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Opting out of publicly provided services: A majority voting result

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Abstract. Our objective in this paper is to examine majority voting in an environment where both public and private alternatives coexist. We construct a model in which households are differentiated by income and have the option of choosing between publicly provided services and private services. Publicly provided services are financed through income tax revenues and made available to all citizens at zero price. Majority voting determines the tax rate. Even though preferences over tax rates are not single peaked, we provide conditions under which a majority voting equilibrium exists. We illustrate our existence result with CES preferences and a Dagum income distribution.

1. Introduction

In most countries, the government expropriates resources from the citizens and uses these resources to provide services at a very low price even though these services are available through private institutions. Examples of such schemes include education, public transportation, healthcare etc. Stiglitz [8] was among the first to study the provision of public education when parents have the ability to opt out of public schools by sending their children to private schools. He examined a static economy in which agents value private

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consumption and the quality of education of their offspring. He concludes that it is difficult, in general, to obtain predictions about funding levels for public schools in such situations. This difficulty arises since preferences over funding for public schools are not single-peaked. Hence, standard theorems guaranteeing existence of a majority voting equilibrium do not apply (see [1]).

Olsen [6] provides a simple example of funding for public schools in a static model when household incomes are distributed uniformly. He also provides examples of economies where voting equilibria do not exist and where individuals have incentives to vote strategically. Ireland [4] analyses the public provision of goods and services when the tax revenues are also used to supplement incomes. In his model, public policy is exogenously specified.

Our objective in this paper is to determine public expenditures through majority voting in an environment where both public and private alternatives coexist. In Sect 2, we consider a static economy populated by a continuum of agents. The only (ex-ante) difference between agents is their income which is exogenously specified. Part of the individual's income is used for own consumption expenditures and the rest for quality of some services. These services are available from the private sector as well as the public sector. A government collects income taxes at a uniform rate from all individuals and uses the tax revenues to make the services available at zero price to everyone. However, all individuals have the option of obtaining private service. Quality of the publicly provided service is assumed to be an increasing function of the tax revenues spent per person demanding the service.

In Sect 3, we endogenize the tax rates. Even though indirect utility over tax rates is not single-peaked in our model, we provide conditions under which a majority voting equilibrium exists. We show that the decisive voter is the agent with median income. We then illustrate our result through an example with CES preferences and Dagum income distribution (see [2]). We conclude in Section 4.

2. The model

Consider an economy with a large number of agents with identical preferences over private consumption, c_i, and the quality of services, q_i. We normalize the population size to unity. Agent i's preferences are represented by $U(c_i, q_i)$ where $U : \mathbb{R}_+ \to \mathbb{R}$ is increasing, strictly quasi-concave, and twice continuously differentiable. Agent i has income y_i and we assume that incomes across agents are distributed according to the cumulative distribution function $F(\cdot)$ with finite mean. We assume that the support of $F(\cdot)$ is $\mathbb{R}_+$ and that $F(\cdot)$ is strictly increasing and twice continuously differentiable.

A government collects taxes from all individuals at a constant rate τ. Tax revenues are used to provide services. Let N be the proportion (or measure) of agents choosing publicly provided services. Public expenditures per agent is then $\tau Y/N$ where Y is total (as well as average) income. The public expenditures are converted into quality of service according to

$$Q = \begin{cases} \frac{\tau Y}{N} & \text{if } \tau > 0 \text{ and } N > 0 \\ 0 & \text{if } \tau = 0 \text{ and } N = 0. \end{cases}$$

We have omitted two cases: (i) $\tau > 0$ and $N = 0$ and (ii) $\tau = 0$ and $N > 0$. Neither case ever occurs, in equilibrium, in our model. If $N = 0$, then, in equilibrium, a positive tax rate will not be supported by any individual. If the tax rate is zero then public expenditures are zero and no individual would prefer the publicly provided services over private services.

All agents are taxed in this economy, but each agent is free to choose between publicly provided services and private services. Quality of the publicly provided service does not vary across the individuals who choose it i.e., $q_i = Q$ for all i who choose the publicly provided services. The quality of private services, however, is specific to the individual. Each individual allocates the after-tax income to consumption expenditures and private services i.e.,

$$q_i = (1 - \tau)y_i - c_i.$$

Note that the technology for converting individual expenditures into quality is the same as in the public sector. However, we have assumed that each individual can choose private services from a wide menu of different quality levels. Thus, the quality of private services may differ across agents who choose the private sector which is not the case for those who choose publicly provided services. The choice between public and private services in our model is non-convex; no agent can choose the publicly provided service and supplement it with some private services.

If agent i with income y_i chooses to obtain the services from the public sector, then the utility maximization problem is trivial: after-tax income is simply spent on the consumption good so that $c_i = (1 - \tau)y_i$. Indirect utility for agent i in such a case is a function of the tax rate τ, own income y_i, aggregate income Y and the public sector enrollment N. Let this indirect utility for agent i be denoted by $V^u(\tau, y_i, Y, N)$.

If agent i opts for private services, then the choice of consumption and quality solves the following maximization problem:

$$\begin{aligned} \text{Max.} \quad & U(c_i, q_i) \\ \text{s.t.} \quad & q_i = (1 - \tau)y_i - c_i \\ & c_i \geq 0, \; q_i \geq 0, \\ & \text{given } \tau \in [0, 1]. \end{aligned}$$

Clearly, the above optimization problem has a unique solution. Let $V^r(\tau, y_i)$ denote i's indirect utility if agent i chooses private services.

Agent i will choose publicly provided services over private services if and only if $V^u(\tau, y_i, Y, N) \geq V^r(\tau, y_i)$. Since each individual takes as given the proportion of agents choosing publicly provided services when making the public-private sector choice, we have to make sure that, in equilibrium, the individual decisions are consistent with the aggregate outcomes. That is, the proportion of agents for whom $V^u(\tau, y_i, Y, N) \geq V^r(\tau, y_i)$ must be exactly equal to N. Formally, the equilibrium fraction of agents choosing publicly provided services, N^*, must solve

$$N = \mu \{i : V^u(\tau, y_i, Y, N) \geq V^r(\tau, y_i)\} \tag{1}$$

where the $\mu\{\cdot\}$ is the probability measure associated with the distribution function $F(\cdot)$. Since $F(\cdot)$ is continuous (i.e., there are no mass points), tie breaking rules in (1) are of no consequence.

All agents in the economy vote on tax rates and the equilibrium tax rate τ^* is the one chosen by a majority of voters.

Definition. A *majority voting equilibrium* is a pair $\{\tau^*, N^*\}$ which satisfies

(i) given τ^*, the solution to equation (1) is N^* and
(ii) there does not exist another pair $\{\tau', N'\}$ such that
 a) given τ', N' solves equation (1) and
 b) τ' is preferred over τ^* by more than half the population.

In the next section we determine the critical income level which separates the individuals who choose publicly provided services from those who choose private services using the indirect utilities $V^u(\cdot)$ and $V^r(\cdot)$. We then use this critical income to determine N^*. We end the section by establishing the existence of a majority voting equilibrium.

3. Majority voting equilibrium

Intuition suggests that individuals with high incomes would be better off choosing private services while those with low incomes would be better off with publicly provided services. We state the result formally in Proposition 1 below. We relegate all proofs to the Appendix.

Proposition 1. *Assume that $U(\cdot)$ is homothetic and that $lim_{c\to\infty} U_c(c, e) = 0$ for all $e > 0$. Given $\tau \in (0, 1)$, $N \in (0, 1]$ and $Y \in \mathbb{R}_{++}$, there exists a unique $\hat{y} > 0$ such that $V^u(\tau, y_i, Y, N) \geq V^r(\tau, y_i)$ if and only if $y_i \leq \hat{y}$.*

In Proposition 1, we excluded the corners $\tau = 0$ and $\tau = 1$. It is clear that if $\tau = 0$ and $N = 0$, the quality of publicly provided services is zero and hence, the critical income is zero i.e., everyone chooses private services. If $\tau = 1$ and $N \in (0, 1]$, then everyone chooses publicly provided services.

Two remarks are in order regarding the critical income $\hat{y}$. First, $\hat{y}$ is a continuous function of $\tau \in (0, 1)$, $Y \in \mathbb{R}_{++}$ and $N \in (0, 1)$. Second, since we determined $\hat{y}$ from each individual's optimization taking τ, Y, and N as given, the income distribution influences $\hat{y}$ only through τ, Y, and N. That is, given τ, Y, and N, the critical income is pinned down. However, as we shall see below, the equilibrium N and τ will depend on the income distribution.

In Lemma 1, we establish some properties of $\hat{y}$ which help us determine N^*. We will assume henceforth that the conditions in Proposition 1 hold.

Lemma 1. *(i) For* $N \in (0, 1)$, $\hat{y}$ *is decreasing in* N. *(ii) For* $Y \in \mathbb{R}_{++}$, $\hat{y}$ *is increasing in* Y. *(iii) For* $\tau \in (0, 1)$, $\hat{y}$ *is increasing in* τ.

Part (i) of Lemma 1 is just a congestion effect. More individuals choosing publicly provided services implies that the quality of publicly provided services is lower. Hence, individuals on the margin would opt out of the public sector into the private sector. Part (ii) of Lemma 1 follows along the same lines since the quality of publicly provided services is increasing in Y.

To understand part (iii), consider Fig. 1. Suppose there are two tax rates τ and τ' with $\tau' > \tau$. Consider an individual with income y' such that his after-tax income $(1 - \tau')y'$ is the same as $(1 - \tau)\hat{y}$. Clearly, y' must be greater than $\hat{y}$. Further, holding quality of publicly provided services constant, this individual would be just indifferent between public and private services at the tax rate τ'. If we account for the fact that the quality of publicly provided services is higher under τ', this individual would prefer publicly provided services to private services. Thus, the critical income under τ' must be greater than $\hat{y}$.

We now turn to the fraction of agents choosing publicly provided services. To determine N^* given the tax rate τ, we have to verify the consistency condition (1). The proportion of agents with income less than or equal to $\hat{y}$ must be the same as N which all individuals take as given. That is, N^* must solve

$$N = F(\hat{y}(\tau, Y, N)). \tag{2}$$

In Proposition 2 below, we establish the existence and uniqueness of N^*.

Proposition 2. *For all* $\tau \in (0, 1)$ *and* $Y \in \mathbb{R}_{++}$, *there exists a unique* $N^* \in (0, 1)$ *which solves equation (2).*

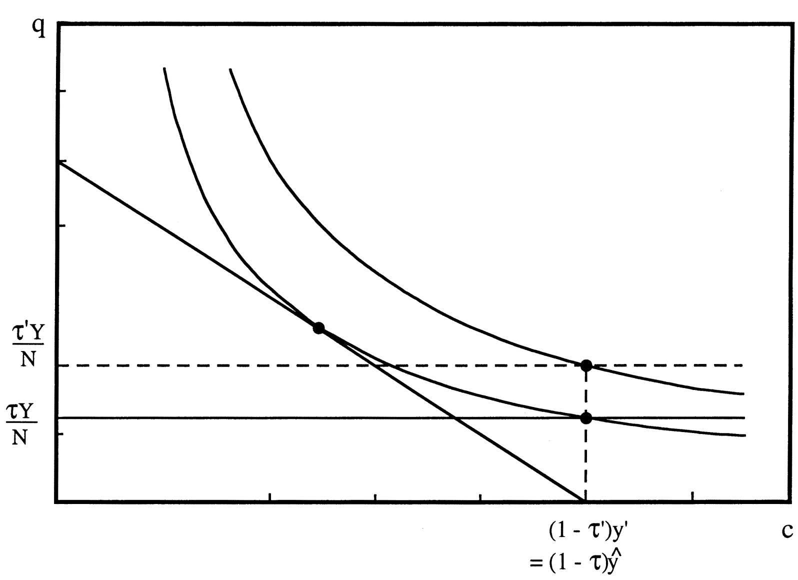

Fig. 1 Critical income and tax rates

In the proof of Proposition 2, we use the fact that the support of $F(\cdot)$ is not compact. The proportion of agents below the critical income, $F(\hat{y})$, is always less than one. This condition is not necessary for the existence of a unique N^* as the following example demonstrates.

Example. Suppose that preferences are logarithmic: $U(c, q) = \ln c + \ln q$. For an agent with income y who chooses private services, $c = q = 1/2(1-\tau)y$. Thus, the critical income is given by, $\hat{y} = 4\tau Y/((1-\tau)N)$. If income is uniformly distributed on $[a, b] \subseteq \mathbb{R}_+$, then $Y = (a+b)/2$ and the equilibrium N^* is the solution to $N = (\hat{y} - a)/(b - a)$ provided that the solution is strictly between zero and one. Substituting for $\hat{y}$ and rearranging, we get

$$(b-a)N^2 + aN - 2\tau(a+b)/(1-\tau) = 0.$$

The positive N^* is given by

$$N^* = \frac{\sqrt{a^2 + \dfrac{8\tau(b^2 - a^2)}{1-\tau}} - a}{2(b-a)}.$$

It is easy to check that N^* is less than one for small tax rates. For tax rates close to one, $N^* = 1$.

As a result of Proposition 2, we can write $N^* = N(\tau)$ where $N(\cdot)$ is a continuous function of τ. In general, N^* depends on the entire income distribution but we have suppressed it in our notation since we are ultimately interested in finding a majority voting equilibrium for a given income distribution.

So far we have described the choice of public versus private services in an environment where the individuals took the tax rate and, hence, the quality of publicly provided services as given. We now endogenize the tax rate through majority voting. We first determine the most preferred tax rate for an individual with income y:

$$\tau^*(y) = \text{argmax } V(\tau, y)$$

subject to $\tau \in [0, 1]$

where $V(\tau, y) = \max.\ \{V^r(\tau, y),\ V^u(\tau, y, N(\tau))\}$ and $N(\tau)$ is the solution to (1). We have suppressed the index *i* for convenience. We have also suppressed the dependence on the average income Y in our notation.

If $V(\cdot, y)$ is single-peaked for each y then we can use Black's theorem to establish the existence of a majority voting equilibrium. In addition, if the most preferred tax rate $\tau^*(\cdot)$ is monotonic in income then the tax rate chosen by the majority would be the one most preferred by the voter with median income. In environments similar to ours, [8] has shown that preferences over tax rates are not single peaked and in general, a majority voting equilibrium may not exist.

Figure 2 illustrates the indirect utility over tax rates for individuals with different income levels. Notice that for low incomes, preferences over tax

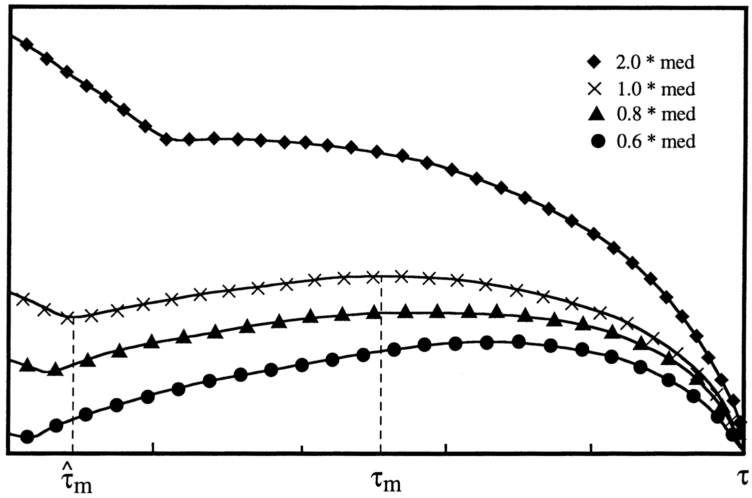

Fig. 2 Preferences over tax rates

rates are not single peaked: if the tax rate is sufficiently close to zero, the quality of publicly provided services is low and a typical individual chooses private services. If the tax rate is increased marginally, private services is still preferred over publicly provided services. But, a small increase in the tax rate lowers utility since private consumption is lower. If the tax rate is increased further the individual becomes indifferent between public and private services. Increasing the tax rate above this level induces the individual to choose public over private services; the utility increases until the most preferred tax rate is reached. Any further increase in the tax rate lowers utility. For sufficiently wealthy individuals preferences over tax rates are single peaked and their most preferred tax rate is zero.

The interior maximum $\tau^u(y)$ for an individual with income y is given by

$$\tau^u(y) = \text{argmax. } V^u(\tau,\ Y,\ N(\tau))$$

Let the interior maximum for the voter with median income be defined as τ_m i.e., $\tau_m = \tau^u(y_m)$. We will demonstrate that τ_m is the tax rate chosen by the majority if preferences over tax rates have certain crucial features (as in Fig. 2). We first define the critical tax rate, $\hat{\tau}(y)$, as a solution to

$$V^r(\tau, y) = V^u(\tau, y, N(\tau)).$$

At $\hat{\tau}(y)$, an agent with income y is indifferent between public and private services. There clearly exists such a tax rate for each y since, for τ close to zero private services is preferred to publicly provided services, and for τ close to one publicly provided services is preferred to private. If there is more than one critical tax rate for each y then interpret $\hat{\tau}(y)$ as the minimum of the critical tax rates.

Two key aspects of Fig. 2 help us establish the existence of a majority voting equilibrium: (i) The critical tax rate $\hat{\tau}(y)$ is increasing in y, and (ii) The interior maximum $\tau^u(y)$ is decreasing in y. The first property is intuitive: the tax rate at which an agent with low income is indifferent between public and private services is less than the critical tax rate for an agent with high income. We prove this formally in Lemma 2.

Lemma 2. *The critical tax rate $\hat{\tau}(y)$ is non-decreasing in y.*

The second property is difficult to obtain. To characterize the properties of $\tau^u(\cdot)$ analytically, we need to know the properties of $N(\tau)$. Restrictions on preferences alone do not pin down the behavior of $N(\tau)$ since the fraction of agents choosing publicly provided services also depends on the income distribution. We provide an example later in the section where one can verify the second property numerically.

Our task now is to show that in a pairwise comparison with τ_m, no tax rate gains more than 50 % vote to beat τ_m. We split the alternatives to τ_m into three regions: (i) $\tau \in [0, \hat{\tau}_m)$, (ii) $\tau \in [\hat{\tau}_m, \tau_m)$ and (iii) $\tau \in (\tau_m, 1]$ where $\hat{\tau}_m \equiv \hat{\tau}(y_m)$. In Lemma 3 and Lemma 4 below, we show that τ_m cannot be beaten by any other tax rate in $[\hat{\tau}_m, 1]$.

Lemma 3. *There does not exist a $\tau \in (\tau_m, 1]$ that is preferred to τ_m by more than 50 % of the population.*

Lemma 4. *There does not exist a $\tau \in [\hat{\tau}_m, \tau_m)$ that is preferred to τ_m by more than 50 % of the population.*

We have used two (more) properties of $V^u(\cdot)$ which are essential for the proofs of Lemma 3 and Lemma 4: (i) $V^u(\cdot)$ is decreasing in τ over the interval $(\tau^u(y), 1]$ and (ii) $V^u(\cdot)$ is increasing in τ over the interval $[\hat{\tau}(y), \tau^u(y))$. In Proposition 3, we eliminate the tax rates $[0, \hat{\tau}_m)$. We first establish a useful monotonicity property in Lemma 5: if the voter with median income prefers the positive tax rate τ_m over zero then the poorest half of the population have the same preference ordering. On the other hand, if the voter with median income prefers zero over τ_m then the richest half have the same preference ordering.

Lemma 5. *Let N_m be the public school enrollment evaluated at the tax rate τ_m i.e., $N_m = N(\tau_m)$.*

(i) *If $V^r(0, y_m) < V^u(\tau_m, y_m, N_m)$ then $V^r(0, y) < V^u(\tau_m, y, N_m)$ for all $y < y_m$.*
(ii) *If $V^r(0, y_m) > V^u(\tau_m, y_m, N_m)$ then $V^r(0, y) > V^u(\tau_m, y, N_m)$ for all $y > y_m$.*

Proposition 3. *If $V^r(0, y_m) < V^u(\tau_m, y_m, N_m)$, then the pair $\{\tau_m, N_m\}$ is a majority voting equilibrium.*

In Proposition 3, we assumed that $V^r(0, y_m) < V^u(\tau_m, y_m, N_m)$. If $V^r(0, y_m)$ is greater than $V^u(\tau_m, y_m, N_m)$, then $\{0, 0\}$ is a majority voting equilibrium. This is because $V^r(\cdot)$ is decreasing in τ and $V^r(0, y) > V^u(\tau, y, N(\tau))$ for all $y > y_m$ and $\tau \in (0, 1]$. Hence, no tax rate in $(0, 1]$ is preferred to zero by more than 50 % of the population. The knife edge case of $V^r(0, y_m) = V^u(\tau_m, y_m, N_m)$ yields both $\{0, 0\}$ and $\{\tau_m, N_m\}$ as majority voting equilibria. This follows from our definition of majority voting equilibrium. We require that any alternative to the candidate pair must beat the candidate pair by more than 50 %.

3.1. Discussion

We illustrate our existence result through a simple example. Suppose that agent i's preferences are represented by

$$U(c_i,\ q_i) = \frac{1}{1-\sigma}\{c_i^{1-\sigma} + q_i^{1-\sigma}\},\ \sigma \in (0,1),$$

and that the income distribution is Dagum. That is,

$$F(y) = \{1 + \lambda y^{-\alpha}\}^{-\beta},\ \alpha > 0,\ \beta > 0,\ \text{and}\ \lambda > 0.$$

The Dagum distribution fits observed income distributions better than the Gamma, the lognormal and the Singh-Maddala distribution (see [5]).[1]

Indirect utility for agent i under publicly provided services is

$$V^u(\tau, y_i, Y, N) = \frac{1}{1-\sigma}\left\{(1-\tau)^{1-\sigma} y_i^{1-6} + \left(\frac{\tau Y}{N}\right)^{1-\sigma}\right\}.$$

If agent i opts for private services, then the choice of consumption and quality of private services are $c_i = 1/2(1-\tau)y_i = q_i$. Then, agent i's indirect utility under private services is

$$V^r(\tau, y_i) = \frac{2^\sigma}{1-\sigma}(1-\tau)^{1-\sigma} y_i^{1-\sigma}.$$

The critical income is given by

$$\hat{y} = (2^\sigma - 1)^{\frac{1}{\sigma-1}}\left\{\frac{\tau Y}{(1-\tau)N}\right\}$$

and the equilibrium N^* solves

$$N = F\left[(2^\sigma - 1)^{\frac{1}{\sigma-1}}\left\{\frac{\tau Y}{(1-\tau)N}\right\}\right].$$

The interior maximum $\tau^u(y)$ is determined according to

$$\tau^u(y) = \text{argmax}\ \frac{1}{1-\sigma}\left\{(1-\tau)^{1-\sigma} y^{1-\sigma} + \left[\frac{\tau Y}{N(\tau)}\right]^{1-\sigma}\right\}.$$

We verify numerically that $\tau^u(y)$ is the unique interior maximum and that it is decreasing in y. The critical tax rate of an individual with income y must solve

$$\frac{\tau Y}{(1-\tau)N(\tau)} = y\{2^\sigma - 1\}^{\frac{1}{1-\sigma}}$$

[1] In fact, Kotz and Johnson [5] report that among the above distributions, the Dagum distribution is the only one that passed the Kolmogorov-Smirnov test.

which is clearly increasing in y. Thus, we satisfy the crucial features of Fig. 2. Proposition 3 guarantees that a majority voting equilibrium exists.

Our median voter result is related to that of [7]. He assumes a *hierarchical adherence* condition on the set of individuals in a society. This condition imposes an ordering of individuals similar to ours. However, his existence result is applicable only when the set of public choices is finite. Epple and Romer [3] impose a *single crossing property* to obtain existence of majority voting equilibrium when preferences are not single peaked. Our approach is different: we construct the majority voting equilibrium by identifying the decisive voter.

It is natural to ask if our result fails for CES preferences with $\sigma \geq 1$ or $\sigma = 0$. Proposition 3 holds for $\sigma = 1$ which is the case of logarithmic preferences. However, the interior maximum τ_m for the voter with median income is the same as that for any $y : \tau^u(y) = \tau_m$ for all y. The proofs of Lemma 3 and Lemma 4 are valid for $\tau^u(\cdot)$ non-increasing in y.

For $\sigma > 1$, $\tau^u(y)$ is strictly increasing in y. We cannot establish the existence of a majority voting equilibrium for this case using our elimination strategies. Lemma 3 still holds; no tax rate in $(\tau_m, 1]$ can beat τ_m since $V^u(\cdot)$ is decreasing over this interval for all $y \leq y_m$. We can also eliminate the tax rates in $[0, \hat{\tau}_m)$ using the proof of Proposition 3. However, Lemma 4 no longer holds. This is because $V^u(\cdot)$ is not increasing in τ over the interval $[\hat{\tau}_m, \tau_m)$ for all $y \leq y_m$. Thus, a majority voting equilibrium may not exist.[2]

For $\sigma = 0$, c and q are perfect substitutes. It is easy to see that for any positive tax rate everyone would prefer publicly provided services i.e., given τ, Y and N, the indirect utility from publicly provided services for an individual with income y is $(1-\tau)y + \tau Y/N$ which is clearly greater than the indirect utility from private services, $(1-\tau)y$. If the tax rate is zero then everyone is indifferent between public and private services. Hence, if the median income is below the average income then the majority voting equilibrium is $\{1, 1\}$.

4. Concluding remarks

Our objective in this paper was to obtain a majority voting equilibrium in an environment where both public and private alternatives coexist. We have presented a simple model where each individual has the choice of opting out of the publicly provided services. Taxes on individuals' income determine the

[2] However, for $\sigma > 1$, we can compute the majority voting equilibrium numerically as follows. For a random sample of individuals drawn from the Dagum distribution and for 100 values of tax rates in [0,1], compute the $N(\tau)$ function. Use the $N(\tau)$ function to compute the indirect utilities over tax rates. Pick a candidate tax rate and compute the number of votes against it for each alternative in [0,1]. This is a majority voting equilibrium if no alternative has more than 50 %. If an alternative has more than 50 % against the candidate tax rate then pick another candidate tax rate and repeat the exercise.

quality of publicly provided services. In our model, preferences over tax rates are not single peaked. However, a majority voting equilibrium does exist and the decisive voter is the agent with median income. We illustrate our existence result through an example where the utility function is of the CES variety and the individual incomes follow a Dagum distribution.

In this paper, opting out of publicly provided services is an exogenously specified institution. Given this institution, we obtain results on the level of public services most preferred by a majority of voters. It would be interesting to extend our model to include endogenous determination of the institution to provide public services.

Appendix

Proof of Proposition 1

It is clear that the critical income $\hat{y}$, if one exists, solves

$$U((1-\tau)y,\ \tau Y/N) = U(c^*((1-\tau)y),\ e^*((1-\tau)y)) \tag{A.1}$$

where $c^*(\cdot)$ and $e^*(\cdot)$ are the optimal choices of the agent with income y under private services. Consider all j such that $(1-\tau)y_j \leq \tau Y/N$. Let c_j^* and e_j^* denote the optimal choices of individual j under private services. Clearly,

$$c_j^* \leq (1-\tau)y_j \text{ and } e_j^* \leq (1-\tau)y_j \leq \tau Y/N.$$

Thus, $V^u(\tau, y_j, Y, N) \geq V^r(\tau, y_j)$ for all j with $(1-\tau)y_j \leq \tau Y/N$.

For y sufficiently small we have shown that the left hand side of (A.1) is greater than the right hand side. We need to show that for y sufficiently large the right hand side of (A.1) exceeds the left hand side.

Since $U(\cdot)$ is homothetic, $\hat{y}$ solves (A.1) if and only if it solves

$$H((1-\tau)y,\ \tau Y/N) = H(c^*((1-\tau)y),\ e^*((1-\tau)y)) \tag{A.2}$$

where $H(\cdot)$ is homogeneous of degree one. The right hand side of (A.2) is linear in y whereas the left hand side is strictly concave in y with $H_y \to 0$ as $y \to \infty$. Thus, if there exists a $\hat{y}$ that solves (A.2) it must be unique.

Now, for sufficiently large y the right hand side of (A.2) exceeds the left hand side. Thus, there exists a $\hat{y} > 0$ which solves (A.2) and hence, (A.1).

Q.E.D

Proof of Lemma 1

The indirect utility $V^r(\cdot)$ does not depend on Y or N. Both (i) and (ii) follow since $V^u(\cdot)$ decreases with N and increases with Y.

To prove (iii), note that the critical income $\hat{y}$ solves

$$(1-\tau)yH(1,\ \tau Y/N(1-\tau)y) = (1-\tau)yH(s_c,\ s_e)$$

where s_c and s_e are the (after-tax) income shares of consumption and quality of education. Since $H(1,\ \tau Y/N(1-\tau)y)$ is increasing in τ for all y, we must have $\hat{y}$ increasing in τ.

Q.E.D

Proof of Proposition 2

For $N = 1, F(\hat{y}) \in (0, 1)$. By continuity, for N close to 1, $F(\hat{y})$ is less than N. From part (i) of Lemma 1, we know $\hat{y}$ is decreasing in N and hence, the right hand side of (2) is continuously decreasing in N. Thus, for N close to zero, $F(\hat{y})$ is greater than N. There exists a unique solution to (2) since the LHS of (2) is strictly increasing in N.

Q.E.D.

Proof of Lemma 2

Let $\hat{\tau}$ and $\hat{\tau}'$ be the critical tax rates for agents with y and y' respectively. Let $y' > y$. Suppose $\hat{\tau}' < \hat{\tau}$. For the agent with income y',

$$V^u(\hat{\tau}', y', N(\hat{\tau}')) = V^r(\hat{\tau}', y')$$

Since $U(\cdot)$ is homothetic, we must have

$$H(1,\ \hat{\tau}'Y/N(\hat{\tau}')(1 - \hat{\tau}')y') = H(s_c,\ s_e)$$

where $H(\cdot)$ is homogeneous of degree one and s_c and s_e are the (after-tax) income shares of consumption and quality of education.

Since $\hat{\tau} > \hat{\tau}'$, for the agent with income y we must have

$$V^u(\hat{\tau}', y, N(\hat{\tau}')) < V^r(\hat{\tau}', y).$$

This implies

$$H(1,\ \hat{\tau}'Y/N(\hat{\tau}')(1 - \hat{\tau}')y) < H(s_c,\ s_e)$$

which is a contradiction since, given tax rates,

$$H(1,\ \hat{\tau}'Y/N(\hat{\tau}')(1 - \hat{\tau}')y) > H(1,\ \hat{\tau}'Y/N(\hat{\tau}')(1 - \hat{\tau}')y').$$

Q.E.D

Proof of Lemma 3

For all $y \geq y_m$, we will show that $V(\tau_m, y) > V(\tau, y)$ for all $\tau \in (\tau_m, 1)$. It is clear from Fig. 2 that this is true for the voter with median income. Let N_m be the public sector enrollment evaluated at the tax rate τ_m i.e., $N_m = N(\tau_m)$. Now, consider any voter with income y for whom $\hat{\tau}(y) < \tau^u(y)$. For any such voter we have

$$V(\tau_m, y) = V^u(\tau_m, y, N_m) > V^u(\tau, y, N(\tau)) = V(\tau, y) \text{ for all } \tau \in (\tau_m, 1].$$

The inequality uses the fact that $\tau^u(\cdot)$ is decreasing in y. For any voter with income y such that $\hat{\tau}(y) \geq \tau^u(y)$ we have

$$V(\tau_m, y) = V^r(\tau_m, y) > V^r(\tau, y) = V(\tau, y) \text{ for all } \tau \in (\tau_m, 1]$$

since $V^r(\cdot)$ is decreasing in τ for each y. Hence, no alternative in $(\tau_m, 1]$ has more than 50 % to beat τ_m.

Q.E.D.

Proof of Lemma 4

Since $\hat{\tau}(\cdot)$ is increasing in y, it is easy to see from Fig. 2 that, for all $y \le y_m$, $V^u(\cdot)$ is increasing in τ over the interval $[\hat{\tau}_m, \tau_m)$ i.e.,
$V(\tau_m, y) = V^u(\tau_m, y, N_m) > V^u(\tau, y, N(\tau)) = V(\tau, y,)$ for all $y \le y_m$, $\tau \in [\hat{\tau}_m, \tau_m)$. Hence, no tax rate in $[\hat{\tau}_m, \tau_m)$ can beat τ_m.
Q.E.D.

Proof of Lemma 5

(i) Homotheticity of $U(\cdot)$ implies that $V^r(0, y_m) < V^u(\tau_m, y_m, N_m)$ if and only if $H(s_c, s_e) < H(1 - \tau_m, \tau_m Y / N_m y_m)$. The result follows since

$$H(1 - \tau_m,\ \tau_m Y / N_m y_m) < H(1 - \tau_m,\ \tau_m Y / N_m y) \text{ for all } y < y_m.$$

(ii) Proof is similar to (i).
Q.E.D.

Proof of Proposition 3

By Lemma 3 and Lemma 4 we have shown that no tax rate in $[\hat{\tau}_m, \tau_m) \cup (\tau_m, 1]$ beats τ_m. We have to show that if $\tau \in [0, \hat{\tau}_m)$ then τ is not preferred to τ_m by more than 50 % of the population. Consider an arbitrary voter with income $y \le y_m$. He would prefer τ_m to any tax rate in $[0, \hat{\tau}(y)]$ since

$$V(\tau, y) = V^r(\tau, y) < V^r(0, y) < V^u(\tau_m, y, N_m) = V(\tau_m, y) \text{ for all } [0, \hat{\tau}(y)].$$

The first inequality holds since $V^r(\cdot)$ is decreasing in τ and the second inequality follows from part (i) of Lemma 5. For any tax rate in $(\hat{\tau}(y), \hat{\tau}_m)$, this individual would prefer τ_m since $V^u(\cdot)$ is increasing over the interval $(\hat{\tau}(y), \hat{\tau}_m)$. Hence, the pair $\{\tau_m, N_m\}$ is a majority voting equilibrium.
Q.E.D.

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