Math 271: Mathematical Methods | Tridip Ray
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Homework 1 (Class Test on 19 August 2024)

1. Check in each of the following cases whether the given set of vectors is linearly independent:

   (a) $x^1 = (1, 2)\,,\; x^2 = (0, 0)\,;$

   (b) $y^1 = (0, 1, -2)\,,\; y^2 = (1, 1, 1)\,;\; y^3 = (1, 2, 3)\,.$

2. Let x, y and z be 3 linearly independent vectors in $\mathbb{R}^n$.

   (a) Are the vectors $(x - y)\,,\; (y - z)\,,\; (z - x)$ linearly independent? Explain clearly.

   (b) Are the vectors $(x + y)\,,\; (y + z)\,,\; (z + x)$ linearly independent? Explain clearly.

3. Recall the Fundamental Theorem on Vector Spaces:

   If each of the $(m + 1)$ vectors $y^0, y^1, ..., y^m$ in $\mathbb{R}^n$ can be expressed as a linear combination of the m vectors x^1, x^2, ..., x^m in $\mathbb{R}^n$, then the vectors $y^0, y^1, ..., y^m$ are linearly dependent.

   In what follows we will develop a proof of this theorem using the method of induction on m.

(a) Initial Step: Prove that the theorem is true for $m = 1$.

(b) Inductive Step: Assume that the theorem holds for $m = k - 1$, and let us prove it for $m = k$.

   – By hypothesis we have each of the $(k + 1)$ vectors, $y^0, y^1, ..., y^k$, can be expressed as a linear combination of the k vectors x^1, x^2, ..., x^k:

$$\begin{aligned}
y^0 &= a_{10}x^1 + a_{20}x^2 + ... + a_{k0}x^k, \\
y^1 &= a_{11}x^1 + a_{21}x^2 + ... + a_{k1}x^k, \\
y^2 &= a_{12}x^1 + a_{22}x^2 + ... + a_{k2}x^k, \\
&\;\;...\\
y^k &= a_{1k}x^1 + a_{2k}x^2 + ... + a_{kk}x^k.
\end{aligned}$$

We have to prove that the $(k+1)$ vectors, $y^0, y^1, ..., y^k$ are linearly dependent. If all a_{ij} are zero, the proof is immediate; so assume this is not the case. Then at least one number a_{ij} is not zero, say $a_{10} \neq 0$. Define

$$\begin{aligned} z^1 &\equiv y^1 - \left(\tfrac{a_{11}}{a_{10}}\right) y^0 = \left(a_{21} - \tfrac{a_{11}}{a_{10}} a_{20}\right) x^2 + ... + \left(a_{k1} - \tfrac{a_{11}}{a_{10}} a_{k0}\right) x^k, \\ z^2 &\equiv y^2 - \left(\tfrac{a_{12}}{a_{10}}\right) y^0 = \left(a_{22} - \tfrac{a_{12}}{a_{10}} a_{20}\right) x^2 + ... + \left(a_{k2} - \tfrac{a_{12}}{a_{10}} a_{k0}\right) x^k, \\ & ... \\ z^k &\equiv y^k - \left(\tfrac{a_{1k}}{a_{10}}\right) y^0 = \left(a_{2k} - \tfrac{a_{1k}}{a_{10}} a_{20}\right) x^2 + ... + \left(a_{kk} - \tfrac{a_{1k}}{a_{10}} a_{k0}\right) x^k. \end{aligned}$$

- **Question:** *Use the above construction to complete the inductive step.*

4. Let S be a set of vectors in $\mathbb{R}^2$ defined as follows:

$$S = \left\{(x_1, x_2) \text{ in } \mathbb{R}^2 \text{ such that } x_1 + x_2 = 4\right\}.$$

   (a) What is the rank of S? Explain clearly.

   (b) Find a basis of S, showing your procedure clearly.

5. Consider the set $S \subset \mathbb{R}^2$ defined as follows:

$$S = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 5\right\}.$$

   What is the rank of S? Explain clearly.

6. Let $S = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 = x_3\}$. Find a basis of S, showing your procedure clearly.

7. Let $x, y \in \mathbb{R}^n$.

   (a) Prove that $|xy| \leq \|x\| \cdot \|y\|$ (Cauchy-Schwarz Inequality).

   (b) Use (a) to prove that $\|x + y\| \leq \|x\| + \|y\|$ (Triangle Inequality).

8. Let A be an $n \times n$ matrix, and let $A^1, A^2, ..., A^n$ denote the n column vectors of A. Define an $n \times n$ matrix B as follows:

$$b_{ij} = A^i A^j \qquad i = 1, ..., n; \; j = 1, ..., n$$

   where b_{ij} is the element corresponding to the ith row and jth column of matrix B, and $A^i A^j$ denotes the inner product of the vectors A^i and A^j. Suppose A is non-singular. Does it follow that B is non-singular? Explain clearly.

9. (a) Suppose A is an $m \times n$ matrix and B is an $n \times r$ matrix. Show that

$$\text{rank }(AB) \leq \min\{\text{rank }(A),\ \text{rank }(B)\}.$$

(b) Give an example of 2×2 matrices A and B such that

$$\text{rank }(AB) \neq \min\{\text{rank }(A),\ \text{rank }(B)\}.$$

(c) Prove that if a matrix of rank k is multiplied in either order (that is, either premultiplied or postmultiplied) by a nonsingular matrix, the rank of the product is k.

10. Let A be an $m \times n$ matrix. *Row rank* of A is the rank of the set of row vectors, $\{A_1, A_2, ..., A_m\}$. *Column rank* of A is the rank of the set of column vectors, $\{A^1, A^2, ..., A^n\}$. Let $r =$ *row rank* of A, $c =$ *column rank* of A.

(a) Suppose that $\boldsymbol{r < c}$.

Reordering rows or columns of A does not affect its row or column rank. So choose a row basis for A which we may assume consists of the rows $\{A_1, A_2, ..., A_r\}$, and a column basis which we may assume consists of the columns $\{A^1, A^2, ..., A^c\}$.

(i) Let $\hat{A}_i = (a_{i1}, a_{i2}, ..., a_{ic})$ and consider the system of equations

$$\hat{A}_i \cdot y = 0,\ \ i = 1, 2, ..., r.$$

Prove that this system of equations has a *nonzero* solution $\bar{y}$.

(ii) Since $\{A_1, A_2, ..., A_r\}$ is a row basis, it follows from the Basis Theorem that, for all $k = 1, 2, ..., m$, $A_k = \sum_{i=1}^{r} \mu_{ik} A_i$ for some real numbers μ_{ik}. Hence

$$\hat{A}_k = (a_{k1}, a_{k2}, ..., a_{kc}) = \sum_{i=1}^{r} \mu_{ik} \hat{A}_i,$$

and so

$$\hat{A}_k \cdot \bar{y} = \sum_{i=1}^{r} \mu_{ik} \left(\hat{A}_i \cdot \bar{y}\right) = 0,\ \text{ for all } k = 1, 2, ..., m.$$

Prove that this leads to a *contradiction* which shows that our supposition $r < c$ is wrong, that is, it must be that $r \geq c$.

(b) Briefly sketch the argument that $r = c$.