s)$ is the characteristic function of the random variable $g(X)$ and hence determines the distribution function of the random variable $g(X)$ uniquely.

Examples of such binary operations are given in Castagnoli (1974, 1978, 1982), Muliere (1984) and Castagnoli and Muliere (1984, 1986, 1988). For instance (i) if $A = (-\infty, \infty)$ and $x * y = x + y$, then $g(x) = x$, (ii) if $A = (0, \infty)$ and $x * y = xy, x > 0, y > 0$ then $g(x) = \log x$, (iii) if $A = (0, \infty)$ and $x * y = (x^\alpha + y^\alpha)^{1/\alpha}, x > 0, y > 0$ for some $\alpha > 0$, then $g(x) = x^\alpha$, (iv) if $A = (-1, \infty)$ and $x * y = x + y + xy + 1, x > -1, y > -1$, then $g(x) = \log(1 + x)$ (v)if $A = (0, \infty)$ and $x * y = xy/(x + y), x > 0$, then $g(x) = 1/x$ and (vi)if $A = (0, \infty)$ and $x * y = (x + y)/(1 + xy), x > 0, y > 0$, then $g(x) = arth\ x$.

A characterization of the multivariate normal distribution through a binary operation which is associative is given in Prakasa Rao (1974) and in Prakasa Rao (1977) for Gaussian measures on locally compact abelian groups. Muliere and Scarsini (1987) characterize a class of bivariate distributions that generalize the Marshall-Olkin bivariate exponential distribution through a functional equation involving two associative operations.

Let * be a binary operation over an interval $A$ contained in $R$ as described above. Suppose $X_i, 1 \leq i \leq 3$ are independent real valued random variables with probability distributions with support $A$. Define $Z_1 = X_1 * X_3$ and $Z_2 = X_2 * X_3$. Suppose the joint distribution of $(Z_1, Z_2)$ is specified. We give a characterization of probability distributions of $X_i, 1 \leq i \leq 3$ up to changes in location with respect to the binary operation * under certain conditions in Section 2. Explicit determination of the distributions of individual components is discussed in Section 3. The concept of almost lack of memory property under a binary operation * and its relation to periodic failure rate under the binary operation * is investigated in Sections 4 to 6. A representation for distributions with periodic failure rate under the binary operation * is given in Section 7. Distributions with strong Markov property under the binary operation * are characterized in Section 8.

## 2 Characterization via binary operations

Let $*$ be a binary operation over an interval $A$ contained in $R$ as described above. Suppose $X_i, 1 \leq i \leq 3$ are independent real valued random variables with probability distributions with support $A$ contained in $R$. Define

$$\tilde{Z}_1 = X_1 * X_3 \text{ and } \tilde{Z}_2 = X_2 * X_3. \tag{2. 1}$$

**Theorem 2.1:** If the characteristic function of $(\tilde{Z}_1, \tilde{Z}_2)$ does not vanish, then the joint distribution of $(\tilde{Z}_1, \tilde{Z}_2)$ determines the distributions of $X_1, X_2$ and $X_3$ up to changes of locations under the binary operation $*$.

**Proof:** From the comments made earlier, it follows that, given a binary operation $*$ which is reducible and associative, there exists a continuous strictly increasing function $g(.)$ such that $x * y = g^{-1}(g(x) + g(y))$. Let $Z_i = g(\tilde{Z}_i), i = 1, 2$. Denote the characteristic function of the bivariate random vector $(Z_1, Z_2)$ by $\phi(t_1, t_2)$ and let $\phi_X(t)$ denote the characteristic function of a random variable $X$. Then

$$\begin{aligned} \phi(t_1, t_2) &= E[\exp(it_1 Z_1 + it_2 Z_2)] \\ &= E[\exp(it_1 g(X_1 * X_3) + it_2 g(X_2 * X_3))] \\ &= E[\exp(it_1(g(X_1) + g(X_3)) + it_2(g(X_2) + g(X_3)))] \\ &= \phi_{g(X_1)}(t_1)\phi_{g(X_2)}(t_2)\phi_{g(X_3)}(t_1 + t_2)) \end{aligned} \tag{2. 2}$$

by the independence of the random variables $X_i, 1 \leq i \leq 3$. Since $\phi(t_1, t_2) \neq 0$ for all $t_1$ and $t_2$ by hypothesis, it follows that $\phi_{g(X_i)}(t) \neq 0$ for all $t$ for $1 \leq i \leq 3$.

Let $Y_1, Y_2, Y_3$ be another set of three independent random variables and define

$$\tilde{W}_1 = Y_1 * Y_3 \text{ and } \tilde{W}_2 = Y_2 * Y_3. \tag{2. 3}$$

Suppose the conditions stated in the theorem are satisfied by the set of random variables $Y_1, Y_2, Y_3$. Let $W_i = g(\tilde{W}_i), i = 1, 2$ and let $\psi(t_1, t_2)$ denote the characteristic function of the bivariate random vector $(W_1, W_2)$. Further suppose that the joint distributions of $(\tilde{W}_1, \tilde{W}_2)$ and $(\tilde{Z}_1, \tilde{Z}_2)$ are the same. Then it follows that the joint distribution functions of $(Z_1, Z_2)$ and $(W_1, W_2)$ are the same. Hence

$$\psi(t_1, t_2) = \phi(t_1, t_2), -\infty < t_1, t_2 < \infty \tag{2. 4}$$

and it follows from (2.2) that

$$\phi_{g(X_1)}(t_1)\phi_{g(X_2)}(t_2)\phi_{g(X_3)}(t_1 + t_2)) = \phi_{g(Y_1)}(t_1)\phi_{g(Y_2)}(t_2)\phi_{g(Y_3)}(t_1 + t_2)) \tag{2. 5}$$

for $-\infty < t_1, t_2 < \infty$. Further more $\phi_{g(X_i)}(t) \neq 0$ and $\phi_{g(Y_i)}(t) \neq 0$ for all $t$ by hypothesis. Let

$$\gamma_i(t) = \frac{\phi_{g(Y_i)}(t)}{\phi_{g(X_i)}(t)}, 1 \leq i \leq 3. \tag{2. 6}$$

Observe that $\gamma_i(t), 1 \leq i \leq 3$ are continuous complex valued functions with $\gamma_i(0) = 1, 1 \leq i \leq 3$ satisfying the equation

$$\gamma_1(t_1)\gamma_2(t_2)\gamma_3(t_1 + t_2) = 1 \tag{2. 7}$$

for $-\infty < t_1, t_2 < \infty$. Let $t_1 = t$ and $t_2 = 0$. Then

$$\gamma_1(t)\gamma_3(t) = 1 \tag{2. 8}$$

for $-\infty < t < \infty$. Let $t_2 = t$ and $t_1 = 0$. Then

$$\gamma_2(t)\gamma_3(t) = 1 \tag{2. 9}$$

for $-\infty < t < \infty$. Substituting for $\gamma_1(t)$ and $\gamma_2(t)$ in terms of $\gamma_3(t)$ in (2.7), it follows that

$$\gamma_3(t_1 + t_2) = \gamma_3(t_1)\gamma_3(t_2) \tag{2. 10}$$

for $-\infty < t_1, t_2 < \infty$ with $\gamma_3(0) = 1$. It is known that the only measurable solution of the above functional equation is

$$\gamma_3(t) = e^{ct} \tag{2. 11}$$

where $c$ is a complex constant. It follows from (2.8) and (2.9) that

$$\gamma_1(t) = \gamma_2(t) = e^{-ct} \tag{2. 12}$$

for $-\infty < t < \infty$. Equation (2.6) implies that

$$\phi_{g(Y_j)}(t) = \phi_{g(X_j)}(t)e^{-ct}, 1 \leq j \leq 2 \tag{2. 13}$$

and

$$\phi_{g(Y_3)}(t) = \phi_{g(X_3)}(t)e^{ct} \tag{2. 14}$$

From the property that $\phi_{g(X_j)}(t)$ and $\phi_{g(Y_j)}(t)$ are characteristic functions, it follows that $c = i\beta$ where $\beta$ is a real number. Therefore

$$\phi_{g(Y_j)}(t) = \phi_{g(X_j)}(t)e^{-i\beta t}, 1 \leq j \leq 2 \tag{2. 15}$$

and

$$\phi_{g(Y_3)}(t) = \phi_{g(X_3)}(t)e^{i\beta t}. \tag{2. 16}$$

From the uniqueness theoren for characteristic functions, it follows that $g(X_j) - \beta$ and $g(Y_j)$ have the same distribution for $j = 1, 2$ and $g(X_3) + \beta$ and $g(Y_3)$ have the same distribution. Since

$$g(X_j) - \beta = g(X_j * g^{-1}(-\beta)), j = 1, 2 \tag{2. 17}$$

and

$$g(X_3) + \beta = g(X_3 * g^{-1}(\beta)) \tag{2. 18}$$

and $g(.)$ is continuous and strictly increasing, it follows that that the distributions of $X_j * g^{-1}(-\beta)$ and $Y_j$ are the same for $j = 1, 2$ and the distribution $X_3 * g^{-1}(\beta)$ and $Y_3$ are the same.

In other words the distributions of $X_i, i = 1,2,3$ are determined up to changes of locations under the binary operation *.

**Remarks:** (i)The above result was proved by Kotlarski (1967) for real valued random variables when the binary operation was the addition. It was later generalised to other types of operations such as maximum and to random elements taking values in abstract spaces by Kotlarski , Prakasa Rao and others (cf. Prakasa Rao (1992)).

(ii) The above theorem can be extended to $n$ independent random variables $X_i, 1 \leq i \leq n$ in the following form: Define $Z_i = g(X_i * X_n), 1 \leq i \leq n-1$ where $g(.)$ is the function corresponding to the binary operation *. Suppose the characteristic function of $(Z_1, \ldots, Z_{n-1})$ does not vanish. Then the distribution of $(Z_1, \ldots, Z_{n-1})$ determines the distributions of $X_i, 1 \leq i \leq n$ up to changes of locations under the binary operation * .

(iii) As can be seen from the proof given above, the joint distribution of $(Z_1, Z_2)$, where $Z_i = g(\tilde{Z}_i), i = 1,2$ alone will determine the distributions of $X_i, 1 \leq i \leq 3$ up to changes in locations under the binary operation *.

## 3 Explicit determination of the distributions of individual components

Let $X_i, 1 \leq i \leq 3$ be independent random variables and define

$$\tilde{Z}_1 = X_1 * X_3 \text{ and } \tilde{Z}_2 = X_2 * X_3 \tag{3. 1}$$

as defined earlier where * is a binary operation which is continuous, reducible and associative. Such an operation gives rise to a function $g(.)$ which is continuous and strictly increasing such that

$$x * y = g^{-1}(g(x) + g(y)).$$

Let $Z_i = g(\tilde{Z}_i), i = 1,2$. Denote the characteristic function of the bivariate random vector $(Z_1, Z_2)$ by $\phi(t_1, t_2)$ and let $\phi_X(t)$ denote the characteristic function of a random variable $X$. Then

$$\begin{aligned} \phi(t_1,t_2) &= E[\exp(it_1 Z_1 + it_2 Z_2)] \\ &= E[\exp(it_1 g(X_1 * X_3) + it_2 g(X_2 * X_3))] \\ &= E[\exp(it_1(g(X_1) + g(X_3)) + it_2(g(X_2) + g(X_3)))] \\ &= \phi_{g(X_1)}(t_1)\phi_{g(X_2)}(t_2)\phi_{g(X_3)}(t_1 + t_2) \end{aligned} \tag{3. 2}$$

by the independence of the random variables $X_i, 1 \leq i \leq 3$. Suppose the characteristic functions of $g(X_i), 1 \leq i \leq 3$ are nonvanishing everywhere. Then $\phi(t_1, t_2)$ is nonvanishing for $-\infty < t_1, t_2 < \infty$.

Let $t_2 = 0$ in (3.2). Then it follows that

$$\phi(t_1, 0) = \phi_{g(X_1)}(t_1)\phi_{g(X_3)}(t_1), -\infty < t_1 < \infty. \tag{3. 3}$$

Let $t_1 = 0$ in (3.2). Then it follows that

$$\phi(0, t_2) = \phi_{g(X_2)}(t_2)\phi_{g(X_3)}(t_2), -\infty < t_2 < \infty. \tag{3. 4}$$

Equations (3.2)-(3.4) show that

$$\begin{aligned} &\phi_{g(X_1)}(t_1)\phi_{g(X_2)}(t_2)\phi_{g(X_3)}(t_1 + t_2)\phi(t_1, 0)\phi(0, t_2) \\ &= \phi(t_1, t_2)\phi_{g(X_3)}(t_1)\phi_{g(X_1)}(t_1)\phi_{g(X_3)}(t_2)\phi_{g(X_2)}(t_2) \end{aligned} \tag{3. 5}$$

and hence

$$\phi_{g(X_3)}(t_1 + t_2) = \frac{\phi(t_1, t_2)}{\phi(t_1, 0)\phi(0, t_2)}\phi_{g(X_3)}(t_1)\phi_{g(X_3)}(t_2) \tag{3. 6}$$

for $-\infty < t_1, t_2 < \infty$. Let $\gamma_i(t) = \log \phi_{g(X_i)}(t)$ be the continuous branch of the logarithm of $\phi_{g(X_i)}(.)$ with $\gamma_i(0) = 0$. Then it follows that

$$\gamma_3(t_1' + t_2) = \log\{\frac{\phi(t_1', t_2)}{\phi(t_1', 0)\phi(0, t_2)}\} + \gamma_3(t_1') + \gamma_3(t_2) \tag{3. 7}$$

for $-\infty < t_1, t_2 < \infty$. Integrating on both sides of the equation (3.7)with respect to $t_1'$ over the interval $[0, t_1]$, we have

$$\begin{aligned} \int_0^{t_1} \gamma_3(t_1' + t_2)dt_1' &= \int_0^{t_1} \log\{\frac{\phi(t_1', t_2)}{\phi(t_1', 0)\phi(0, t_2)}\}dt_1' + \\ &+ \int_0^{t_1} \gamma_3(t_1')dt_1' + \int_0^{t_1} \gamma_3(t_2)dt_1'. \end{aligned} \tag{3. 8}$$

Let $t = \prime t_1 + t_2$ in the integral on the left side of the above equation. Then we have

$$\begin{aligned} \int_{t_2}^{t_1+t_2} \gamma_3(t)dt &= \int_0^{t_1} \log\{\frac{\phi(t_1', t_2)}{\phi(t_1', 0)\phi(0, t_2)}\}dt_1' + \\ &+ \int_0^{t_1} \gamma_3(t)dt + t_1\gamma_3(t_2). \end{aligned} \tag{3. 9}$$

Rewriting (3.7) in the form

$$\gamma_3(t_1 + t_2') = \log\{\frac{\phi(t_1, t_2')}{\phi(t_1, 0)\phi(0, t_2')}\} + \gamma_3(t_1) + \gamma_3(t_2') \tag{3. 10}$$

and integrating on both sides of this equation with respect to $t_2'$ over the interval $[0, t_2]$, it can be shown that

$$\begin{aligned} \int_{t_1}^{t_1+t_2} \gamma_3(t)dt &= \int_0^{t_2} \log\{\frac{\phi(t_1, t_2')}{\phi(t_1, 0)\phi(0, t_2')}\}dt_2' + \\ &+ \int_0^{t_2} \gamma_3(t)dt + t_2\gamma_3(t_1). \end{aligned} \tag{3. 11}$$

Equating (3.9) and (3.11), we have

$$\begin{aligned} t_1\gamma_3(t_2) - t_2\gamma_3(t_1) &= \int_0^{t_2} \log\{\frac{\phi(t_1, t_2')}{\phi(t_1, 0)\phi(0, t_2')}\}dt_2' \\ &- \int_0^{t_1} \log\{\frac{\phi(t_1', t_2)}{\phi(t_1', 0)\phi(0, t_2)}\}dt_1' \end{aligned} \tag{3. 12}$$

for $-\infty < t_1, t_2 < \infty$. Dividing both sides of the equation (3.12) by $t_1 t_2$, we get that

$$\begin{aligned}\frac{\gamma_3(t_2)}{t_2} - \frac{\gamma_3(t_1)}{t_1} = \frac{1}{t_1 t_2}[\int_0^{t_2} \log\{\frac{\phi(t_1, t_2')}{\phi(t_1,0)\phi(0,t_2')}\}dt_2' \\ - \int_0^{t_1} \log\{\frac{\phi(t_1', t_2)}{\phi(t_1',0)\phi(0,t_2)}\}dt_1']\end{aligned} \tag{3. 13}$$

for $-\infty < t_1, t_2 < \infty$ with $t_1 t_2 \neq 0$. Let $t_2 = t$ and $t_1 \to 0$.

Assume that $E(X_3) = m$ is finite and that the interchange of the limit and the integral is permitted in the following computations. Then we have

$$\lim_{t\to 0} \frac{\gamma_3(t)}{t} = im \tag{3. 14}$$

and, from (3.13),

$$\begin{aligned}\frac{\gamma_3(t)}{t} &= im + \frac{1}{t}\lim_{t_1\to 0}[\int_0^t \frac{1}{t_1}\log\{\frac{\phi(t_1,v)}{\phi(t_1,0)\phi(0,v)}\}dv \\ &\quad -\frac{1}{t_1}\int_0^{t_1} \log\{\frac{\phi(u,t)}{\phi(u,0)\phi(0,t)}\}du] \\ &= im + \frac{1}{t}\lim_{t_1\to 0}[\int_0^t \frac{1}{t_1}\log\{\frac{\phi(t_1,v)}{\phi(t_1,0)\phi(0,v)}\}dv] \\ &\quad -\log\frac{\phi(0,t)}{\phi(0,0)\phi(0,t)} \\ &= im + \frac{1}{t}\lim_{t_1\to 0}[\int_0^t \frac{1}{t_1}\log\{\frac{\phi(t_1,v)}{\phi(t_1,0)\phi(0,v)}\}dv] \\ &= im + \frac{1}{t}\int_0^t \frac{\partial}{\partial u}[\log[\frac{\phi(u,v)}{\phi(u,0)\phi(0,v)}]|_{u=0}dv.\end{aligned} \tag{3. 15}$$

Hence

$$\gamma_3(t) = imt + \int_0^t \frac{\partial}{\partial u}[\log[\frac{\phi(u,v)}{\phi(u,0)\phi(0,v)}]|_{u=0}dv. \tag{3. 16}$$

Using this formula for $\gamma_3(t)$, one can compute $\phi_{g(X_3)}(t)$ and hence $\phi_{g(X_1)}(t)$ and $\phi_{g(X_2)}(t)$ by using the relations

$$\phi_{g(X_1)}(t) = \frac{\phi(t,0)}{\phi_{g(X_3)}(t)}, \phi_{g(X_2)}(t) = \frac{\phi(0,t)}{\phi_{g(X_3)}(t)}, -\infty < t < \infty. \tag{3. 17}$$

Equations (3.16) and (3.17) give explicit formulae for computing the characteristic functions of $g(X_i), 1 \le i \le 3$ given the characteristic function of $(g(X_1 * X_3), g(X_2 * X_3))$. Since the function $g$ is continuous and strictly increasing , one can obtain the distributions of $X_i, 1 \le i \le 3$.

## 4 Almost lack of memory property

Let $X$ be a nonnegative random variable with distribution function $F(x)$. Then $X$ is said to have the *lack of memory property* if

$$P(X > s + t | X > s) = P(X > t) \tag{4. 1}$$

for all $s, t > 0$. If $P(X > s) > 0$ for all $s > 0$, then it follows that

$$\bar{F}(s+t) = \bar{F}(s)\bar{F}(t) \tag{4. 2}$$

for all $s > 0$ and $t > 0$ where $\bar{F}(x) = 1 - F(x)$. It is well known that the only continuous solution of the equation (4.2) is

$$\bar{F}(s) = \exp\{-\lambda s\}, s > 0 \tag{4. 3}$$

for some $\lambda > 0$. This result was generalized by Muliere and Scarsini (1987) in the following manner. Let $*$ be a binary operation as discussed above with an identity $\tilde{e}$. Suppose it is continuous, reducible and associative. Further suppose that $X$ is a random variable with the distribution function $F$ having support $(\tilde{e}, g^{-1}(\infty))$ and satisfying the relation

$$P(X > s * t | X > s) = P(X > t) \tag{4. 4}$$

for all $s > \tilde{e}$ and $t > \tilde{e}$. They proved that the only continuous solution of the equation (4.4) is

$$\bar{F}(s) = \exp\{\alpha g(s)\} \tag{4. 5}$$

for some $\alpha < 0$ and for $\tilde{e} = g^{-1}(0) < s < g^{-1}(\infty)$. The function $g(.)$ is the function corresponding to the binary operation $*$ as discussed in the earlier sections.

**Remarks:** Suppose the equation (4.4) holds. By choosing the binary operation appropriately, we can get different classes of distributions (cf. Muliere and Scarsini (1987), Muliere (1984)). For instance (i) if $x * y = x + y$, then we obtain the characterization of exponential distribution through the lack of memory property; (ii) if $x * y = xy$, then we obtain a characterization of the Pareto distribution; and (iii) if $x * y = (x^\alpha + y^\alpha)^{1/\alpha}$, then we obtain a characterization of the Weibull distribution.

A nonnegative random variable $X$ is said to have the *almost lack of memory property* if the equation (4.1) holds for a sequence $s_n > 0, n \geq 1$ and for all $t \geq 0$. It is known that the equation (4.1) holds for a sequence $s_n > 0, n \geq 1$ and for all $t \geq 0$ if and only if there exists $d > 0$ such that $s_n = nd$ except in case when $P(X \geq d) = 0$ or $P(X \geq d) = 1$. (cf. Ramachandran and Lau (1991); Prakasa Rao (1997)).

Suppose that $*$ is a binary operation with an identity $\tilde{e} \in \bar{R}$ as discussed above and further suppose that the equation (4.4) holds for a random variable $X$, with a continuous distribution function $F$ with support $(\tilde{e}, g^{-1}(\infty))$, for a sequence $g^{-1}(\infty) > s_n > \tilde{e}, n \geq 1$ for all $g^{-1}(\infty) > t > \tilde{e}$. Here $g(.)$ is the continuous strictly increasing function corresponding to the binary associative operation $*$. Equation (4.4) implies that

$$\bar{F}(s_n * t) = \bar{F}(s_n)\bar{F}(t), n \geq 1 \tag{4. 6}$$

for all $t \geq 0$. Let $g(s_n) = u_n, g(t) = v$ and $\bar{F}og^{-1} = H$.

A random variable $X$ satisfying the equation (4.6) is said to have the *almost lack of memory property under the binary operation* $*$.

We now characterize the class of all such distributions. The equation (4.6) shows that

$$\bar{F}(g^{-1}(g(s_n) + g(t)) = \bar{F}(s_n)\bar{F}(t), t \geq 0, n \geq 1 \tag{4. 7}$$

and hence

$$H(s_n + t) = H(s_n)H(t), t \geq 0, n \geq 1. \tag{4. 8}$$

Note that $H(.)$ is a continuous function with $g(\tilde{e}) = 0$ and hence $H(0) = 1$. Applying results from Marsaglia and Tubilla (1975) (cf. Ramachandran and Lau (1991); Prakasa Rao (1997)), it follows that there exists a constant $d > 0$ such that $s_n = nd, n \geq 1$ and further more

$$H(s) = p(s)e^{-\alpha s}, s > 0 \tag{4. 9}$$

for some $\alpha \in R$ and the function $p(s)$ has period $d$. Therefore

$$\bar{F}og^{-1}(s) = p(s)e^{-\alpha s}, s > 0 \tag{4. 10}$$

or equivalently

$$\bar{F}(s) = p(g(s))e^{-\alpha g(s)}, \tilde{e} = g^{-1}(0) < s < g^{-1}(\infty) \tag{4. 11}$$

where $g(.)$ is a continuous strictly increasing function. It easy to see that the constant $\alpha > 0$ from the fact that the function $F(.)$ is a distribution function of the random variable $X$. It is easy to check that if the distribution function of a nonnegative random variable is of the above type, then it has the almost lack of memory property with respect to the binary operation corresponding to the function $g(.)$.

We have the following result.

**Theorem 4.1:** A nonnegative random variable $X$ with a continuous distribution function has the almost lack of memory property under a binary operation * as described above if and only if its distribution function $F$ is of the form

$$\bar{F}(s) = p(g(s))e^{-\alpha g(s)}, \tilde{e} = g^{-1}(0) < s < g^{-1}(\infty) \tag{4. 12}$$

where $\alpha < 0$, $g(.)$ is the continuous strictly increasing function corresponding to the binary operation * and $p(.)$ is a periodic function with period $d$ for some constant $d > 0$.
**Remarks:** It is easy to see that

$$p(g(s) + d) = p(g(s))$$

which implies that

$$p(g(s) + g(g^{-1}(d)) = p(g(s))$$

which in turn shows that

$$p(g(s * g^{-1}(d))) = p(g(s))$$

for all $\tilde{e} = g^{-1}(0) < s < g^{-1}(\infty)$. Let us denote $p(g(s))$ by $(pog)(s)$. Hence

$$(pog)(s * \rho) = (pog)(s)$$

for all $\tilde{e} = g^{-1}(0) < s < g^{-1}(\infty)$ for some constant $\rho > \tilde{e}$. In other words the function $pog(.)$ is periodic under the operation * with period $\rho$.

## 5 Distributions with periodic failure rate under the binary operation *

Consider a binary operation * with an identity $\tilde{e}$ as described earlier. Let $g(.)$ be the corresponding continuous strictly increasing function such that

$$x * y = g^{-1}(g(x) + g(y)).$$

Let $X$ be a random variable with a continuous distribution function of the form

$$\bar{F}(s) = p(g(s))e^{-\alpha g(s)}, \tilde{e} = g^{-1}(0) < s < g^{-1}(\infty) \tag{5. 1}$$

where $\alpha > 0$ and $pog(.)$ is periodic under the operation * with period $\rho > \tilde{e}$. It is obvious that the function $p(g(s))$ is nonnegative for $\tilde{e} = g^{-1}(0) < s < g^{-1}(\infty)$ and $p(g(\tilde{e})) = p(0) = 1$. Suppose the function $p(g(.))$ is differentiable with respect to $s$. Then the probability density function of $X$ is given by

$$\begin{aligned} f(s) &= e^{-\alpha g(s)} g'(s)(\alpha p(g(s)) - p'(g(s)), \tilde{e} = g^{-1}(0) < s < g^{-1}(\infty) \\ &= 0 \text{ otherwise.} \end{aligned} \tag{5. 2}$$

It follows ,from the properties of a probability density function and the fact that $g(.)$ is strictly increasing, that

$$\alpha p(g(s)) - p'(g(s)) \geq 0 \quad \text{a.e. on } \tilde{e} = g^{-1}(0) < s < g^{-1}(\infty). \tag{5. 3}$$

Let

$$\lambda(s) = \frac{f(s)}{\bar{F}(s)} = \frac{g'(s)(\alpha p(g(s)) - p'(g(s)))}{p(g(s))} \tag{5. 4}$$

for $s > \tilde{e} = g^{-1}(0)$ with $F(s) < 1$. It is easy to see that

$$(pog)(s * \rho) = (pog)(s), \quad \text{and} \quad (p'og)(s * \rho)g'(s * \rho) = (p'og)(s)g'(s), \tag{5. 5}$$

for $\tilde{e} = g^{-1}(0) < s < g^{-1}(\infty)$ from the periodicity of the function $pog$ under the binary operation * with period $\rho$. Further more

$$g(s * \rho) = g(s) + g(\rho), \tilde{e} = g^{-1}(0) < s < g^{-1}(\infty) \tag{5. 6}$$

and hence

$$g'(s * \rho) = g'(s), \tilde{e} = g^{-1}(0) < s < g^{-1}(\infty). \tag{5. 7}$$

Therefore

$$\lambda(s * \rho) = \lambda(s), \tilde{e} = g^{-1}(0) < s < g^{-1}(\infty) \tag{5. 8}$$

with $F(s * \rho) < 1$. This shows that the distribution function $F$ has periodic failure rate with period $\rho$ under the binary operation *.

## 6 Distributions with periodic failure rate and "almost lack of memory property" under the binary operation *

Suppose $X$ is a random variable with periodic failure rate $\lambda(.)$ with period $\rho$ under a binary operation $*$. Further suppose that the support of $X$ is $(\tilde{e}, g^{-1}(\infty))$ where $g(.)$ is the continuous strictly increasing function corresponding to $*$. Let $F$ and $f$ denote the probability distribution function and the probability density function of the random variable $X$. Then

$$\lambda(s) = \lambda(s * \rho), s \in (\tilde{e}, g^{-1}(\infty)), F(s * \rho) < 1. \tag{6. 1}$$

and hence

$$\frac{f(s)}{\bar{F}(s)} = \frac{f(s * \rho)}{\bar{F}(s * \rho)}, s \in (\tilde{e}, g^{-1}(\infty)), F(s * \rho) < 1. \tag{6. 2}$$

Integrating both sides of (6.2) with respect to $s$ on the interval $(\tilde{e}, u)$, we have

$$\int_{\tilde{e}}^{u} \lambda(s) ds = \int_{\tilde{e}}^{u} \lambda(s * \rho) ds, \tilde{e} \leq u < g^{-1}(\infty) \tag{6. 3}$$

or equivalently

$$-\log \bar{F}(u) + \log \bar{F}(\tilde{e}) = -\log \bar{F}(u * \rho) + \log \bar{F}(\tilde{e} * \rho), \tilde{e} \leq u < g^{-1}(\infty). \tag{6. 4}$$

Therefore

$$\frac{\bar{F}(\tilde{e})}{\bar{F}(u)} = \frac{\bar{F}(\rho)}{\bar{F}(u * \rho)}, \tilde{e} \leq u < g^{-1}(\infty). \tag{6. 5}$$

This implies that

$$\bar{F}(u * \rho)\bar{F}(\tilde{e}) = \bar{F}(u)\bar{F}(\rho), \tilde{e} \leq u < g^{-1}(\infty). \tag{6. 6}$$

Note that $\bar{F}(\tilde{e}) = 1$ and hence

$$\bar{F}(u * \rho) = \bar{F}(u)\bar{F}(\rho), \tilde{e} \leq u < g^{-1}(\infty). \tag{6. 7}$$

Note that $g(u * \rho) = g(u) + g(\rho)$ and hence

$$\bar{F}(g^{-1}(g(u) + g(\rho))) = \bar{F}(u)\bar{F}(\rho), \tilde{e} \leq u < g^{-1}(\infty). \tag{6. 8}$$

Let $g(u) = v, g(\rho) = d$ and $\bar{F}og^{-1} = H$. Then the equation (6.8) reduces to

$$H(v + d) = H(v)H(d), 0 \leq v < \infty. \tag{6. 9}$$

Repeated application of the relation (6.9) shows that

$$H(v + nd) = H(nd)H(v), 0 \leq v < \infty, n \geq 1. \tag{6. 10}$$

This in turn shows that

$$\bar{F}(u * n\rho) = \bar{F}(u)\bar{F}(n\rho), \tilde{e} \leq u < g^{-1}(\infty), n \geq 1 \tag{6. 11}$$

where $\rho = g^{-1}(d)$ which proves that the the random variable $X$ has the almost lack of memory property under the binary operation $*$. Further more, it follows from the Lemma of Marsaglia and Tubilla (1975) (cf. Lemma 22.2.1, Prakasa Rao (1997)) that

$$H(v) = p(v)e^{-\alpha v}, v \geq 0 \tag{6. 12}$$

for some $\alpha > 0$ and $p(.)$ a periodic function with period $d > 0$. Note that $p(0) = 1$. Hence

$$\bar{F}(s) = p(g(s))e^{-\alpha g(s)}, \tilde{e} \leq s < g^{-1}(\infty) \tag{6. 13}$$

where $\alpha > 0$ and $p(.)$ is a periodic function with period $d > 0$. We have seen above that if a random variable $X$ has a periodic failure rate under a binary operation $*$, then it has the almost lack of memory property under the same binary operation $*$.

Suppose that a random variable $X$ has the almost lack of memory property under a binary operation $*$. Then

$$P(X > s * n\rho | X > n\rho) = P(X > n\rho), \tilde{e} \leq s < g^{-1}(\infty), n \geq 1 \tag{6. 14}$$

and hence

$$\bar{F}(s * n\rho) = \bar{F}(s)\bar{F}(n\rho), \tilde{e} \leq s < g^{-1}(\infty), n \geq 1. \tag{6. 15}$$

Let $H(s) = \bar{F}og^{-1}$. It is easy to see that

$$H(v + nd) = H(nd)H(v), 0 \leq v < \infty, n \geq 1. \tag{6. 16}$$

Since

$$\lambda(s) = \frac{f(s)}{\bar{F}(s)}, \tilde{e} \leq s < g^{-1}(\infty) \tag{6. 17}$$

we have

$$\bar{F}(\tilde{e}) - \bar{F}(s) = \exp\{-\int_{\tilde{e}}^{s} \lambda(t)dt\}. \tag{6. 18}$$

Let $s = g^{-1}(u)$. Then $\tilde{e} = g^{-1}(0)$ and we have

$$H(u) = \exp\{-\int_{\tilde{e}}^{g^{-1}(u)} \lambda(t)dt\}. \tag{6. 19}$$

Combining the relations (6.16) and (6.19), we get that

$$\int_{\tilde{e}}^{g^{-1}(v)} \lambda(t)dt + \int_{\tilde{e}}^{g^{-1}(nd)} \lambda(t)dt = \int_{\tilde{e}}^{g^{-1}(v+nd)} \lambda(t)dt, v \geq 0, n \geq 1. \tag{6. 20}$$

Therefore

$$\int_{\tilde{e}}^{g^{-1}(v)} \lambda(t)dt = \int_{g^{-1}(nd)}^{g^{-1}(v+nd)} \lambda(t)dt, v \geq 0, n \geq 1. \tag{6. 21}$$

Applying the transformation $u = g(t)$, we have

$$\begin{aligned} \int_{0}^{v} \lambda(g^{-1}(u))\gamma(u)du &= \int_{nd)}^{v+nd} \lambda(g^{-1}(u))\gamma(u)dt, v \geq 0, n \geq 1 \\ &= \int_{0}^{v} \lambda(g^{-1}(u + nd))\gamma(u)du \end{aligned} \tag{6. 22}$$

where $\gamma(u) = \frac{1}{g'(g^{-1}(u))}$. Suppose $f$ is continuous. Then $(\lambda o g^{-1})(.)$ is continuous and it follows that

$$\lambda(g^{-1}(v)) = \lambda(g^{-1}(v + nd)), v \geq 0, n \geq 1. \tag{6. 23}$$

Even otherwise, we can conclude that

$$\lambda(g^{-1}(v)) = \lambda(g^{-1}(v + nd)) \text{ a.e.} \tag{6. 24}$$

with respect to the Lebesgue measure for every $n \geq 1$. Equation (6.24) proves that

$$\lambda(s) = \lambda(s * n\rho) \text{ a.e.} \tag{6. 25}$$

and we have the following result.

**Theorem 6.1:** A random variable $X$ with a continuous probability density function has a periodic failure rate under a binary associative, reducible operation * if and only if it has the almost lack of memory property under that operation.

# 7 Representation for distributions with periodic failure rate under a binary associative operation *

Suppose $X$ is a random variable with periodic failure rate $\lambda(.)$ with period $\rho$ under a binary associative operation *. Suppose the support of $X$ is $(\tilde{e}, g^{-1}(\infty))$ where $g(.)$ is the continuous strictly increasing function corresponding to *. Let $F$ and $f$ denote the probability distribution function and the probability density function of the random variable $X$ respectively.

It is easy to see that the random variable $g(X)$ has the distribution function

$$\tilde{H}(y) = 1 - H(y) = (F o g^{-1})(y), 0 \leq y < \infty \tag{7. 1}$$

and it has the periodic failure rate $(\lambda o g^{-1})(y)$ with period $d = g(\rho)$. For convenience, let us denote the derivative of $\tilde{H}(y)$ by $\tilde{h}(y)$. Define a new random variable $Y$ with probability distribution function $\tilde{H}_Y(y)$ and the probability density function

$$\begin{aligned} \tilde{h}_Y(y) &= \frac{\tilde{h}(y)}{\tilde{H}(d)}, \quad 0 \leq y \leq d \\ &= 0 \text{ otherwise.} \end{aligned} \tag{7. 2}$$

Note that $\tilde{h}_Y(.)$ is the probability density function of the conditional distribution of the random variable $g(X)$ restricted to the interval $[0, d]$. Let $\gamma = 1 - \tilde{H}(d) = H(d)$. It follows from the arguments given in Prakasa Rao (1997) that the density function $\tilde{h}(y)$ of $g(X)$ can be represented in the form

$$\tilde{h}(y) = \tilde{h}_Y(y - [\frac{y}{d}]d)(1-\gamma)\gamma^{[\frac{y}{d}]}, 0 \leq y < \infty \tag{7. 3}$$

where $[x]$ denotes the greatest integer less than or equal to $x$ and $\gamma = H(d)$. Further more the distrbution function of $g(X)$ is given by

$$\tilde{H}(y) = 1 - \gamma^{[\frac{y}{d}]} + (1-\gamma)\gamma^{[\frac{y}{d}]}\tilde{H}_Y(y - [\frac{y}{d}]d), 0 \leq y < \infty. \tag{7. 4}$$

Equations (7.3) and (7.4) give representations for the density and distribution functions of the random variable $g(X)$ with a periodic failure rate with period $d$ and $\gamma = H(d)$. Recall that $H = \bar{F}og^{-1}$ and $d = g(\rho)$. One can obtain a representation of the probability distrbution function $F$ and the probability density function $f$ from the above equations observing that $g(.)$ is continuous and strictly increasing.

Let $Y$ be a random variable as defined above and $Z$ be a random variable independent of $Y$ with

$$P(Z = k) = (1-\gamma)\gamma^k, k \geq 0. \tag{7. 5}$$

It is easy to check that the the random variable $g(X)$ can be represented in the form $Y + dZ$. For details, see Prakasa Rao (1997). In particular $X$ can be represented in the form $g^{-1}(Y + g(\rho)Z)$.

# 8 Distributions with strong Markov property under a binary associative operation *

Suppose $X$ is a random variable with an exponential distribution and $Y$ be a nonnegative random variable *independent* of $X$. Further suppose that $P(X > Y) > 0$. Then it is known that

$$P(X > Y + x | X > Y) = P(X > x), x \geq 0. \tag{8. 1}$$

This property is known as the *strong lack of memory property* or the *strong Markov property* of the exponential distribution (cf. Ramachandran and Lau (1991); Prakasa Rao (1997)). It is obvious that this property is stronger than the lack of memory property. Let $G$ be the distribution function of $Y$. Then it follows that

$$c\bar{F}(x) = \int_0^\infty \bar{F}(x+y)dG(y), x \geq 0 \tag{8. 2}$$

where $c = P(X > Y)$.

Ramachandran and Lau (1991) proved the following theorem.

**Theorem 8.1:** Suppose the equation (8.2) holds and further suppose that $G(0) < c < 1$. Let $\alpha > 0$ be defined by the relation

$$\int_0^\infty e^{-\alpha y} dG(y) = c, \tag{8. 3}$$

and let, for $d > o, A(d) = \{nd, n \geq 1\}$. Then

(i) $F$ is an exponential distribution with parameter $\alpha$ if the support of $G$ is not contained in the set $A(d)$ for any $d > 0$, and

(ii) $F(x) = 1 - p(x)e^{-\alpha x}, x \geq 0$ where $p(.)$ is right continuous and has period $d$ if the support of $G$ is contained in $A(d)$ for some $d > 0$ which we take it to be the largest such $d$.

Suppose $X$ is a random variable with the distribution function $F$ and the support of $X$ is $(\tilde{e}, g^{-1}(\infty))$ where $\tilde{e}$ is the identity corresponding to a binary operation $*$ which is continuous, reducible and associative with an identity and $g(.)$ is the continuous strictly increasing function corresponding to the binary operation $*$. Let $Y$ be a nonegative random variable *independent* of $X$. Further suppose that $P(X > Y) > 0$ and $P(X > s) > 0$ for all $s > \tilde{e}$ and

$$P(X > Y * s | X > Y) = P(X > s), \tilde{e} = g^{-1}(0) < s < g^{-1}(\infty). \tag{8. 4}$$

Let $K$ be the distribution function of $Y$. Then it follows that

$$c\bar{F}(s) = \int_{\tilde{e}}^{g^{-1}(\infty)} \bar{F}(s * y) dK(y), \tilde{e} = g^{-1}(0) < s < g^{-1}(\infty) \tag{8. 5}$$

where $c = P(X > Y)$. Let $H = \bar{F}og^{-1}$. Then it follows that

$$cH(x) = \int_0^\infty H(x + u) \;\; d(Kog^{-1})(u), x \geq 0 \tag{8. 6}$$

Let $G(u) = (Kog^{-1})(u)$. Suppose that $G(0) < c < 1$. Let $\alpha > 0$ be defined by the relation

$$\int_0^\infty e^{-\alpha y} dG(y) = c \tag{8. 7}$$

and define the set $A(d)$ as given above. Applying the above theorem, we obtain that (i) $H(x) = e^{-\alpha x}$ with parameter $\alpha > 0$ if the support of $G$ is not contained in $A(d)$ for any $d > 0$; and

(ii)$H(x) = p(x)e^{-\alpha x}$ for all $x \geq 0$ where $p(.)$ is right continuous and has period $d$ if the support of G is contained in $A(d)$ for some $d > 0$ which we take it to be the largest such $d$.

This implies that

(i) $\bar{F}(s) = e^{-\alpha g(s)}$ for all $s \in (\tilde{e}, g^{-1}(\infty))$ with parameter $\alpha > 0$ if the support of the random variable $g(Y)$ is not contained in $A(d)$ for any $d > 0$; and

(ii)$\bar{F}(s) = p(g(s))e^{-\alpha g(s)}$ for all $s \in (\tilde{e}, g^{-1}(\infty))$ where $p(.)$ is right continuous and has period $d$ if the support of $g(Y)$ is contained in $A(d)$ for some $d > 0$ which we take it to be the largest such $d$.

This result gives a characterization of the class of all distributions satisfying the equation (8.4) subject to the conditions stated above.

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