The Economics of Rotating Savings and Credit Associations
By T. Besley, S. Coate and G. Loury
AER (September,1993)

Introduction

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- *random Rosca* and *bidding Rosca*

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- We assume all individuals have identical and inter-temporally additive preferences. We take discount factor $\beta = 1$.

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- Each individuals instantaneous utility depends on nondurable consumption, c, and on whether or not he owns the durable or not. So it is $v(1, c)$ when individual owns the durable and it is $v(0, c)$ when he does not own the durable. v is assumed to be increasing, strictly concave and three times continuously differentiable in the second argument.

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- Define

$$\mu(\alpha) = \underset{0 \leq c \leq y}{\text{minimize}} \left[\frac{v(1,y) - v(\alpha,c)}{y-c}\right] \tag{3}$$

Lemma

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Under the assumptions on preferences set out above,
a) the minimized cost $\mu(.)$ *is a decreasing, concave function of* α*, and the cost-minimizing consumption rate* $c^*(.)$ *is an increasing function of* α*.*
b) Both are twice continuously differentiable on [0,1], where they satisfy the identity $\mu(\alpha) \equiv v'(\alpha, c^*(\alpha))$*.*
c) Moreover, if $v'''(i, c) > 0$ *for* $i = 0 and 1$*, and if* $\Delta v''(c) \geq 0$*, then* $c^*(.)$ *is strictly convex.*

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- Each time the Rosca meets, an individual is randomly selected to receive the pot of B, allowing him to buy the durable.
- Each individual continues to save at the rate B/t_a over the interval $[0, t_a]$, as under the autarky. But expected time to get the durable is $t_a(n+1)/2n$.

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- Each member saves at the rate B/t over the life of the Rosca and thus nondurable consumption is $c = y - B/t$ during this period.

Theorem 1

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a) By forming a random Rosca, group members raise their expected lifetime utilities.
b) The optimal random Rosca involves members saving at a lower rate over longer interval than under Autarky. Nevertheless, if $v'''(i, c) > 0$ for $i = 0$ and 1, and if $\Delta v''(c) \geq 0$, then individuals expect to receive the durable good sooner in the optimal random Rosca than under Autarky (i.e $t_r > t_a > (n+1)t_r/2n$).

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- Since individuals are identical in preferences this bidding protocol should give rise to indifference of individuals regarding bid/ receipt-date pairs.
- Also at each time, the contribution of all agents should sum up to the cost of durable good i.e. B. Efficiency dictates that auction procedure should be structured such that the total bids are just adequate to finance acquisition of the durable by recipient of the pot at each meeting date. (No savings within or outside Rosca)

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- Notice that this auctioning protocol is indirect form of second price auction.

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- If a Rosca member i bids b_i, he will have nondurable consumption $c_i = y_i - (n/t)b_i$ at each moment during the Rosca's life. Thus we can characterize the Rosca in terms of the consumption rates: $\{c_i\}_{i=1}^n$.

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- Condition (ii) implies that individual i's equilibrium utility level is

$$t\left[\frac{i}{n}.v(0, c_i) + \frac{n-i}{n}.v(1, c_i)\right] + (T-t).v(1, y) \tag{4}$$

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- $v(\alpha_i, c_i) = x \quad i = 1, 2, ..., n$ (common average utility during the life of Rosca of length t)
- $t(y - \overline{c}) = B$ where $\overline{c} \equiv (1/n)\sum_{i=1}^{n} c_i$ (average nondurable consumption rates during the life of Rosca)

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- $$\bar{c}(x) \equiv \left(\frac{1}{n}\sum_{i=1}^{n}\hat{c}(\alpha_i, x)\right) \tag{6}$$
- $$W_b = T.v(1,y) - B.\mu_b \tag{7}$$
- Where
$$\mu_b = \underset{x}{\text{minimize}} \left[\frac{v(1,y)-x}{y-\bar{c}(x)}\right] \tag{8}$$

Theorem 2

Theorem

a) By forming a bidding Rosca, group members raise their expected lifetime utilities relative to autarky.
b) Moreover if $1/v'(0,.)$ is concave, the optimal bidding Rosca involves group members saving at a lower average rate and over a longer interval than under autarky.

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- Theorem 2 also reveals that the last individual to acquire the durable in the bidding Rosca must have greater nondurable consumption during accumulation than under autarky i.e. $c_a < c_n$

Theorem 3

Theorem

a) Group members' expected utility will be higher if they use a random rather than a bidding Rosca.
b) If the value of the durable is independent of the nondurable consumption rate [i.e. $\Delta v^{'}(c) \equiv 0$] and if, $1/v'(0,.)$ is convex function, then the optimal random Rosca involves members saving at a lower rate over a longer interval than the optimal bidding Rosca.

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- Consider a more general scheme which randomly assigns members an order of receipt i, $1 \leq i \leq n$ and a consumption rate c, $0 \leq c \leq y$, but which requires neither equal consumption rates nor equal ex post utilities.
- If such a scheme is designed to maximize ex ante expected welfare, it would would equate individuals' marginal utilities: $v^{'}(\alpha_i, c_i) = v'(\alpha_j, c_j), 1 \leq i, j \leq n$.

*

Proof of Lemma

Lifetime utility under autarky can be written as

$$W_a = T \cdot v(1, y) - B \cdot \mu(0)$$

We assume that buying the durable good is profitable than non buying it i.e.

$$W_a = T \cdot v(1, y) - B \cdot \mu(0) \geq T \cdot v(0, y)$$

$$\Rightarrow \quad T\big(v(1, y) - v(0, y)\big) \geq B \cdot \mu(0)$$

$$\Rightarrow \quad \frac{T \Delta v(y)}{B} \geq \mu(0)$$

If this condition is satisfied solution exists (interior soln) for following optimisation problem.

$$\max_{0 \leq c \leq y} \quad T \cdot v(1, y) - B\left[\frac{v(1, y) - v(\alpha, c)}{y - c}\right]$$

where $\alpha \in [0, 1]$ is the probability of getting the durable good.

FOC $\Rightarrow$

$$B\left[\frac{-v'(\alpha, c^*)}{y - c^*} + \frac{\big(v(1, y) - v(\alpha, c^*)\big)}{(y - c^*)^2}\right] = 0$$

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$$\Rightarrow v'(\alpha, c^*) = \frac{v(1,y) - v(\alpha, c^*)}{y - c^*} \quad \text{—— (I)}$$

$$= \mu(\alpha)$$

as $\alpha \uparrow$ $v(\alpha, c^*) \uparrow$

$\Rightarrow \mu(\alpha) \downarrow$ in α

We now prove that $\mu(\alpha)$ is concave in α

Let $\alpha_3 = \lambda\alpha_1 + (1-\lambda)\alpha_2$ where $\lambda \in (0,1)$

$$\mu(\alpha_3) = \frac{v(1,y) - v(\alpha_3, c^*)}{y - c^*}$$

-- [c^* - optimal consumption at α_3]

$$= \frac{v(1,y) - \cancel{\lambda\alpha_1}\, \alpha_3 v(1,c^*) - (1-\alpha_3) v(0,c^*)}{y - c^*}$$

$$= \frac{\left[v(1,y) - \lambda\alpha_1 v(1,c^*) - (1-\lambda)\alpha_2 v(1,c^*) - (\lambda + (1-\lambda) - \lambda\alpha_1 - (1-\lambda)\alpha_2) v(0,c^*)\right]}{y - c^*}$$

$$= \frac{\lambda v(1,y) - \lambda\alpha_1 v(1,c^*) - \lambda(1-\alpha_1) v(0,c^*) + (1-\lambda) v(1,y) - (1-\lambda)\alpha_2 v(1,c^*) - (1-\lambda)(1-\alpha_2) v(0,c^*)}{y - c^*}$$

2

$$= \lambda\left[\frac{v(1,y)-v(\alpha_1,c^*)}{y-c^*}\right] + (1-\lambda)\left[\frac{v(1,y)-v(\alpha_2,c^*)}{y-c^*}\right]$$

$$\geq \lambda\, u(\alpha_1) + (1-\lambda)\, u(\alpha_2)$$

$\cdots$ [as c^* need not be optimal at α_1, α_2]

This proves $u(\alpha)$ is concave in α.

Now using envelope theorem

$$\frac{du}{d\alpha} = \frac{\partial v'(\alpha, c^*)}{\partial \alpha}$$

$$= \frac{\partial}{\partial \alpha}\left[\frac{v(1,y)-v(\alpha,c^*)}{y-c^*}\right] \quad \cdots \text{from (I)}$$

$$\frac{du}{d\alpha} = \frac{-\Delta v(c^*)}{y-c^*} \quad \text{——— (II)}$$

Using implicit function theorem we get desired differentiability of $u(\cdot)$ and $c^*(\cdot)$

Now differentiate $v'(\alpha, c^*) = u(\alpha)$ wrt α

$$\Rightarrow \frac{dv'(\alpha,c^*)}{d\alpha} = \frac{du(\alpha)}{d\alpha}$$

Using (II) and chain rule.

$$\frac{\partial v'(\alpha, c^*)}{\partial c^*} \cdot \frac{\partial c^*}{\partial \alpha} + \frac{\partial v'(\alpha, c^*)}{\partial \alpha} = u'(\alpha)$$

$$\Rightarrow v''(\alpha, c^*) \frac{\partial c^*}{\partial \alpha} + \left[v'(1, c^*) - v'(0, c^*) \right] = \frac{-\Delta v(c^*)}{y - c^*} \quad \text{—— III}$$

$$\Rightarrow \frac{\partial c^*}{\partial \alpha} = -\frac{1}{v''(\alpha, c^*)} \left[\frac{\Delta v(c^*)}{y - c^*} + \Delta v'(c^*) \right]$$

given concavity of $v(i, \cdot)$ and $\Delta v' \geq 0$

we get $\dfrac{\partial c^*}{\partial \alpha} > 0$

differentiating III w.r.t to α

$$\frac{\partial^2 c^*}{\partial \alpha^2} v''(\alpha, c^*) + \frac{\partial c^*}{\partial \alpha} \left[\frac{\partial v''(\alpha, c^*)}{\partial c^*} \cdot \frac{\partial c^*}{\partial \alpha} + \frac{\partial v''(\alpha, c^*)}{\partial \alpha} \right]$$

$$+ \frac{\partial}{\partial c^*} \left[v'(1, c^*) - v'(0, c^*) \right] \frac{\partial c^*}{\partial \alpha} = u''(\alpha)$$

$$\Rightarrow \frac{\partial^2 c^*}{\partial \alpha^2} = \frac{1}{v''(\alpha, c^*)} \left[u''(\alpha) - 2 \Delta v''(c^*) \frac{d c^*}{d \alpha} \cancel{v'''(\alpha, c^*)} - v'''(\alpha, c^*) \left(\frac{\partial c^*}{\partial \alpha} \right)^2 \right]$$

Now $u''(\alpha) < 0$.

If assume $\Delta v''(c^*) \geq 0$

and $v'''(\alpha, c^*) \geq 0$

$$\Rightarrow \frac{d^2 c^*}{d\alpha^2} > 0$$

$\Rightarrow$ c^* increasing in α and strictly convex in α

Proof of Proposition 1 (Reverse Argument)

Here we prove only the (b) part of statement (as per the presentation)

We need to prove that the expected receipt date under the optimal random Rosca is sooner than that under Autarky

$$\Leftrightarrow \quad t_a > \frac{(n+1)t_r}{2n} = (1-\bar{\alpha})\, t_r$$

where $\bar{\alpha} = \dfrac{n-1}{2n}$

and $\dfrac{(n+1)t_r}{2n}$ is the expected receipt date under optimal random Rosca.

using the identity $t_a = B/y - c^*(0)$ &

$t_r = B/y - c^*(\bar{\alpha})$

it is sufficient to show that

$$y - c^*(\bar{\alpha}) > (1-\bar{\alpha})\left[y - c^*(0)\right]$$

Notice that $y = c^*(1)$

$$\bar{\alpha}\, c^*(1) + (1-\bar{\alpha})\, c^*(0) > c^*(\bar{\alpha})$$

which is true due to concavity of $c^*(\,)$ in α .. (by lemma)

Proof of proposition 2

This proposition compares bidding Rosca with autarky.

Proof of part (a) of proposition (As per ppt).

$$W_b = T \cdot v(1,y) - B\mu_b$$

$$\text{where} \quad \mu_b = \min_{x} \left[\frac{v(1,y) - x}{y - \bar{c}(x)} \right]$$

$$W_a = T \cdot v(1,y) - B\mu(0)$$

$$W_b - W_a = B\left[\mu(0) - \mu_b\right]$$

If we show $\mu(0) > \mu_b$ then $W_b > W_a$ is proven

$$v(\alpha, \hat{c}(\alpha, x)) = x$$

differentiate wrt to α

$$\frac{\partial v(\alpha, \hat{c}(\alpha, x))}{\partial \hat{c}} \frac{\partial \hat{c}}{\partial \alpha} + \frac{\partial v(\alpha, \hat{c})}{\partial \alpha} = 0$$

$$\Rightarrow \frac{\partial \hat{c}}{\partial \alpha} v'(\alpha, \hat{c}) = \Delta v(\hat{c})$$

$$\Rightarrow \frac{\partial \hat{c}}{\partial \alpha} = \frac{-\Delta v(\hat{c})}{v'(\alpha, \hat{c})} < 0$$

[Also used in proof of proposition 3]

$\hat{c}$ decreases in α —— (I)

Consider any common average utility level of the member during the life of bidding Rosca .... Let it be x

$$\bar{c}(x) = \frac{1}{n} \sum_{i=1}^{n} \hat{c}(\alpha_i, x)$$

pick $x = v(0, c)$

$$\bar{c}(v(0,c)) = \frac{1}{n} \sum_{i=1}^{n} \hat{c}(\alpha_i, v(0,c))$$

$$< \frac{1}{n} \sum_{i=1}^{n} \hat{c}(0, v(0,c)) = c$$

(inequality follows from (I))

(equality follows from defⁿ of $\hat{c}$)

pick $c = c_a$

$$\bar{c}(\nu(0, c_a)) < c_a$$

$$\Rightarrow \quad y - c_a < y - \bar{c}(\nu(0, c_a))$$

$$\Rightarrow \quad \frac{\nu(1,y) - \nu(0,c_a)}{y - \bar{c}(\nu(0,c_a))} < \frac{\nu(1,y) - \nu(0,c_a)}{y - c_a} = \mu(0)$$

but

$$\mu_b = \frac{\nu(1,y) - x^*}{y - \bar{c}(x^*)} \leq \frac{\nu(1,y) - \nu(0,c_a)}{y - \bar{c}(\nu(0,c_a))}$$

$\because$ by defⁿ of μ_b

$$\Rightarrow \quad \mu_b < \mu(0)$$

This proves part (a)

Proof of part (b)

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Part b.

$$\mu(0) = v'(0, c_a) > u_b$$

$$u_b = \min_x \left[\frac{v(1,y) - x}{y - \bar{c}(x^*)} \right]$$

FOC

$$\frac{\partial}{\partial x}\left[\frac{v(1,y) - x}{y - \bar{c}(x)} \right] = 0$$

$$\Rightarrow \frac{-\cancel{[y - \bar{c}(x^*)]}}{[y - \bar{c}(x^*)]^2} + \frac{[v(1,y) - x^*] \left.\frac{\partial \bar{c}(x)}{\partial x}\right|_{x^*}}{[y - \bar{c}(x^*)]^2} = 0$$

$$\Rightarrow \quad \mu_b = \frac{1}{\left.\frac{\partial \bar{c}(x^*)}{\partial x}\right|_{x^*}} \quad \text{—— (I)}$$

$$\frac{\partial \bar{c}(x^*)}{\partial x} = \frac{\partial}{\partial x}\left[\frac{1}{n} \sum_{i=1}^{n} \hat{c}(\alpha_i, \cancel{x}\ \cancel{v(\alpha_i, x)} x) \right]$$

$$= \frac{1}{n} \sum_{i=1}^{n} \frac{\partial \hat{c}(\alpha_i, x)}{\partial x} \quad \text{—— (II)}$$

Now ~~kn~~ we know that for any agent i

$$v(\alpha_i, \hat{c}(\alpha_i, x)) = x$$

$$\frac{\partial v(\alpha, \hat{c})}{\partial \hat{c}} \cdot \frac{\partial \hat{c}}{\partial x} = 1 \Rightarrow \frac{\partial \hat{c}}{\partial x} = [v'(\alpha_i, \hat{c})]^{-1}$$

$$\left.\frac{\partial \overset{*}{c}}{\partial x}\right|_{x^*} = \frac{1}{n}\sum_{i=1}^{n}\left[v'(\alpha_i, \hat{c})\right]^{-1}$$

$$\Rightarrow \quad \mu_b = \left[\frac{1}{n}\sum_{i=1}^{n}\left[v'(\alpha_i, \hat{c}(\alpha_i, x^*))\right]^{-1}\right]^{-1}$$

<u>or</u>

$$v'(0, c_a) = \left[\frac{1}{n}\sum_{i=1}^{n}\left[v'(\alpha_i, \hat{c}(\alpha_i, x^*))\right]^{-1}\right]^{-1} \quad \text{— III}$$

Now $\dfrac{1}{v'(0,\cdot)}$ is concave

$$\Rightarrow \quad \frac{1}{v'(0, \bar{c})} \geq \left[\sum_{i=1}^{n}\frac{1}{v'(0, \hat{c}(\alpha_i, x^*))}\right]\frac{1}{n}$$

$$\Rightarrow \left[\frac{1}{n}\sum_{i=1}^{n}\left[v'(0, \hat{c}(\alpha_i, x^*))\right]^{-1}\right]^{-1} \geq v'(0, \bar{c}(x^*)). \quad \text{— IV}$$

~~Nt~~ <u>Note</u>, $\Delta v'(c) \geq 0$

$$\Rightarrow \quad \alpha\left[v'(1,c) - v'(0,c)\right] \geq 0 \qquad 0 \leq \alpha \leq 1$$

$$\Rightarrow \quad \alpha v'(1,c) + (1-\alpha) v'(0,c) \geq v'(0,c)$$

$$\Rightarrow \quad v'(\alpha, c) \geq v'(0,c)$$

Using this & III ~~can~~ ~~to~~ we get

$$v'(0,c_a) > \left[\frac{1}{n}\sum_{i=1}^{n}\left[v'(0,\hat{c}(\alpha_i,x^*))\right]^{-1}\right]^{-1}$$

By IV

$$v'(0,c_a) > v'(0,\bar{c}(x^*))$$

Now by concavity of $v(0,\cdot)$ we get

$$\boxed{\bar{c}(x^*) > c_a}$$

∎

Revise

$$\max_{t,c} \; t\,v(0,c) + (T-t)\,v(1,y)$$

s.t. $t(y-c) = B \longrightarrow$ cost of ~~non~~ durable good

$0 \le c \le y$

$$\iff \max_{c} \; T\cdot v(1,y) - B\left[\frac{v(1,y)-v(0,c)}{y-c}\right]$$

s.t. $0 \le c \le y$

or $$\min_{0\le c\le y}\left[\frac{v(1,y)-v(0,c)}{y-c}\right] = u(0)$$

Proof of proposition 3.

Here we compare bidding Rosca with the Random Rosca

Proof of part (a).

$$W_r - W_b = B\left[\mu_b - \mu(\bar{\alpha})\right]$$

from

$$W_r = T\cdot\nu(1,y) - B\mu(\bar{\alpha})$$

where $\bar{\alpha} = \dfrac{n-1}{2n}$ and

$$W_b = T\cdot\nu(1,y) - B\mu_b ;$$

We need to prove $\mu_b > \mu(\bar{\alpha})$.

We re-write

$$\mu(\bar{\alpha}) = \min_{0\le c\le y}\left[\frac{\nu(1,y)-\nu(\bar{\alpha},c)}{y-c}\right]$$

$$= \min_{x}\left[\frac{\nu(1,y)-x}{y-\hat{c}(\bar{\alpha},x)}\right].$$

Now, $\mu_b > \mu(\bar{\alpha})$.

$$\iff \frac{\nu(1,y)-x}{y-\bar{c}(x)} > \frac{\nu(1,y)-x}{y-\hat{c}(\alpha,x)} \quad \forall x .$$

Date: / /

$$\Longleftrightarrow \quad \bar{c}(x) > \hat{c}(\bar{\alpha}, x) \quad \forall x.$$

$$\Leftrightarrow \left[\frac{1}{n} \sum_{i=1}^{n} c(\alpha_i, x) > \hat{c}\left(\frac{1}{n} \sum_{i=1}^{n} \alpha_i, x \right) \quad \forall x \right] \quad \text{I}$$

where $\alpha_i = \dfrac{n-i}{n}$

Notice that

$$\bar{\alpha} = \frac{n-1}{2n} = \frac{1}{n} \sum_{i=1}^{n} \frac{n-i}{n}$$

$$= \frac{1}{n} \left[n - \frac{n(n+1)}{n \times 2} \right]$$

$$= \frac{1}{\cancel{n}} \cancel{n} \left[\frac{n-1}{2n} \right]$$

i.e. $\bar{\alpha}$ is convex combination of $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$.

~~by convexity of c() proven~~.

so I will hold for ~~Ɐx~~ if $\hat{c}(\cdot, x)$ is convex in α.

We now prove that $\hat{c}(\cdot, x)$ is convex in its first argument

we need to show

$$\frac{d\hat{c}}{d\alpha} < 0 \quad \text{and} \quad \frac{d^2\hat{c}}{d\alpha^2} > 0$$

$$\frac{d\hat{c}}{d\alpha} = \frac{-\Delta v(\hat{c})}{v'(\alpha, \hat{c})} < 0$$

$$\Leftrightarrow \quad v'(\alpha, \hat{c}) \frac{d\hat{c}}{d\alpha} + \Delta v(\hat{c}) = 0$$

[we get this by differentiating $v(\alpha, \hat{c}(\alpha, x)) = x$ wrt. α]

Again differentiating wrt to α

$$0 = \Delta v'(\hat{c}) \frac{\partial \hat{c}}{\partial \alpha} + v'(\alpha, \hat{c}) \frac{d^2\hat{c}}{d\alpha^2} + \frac{\partial \hat{c}}{\partial \alpha}\left[v''(\alpha, \hat{c}) \frac{\partial \hat{c}}{\partial \alpha} + \Delta v'(\hat{c}) \right]$$

$$\Rightarrow \frac{\partial^2 \hat{c}}{\partial \alpha^2} = \frac{-1}{v'(\alpha, \hat{c})}\left[\underbrace{\Delta v'(c)\frac{\partial \hat{c}}{\partial \alpha}}_{<0} + \underbrace{v''(\alpha, \hat{c})\left(\frac{\partial \hat{c}}{\partial \alpha}\right)^2}_{<0} + \underbrace{\Delta v'(\hat{c})\left(\frac{\partial \hat{c}}{\partial \alpha}\right)}_{<0} \right]$$

$$\frac{\partial^2 \hat{c}}{\partial \alpha^2} > 0$$

$\Rightarrow \hat{c}(\cdot, x)$ is convex in α

Hence part (a) is proven

Proof of part (b)

We need to prove that if $\Delta v'(c) \equiv 0$
(i.e. the value durable is independent of the non-durable consumption rate)
and $1/v'(0,\cdot)$ is a convex function, then optimal random rosca involves members saving at a lower rate over a longer interval that the optimal bidding Rosca

i.e. $t_r > t_b$

$$\Leftrightarrow \quad \frac{B}{y - c_r} > \frac{B}{y - \bar{c}(\alpha^*)}$$

$$\Leftrightarrow \quad c_r > \bar{c}(\alpha^*)$$ — [This ~~is~~ is what we will prove now]

Before we proved $\mu_b > \mu(\bar{\alpha})$ — I

$\Delta v'(c) = 0 \Rightarrow v'(\alpha_1, c) = v'(\alpha_2, c)$ — II

$\frac{1}{\nu'(0,\cdot)}$ is convex $\Rightarrow$ By Jensen's Inequality we get

$$\frac{1}{\nu'\left(0, \frac{1}{n}\sum_{i=1}^{n} c(\alpha_i, x^*)\right)} < \frac{1}{n}\sum_{i=1}^{n} \frac{1}{\nu'(0, c(\alpha_i, x^*))}.$$

$$\Rightarrow \left[\frac{1}{n}\sum_{i=1}^{n}\left[\nu'(0, c(\alpha_i, x^*))\right]^{-1}\right]^{-1} < \nu'(\bar{\alpha}, \bar{c}(x^*)).$$

we replace $\nu'\left(0, \frac{1}{n}\sum_{i=1}^{n} c(\alpha_i, x^*)\right) = \nu'(0, \bar{c}(x^*))$

with $\nu'(\bar{\alpha}, \bar{c}(x^*))$ ... from (II)

$\mu(\bar{\alpha}) = \nu'(\bar{\alpha}, c_\gamma)$ ... <u>By lemma</u>

$$< \mu_b = \left\{\frac{1}{n}\sum_{i=1}^{n}\left(\nu'(\bar{\alpha}, \hat{c}(\alpha_i, x^*))\right)^{-1}\right\}^{-1}$$

[... by (I). we get $\mu(\bar{\alpha}) < \mu_b$

$$\nu'(\bar{\alpha}, c_\gamma) < \nu'(\bar{\alpha}, \bar{c}(x^*))$$

$\Rightarrow \bar{c}(x^*) < c_\gamma$ ... by concavity of $\nu(\alpha, \cdot)$